Picard-Lefschetz theory

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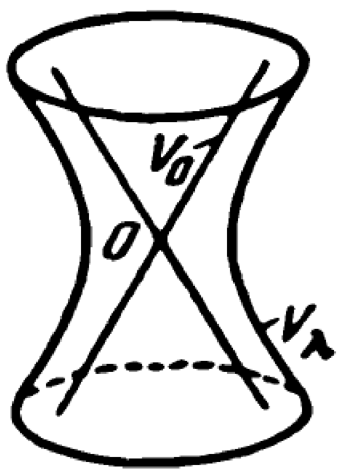

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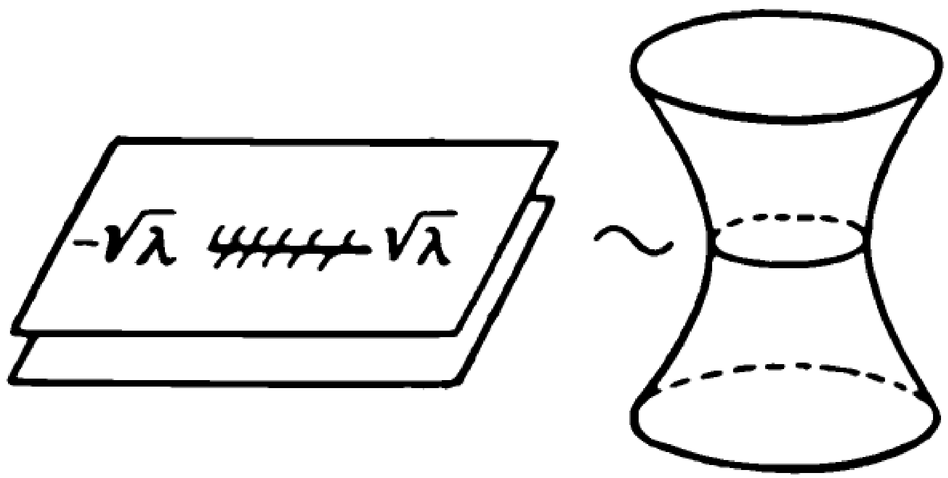
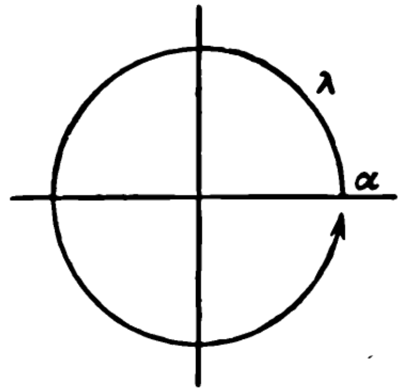
Another proof:



Monodromy 1

Consider the path $\lambda(t) = \exp(2\pi i t)\alpha\,,\ 0 \leq t \leq 1\,,\ \alpha > 0$



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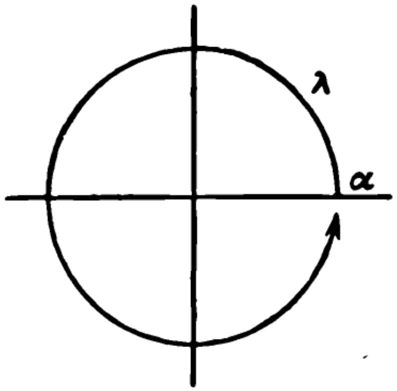

Trace how $V_\lambda(t) = \{(z, w) \mid w = \sqrt{\lambda(t) - z^2}\}$ changes as t goes from 0 to 1.

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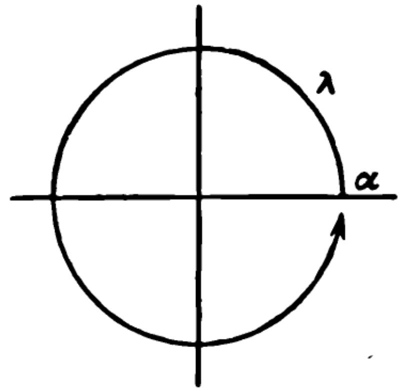

