Lecture 2

Quantifiers

Introduction to quantifiers

In mathematics, many statements depend on variables
that can take values from a certain set (the **universe** or domain of discourse).

To express properties about "all" or "some" elements of this universe,
we use **quantifiers**.

These quantifiers allow us to convert **predicates** –
statements whose truth depends on variables –
into **propositions** – statements that are either true or false.

In this lecture, we will explore the two most common quantifiers:

- The universal quantifier $\forall$, meaning "for all"
- The existential quantifier $\exists$, meaning "there exists."

We will study how to read, interpret, and manipulate
statements involving quantifiers,
and understand their crucial role in mathematical logic.

Quantifiers: first examples

Example. Let x be a variable that may take any real value.
The set of all real numbers is denoted by $\mathbb{R}$. For short, we write $x \in \mathbb{R}$.

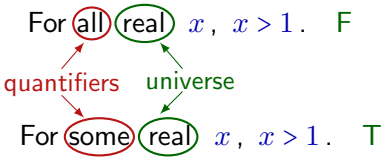
The set of all admissible values of a variable is called the **universe** (of discourse).

In our example, the universe is the set of real numbers $\mathbb{R}$.

Consider the statement "$x > 1$." Is it a proposition?
No, because its truth value is not fixed, it depends on x.
Such statement is called a **predicate** (**open sentence**).

To turn a predicate into a **proposition**, we must say something about x.

Compare:



What do quantifiers do?

A **quantifier** is placed in front of a variable
to bind it, turning the statement into a proposition.

There are two quantifiers,
a **universal quantifier**, denoted by the symbol $\forall$,
and **existential quantifier**, denoted by the symbol $\exists$.

The sentence "**For all real** x**,** $x > 1$" is written in symbols as

$$\forall\, x \in \mathbb{R}\ \ x > 1\,.$$

universal quantifier

The sentence "**For some real** x**,** $x > 1$" is written in symbols as

$$\exists\, x \in \mathbb{R}\ \ x > 1\,.$$

existential quantifier

The two sentences above are called the **quantified sentences**.
A variable placed next to a quantifier is said to be **bound** by the quantifier.
A variable which is **not** bound by any quantifier is called **free**.

How to read quantified sentences

Let $P(x)$ be an open sentence depending on a free (non-bound) variable x.

We may bound x by a quantifier and get $\forall x\, P(x)$ or $\exists x\, P(x)$.

The quantified sentence $\forall x\, P(x)$ has the following wordings:

- For **any** x, $P(x)$
- For **all** x, $P(x)$
- For **each** x, $P(x)$
- For **every** x, $P(x)$
- For **arbitrary** x, $P(x)$

The quantified sentence $\exists x\, P(x)$ has the following wordings:

- For **some** x, $P(x)$
- There **exists** x such that $P(x)$

Unique existential quantifier $\exists!$

Consider the statement

There exists a unique integer x such that $1 < x < 3$.

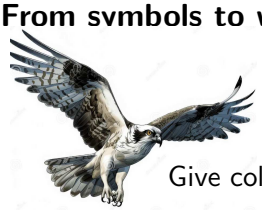
Symbolically this statement is written as $\exists!\, x \in \mathbb{Z} \;\; 1 < x < 3$.

The symbol $\exists!$ stands for the **unique existential quantifier**.

The unique existential quantifier
can be expressed in terms of the universal and existential quantifiers:

$$\exists!\, x\, P(x) \iff (\exists\, x\, P(x)) \wedge (\forall x_1 \forall x_2\, P(x_1) \wedge P(x_2) \implies x_1 = x_2)$$

From symbols to words



Example. Let B be the set of all birds, and $F(x)$ be the predicate "x can fly."
Give colloquial English equivalents for the following phrases:

$\forall\, x \in B \;\; F(x)$	**All** birds can fly.	F
$\exists\, x \in B \;\; F(x)$	**Some** birds can fly.	T
$\forall\, x \in B \;\; \neg F(x)$	All birds are unable to fly. No birds can fly.	F
$\exists\, x \in B \;\; \neg F(x)$	Some birds can't fly.	T

From symbols to words

Example. Let the universe (where the variable take values) be the set of people.