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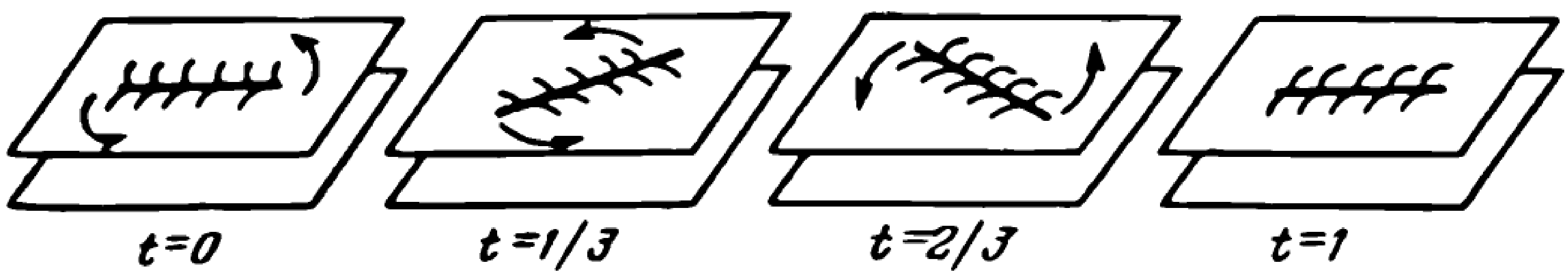

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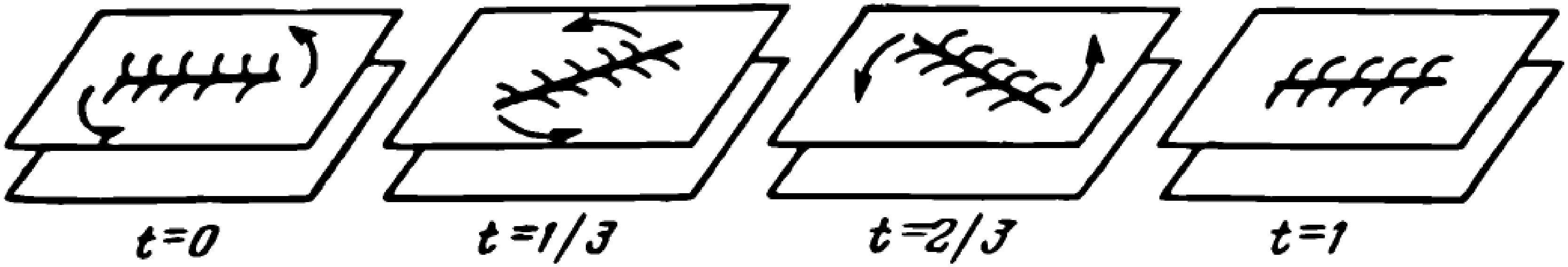

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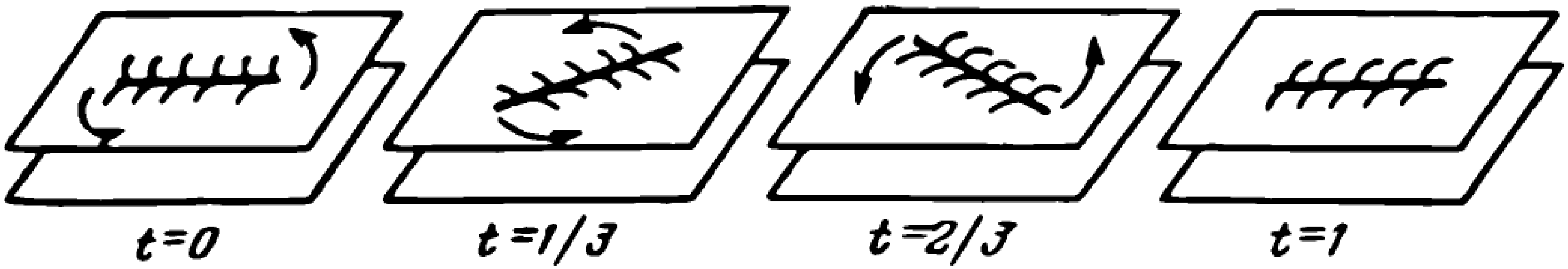

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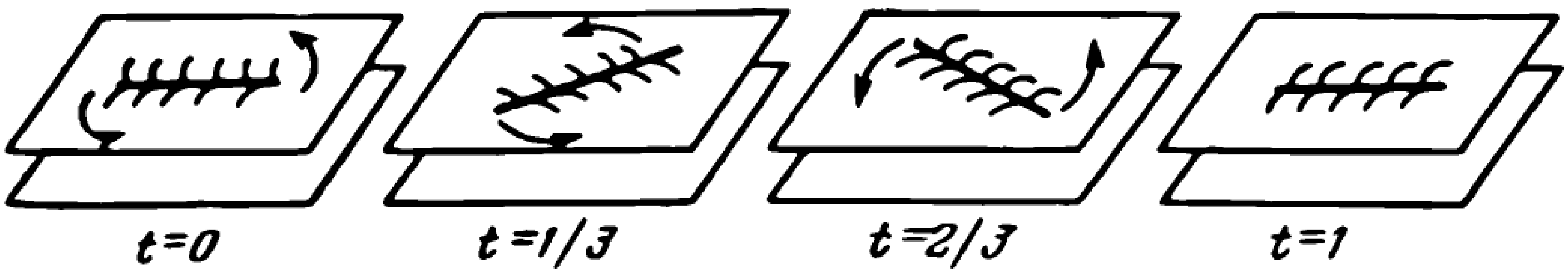

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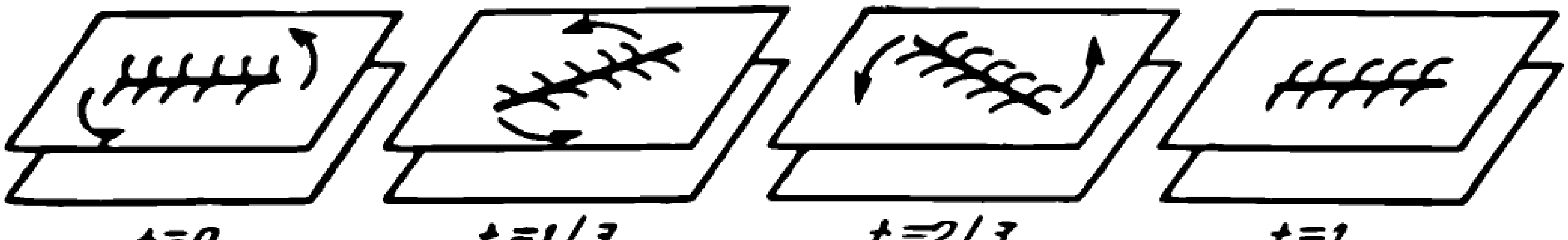

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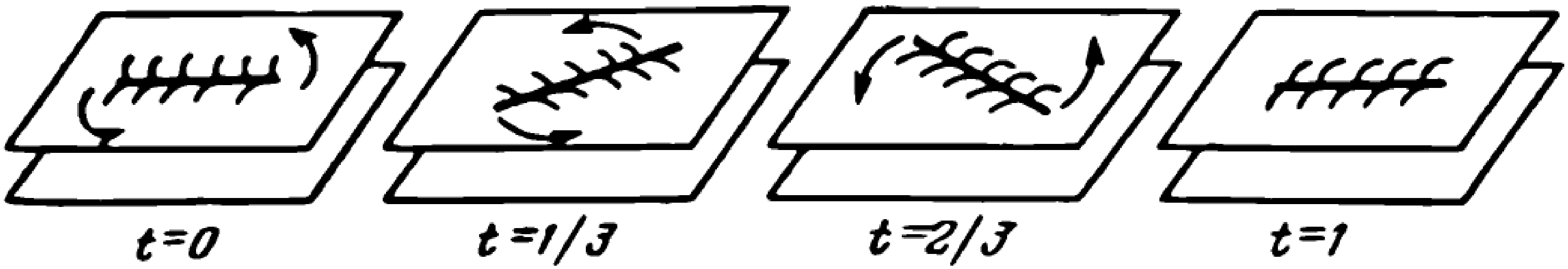

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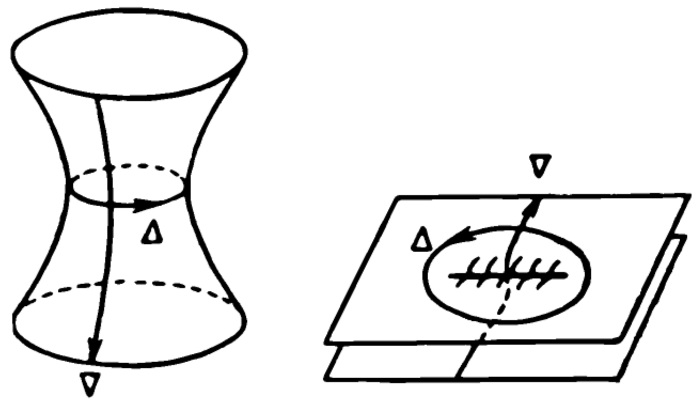
Let $g_t(z) = \exp\{\pi i t \chi(|z|)\} \cdot z$. This isotopy rotates the disk $\{z \mid |z| \le 2\sqrt{\alpha}\}$ and is identity outside $\{z \mid |z| \le 3\sqrt{\alpha}\}$.

Let Γ_t covers g_t.

Vanishing cycle

In order to see how Γ_1 acts, draw curves:



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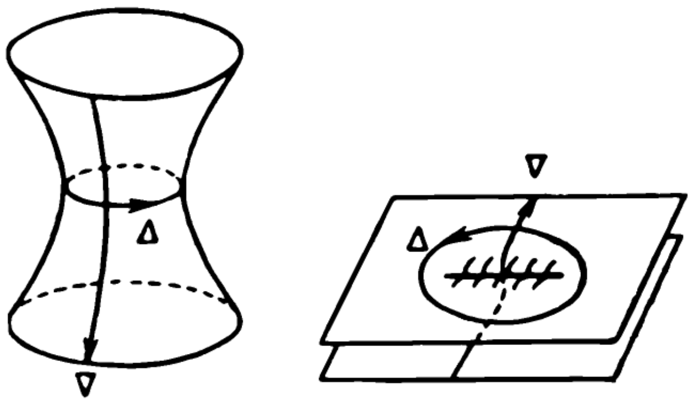

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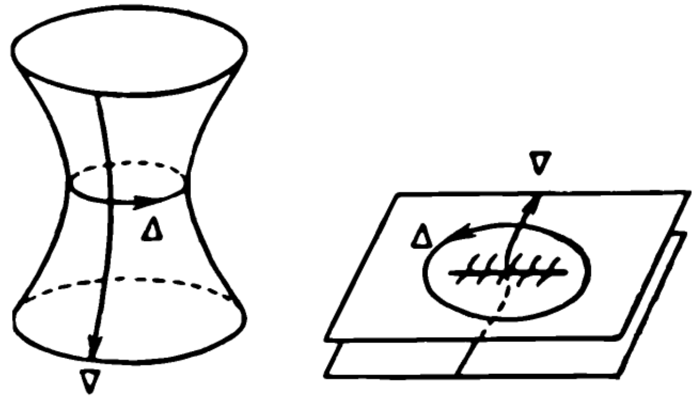

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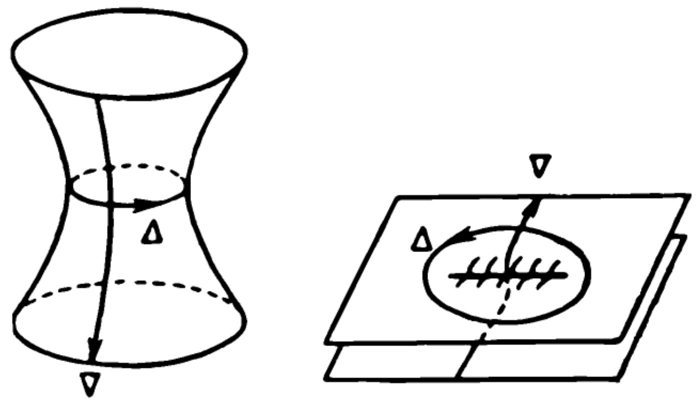

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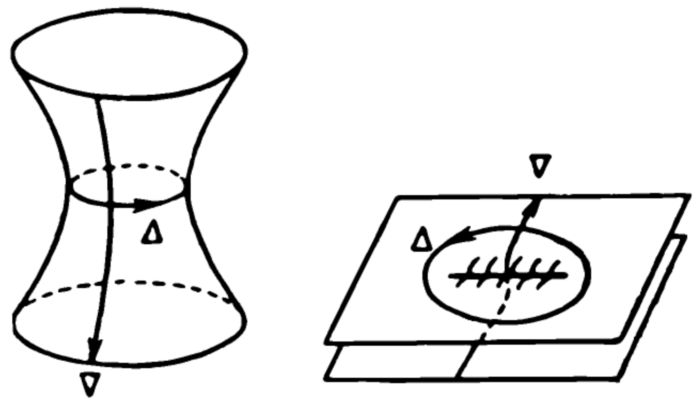

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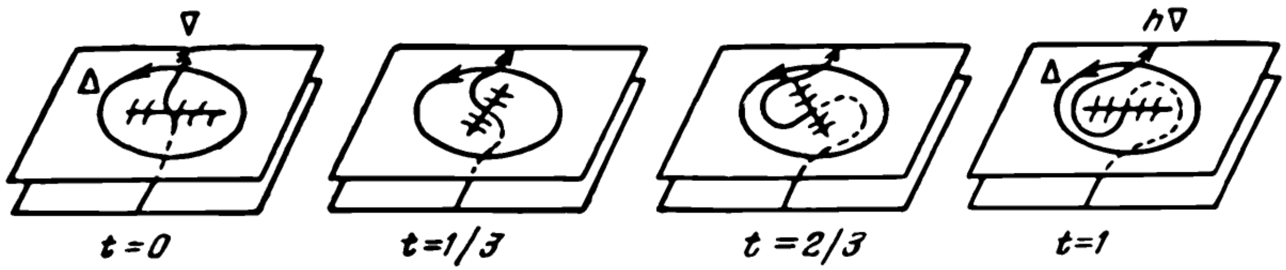

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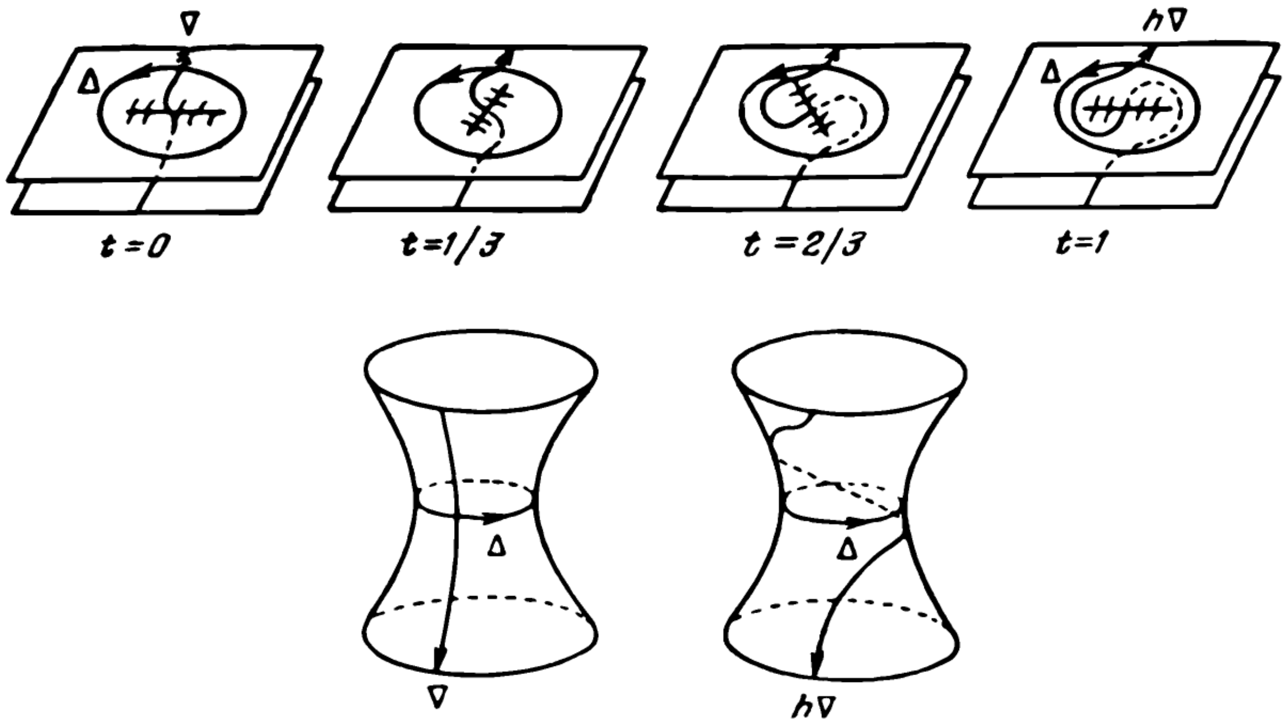

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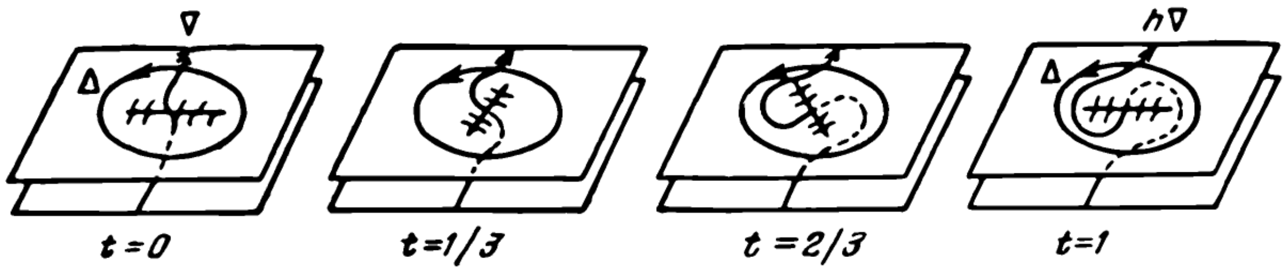

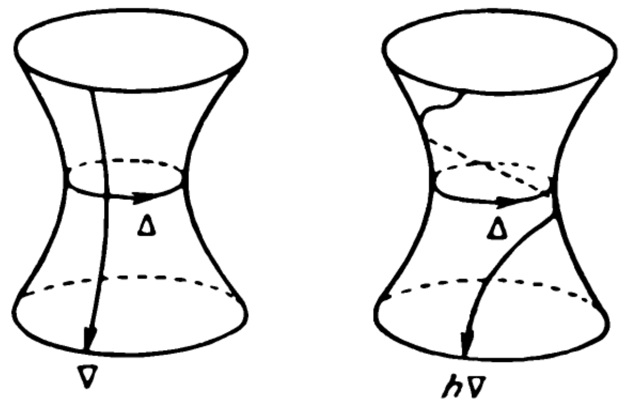

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$$h(\delta) = \delta + (\Delta \circ \delta)\Delta$$

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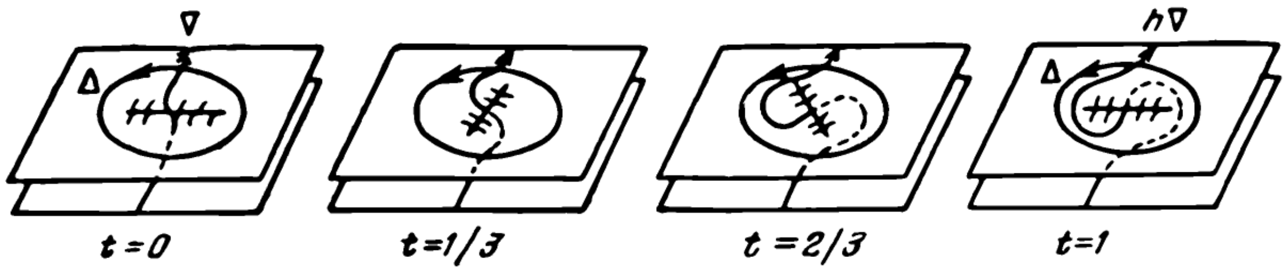

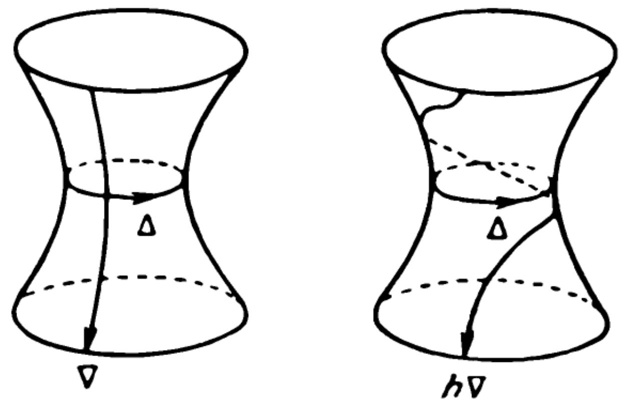

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In dim n $h(a) = a + (-1)^{(n-1)n/2}(a \circ \Delta)\Delta$