Let $W(x)$ be the predicate "x likes whisky" and
$S(x)$ be the predicate "x is a Scot."

Give a colloquial English expressions for the following quantified sentences:

$\forall x \ (S(x) \implies W(x))$	All Scots like whisky.
$\forall x \ (W(x) \implies S(x))$	Each person who likes whisky is a Scot.
$\exists x \ (S(x) \wedge \neg W(x))$	Some Scots don't like whisky.
$\forall x \ (\neg W(x) \implies \neg S(x))$	Anyone who doesn't like whisky can't be a Scot.

From words to symbols

Example 1. Write the following sentence in a **symbolic** form:

Some integers are even.

$\exists$ universe $\mathbb{Z}$ divisible by 2

We need to introduce a **variable** for an integer. Call it n.

Then the phrase "**Some integers are even**" gets the following symbolic writing:

$\exists n \in \mathbb{Z} \quad 2|n$. It may be written also as $\exists n \ (n \in \mathbb{Z} \wedge 2|n)$.

Example 2. Write the following sentence in a **symbolic** form:

Every rational number is a real number.

$\forall$ $\mathbb{Q}$ $\stackrel{?}{\Longrightarrow}$ $\mathbb{R}$

Introduce a variable x. Then the sentence will be written as

$$\forall x \, (x \in \mathbb{Q} \implies x \in \mathbb{R})$$

Do we understand quantified sentences?

Determine the **truth value** for each of the following statements:

$\exists\, n \in \mathbb{Z} \quad n^2 = 9$ T Indeed, there exists such n, namely $n = 3$.

$\exists!\, n \in \mathbb{Z} \quad n^2 = 9$ F Such n is **not** unique: $n = 3$ and $n = -3$ will work.

$\forall\, n \in \mathbb{Z} \quad n^2 = 9$ F **Not all** n posses this property.

For example, $n = 1$ doesn't work.

$\exists\, n \in \mathbb{Z} \quad n^2 = 5$ F There is **no** integer whose square is 5.

$\forall\, n \in \mathbb{Z} \quad n^2 \neq 5$ T Whatever integer you take, its square is not 5.

$\exists\, n \in \mathbb{Z} \quad 3|(n^2 - 1)$ T Indeed, there exists such n, for example, $n = 2$.

$\forall\, n \in \mathbb{Z} \quad 3|(n^2 - 1)$ F **Not all** n posses this property.

For example, $n = 3$ doesn't work.

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Precaution

How are the quantified sentences $\forall\, x\, P(x)$ and $\exists\, x\, P(x)$ related?

$$\forall\, x\, P(x) \implies \exists\, x\, P(x)\text{, provided the universe is not empty.}$$

The **converse** is **not** true:

$$\forall\, x\, P(x) \not\Longleftarrow \exists\, x\, P(x)$$

The erroneous implication $\exists\, x\, P(x) \implies \forall\, x\, P(x)$

is a familiar mistake in students' writing. ☹

Example. Prove that $n^3 - n$ is divisible by 3 for all integers n.

"Solution." $n^3 - n\big|_{n=1} = 0$, $n^3 - n\big|_{n=2} = 8 - 2 = 6$, $n^3 - n\big|_{n=3} = 27 - 3 = 24$.

And the pattern will go on and on. Therefore, $n^3 - n$ is divisible by 3 for all integer n.
What's wrong in this reasoning?
What is actually proven above: $\exists\, n$ (namely, $n = 1, 2, 3$) such that $3|(n^3 - n)$.
What we need to prove: $\forall\, n\ 3|(n^3 - n)$.
But the implication $\exists\, n\, P(n) \implies \forall\, n\, P(n)$ is **incorrect**.
Therefore, the proof is incorrect.

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How to negate quantified sentences

$$\neg\,(\forall\, x\; P(x)) \iff \exists\, x\;\neg P(x)$$

$$\neg\,(\exists\, x\; P(x)) \iff \forall\, x\;\neg P(x)$$

Example. For the sentence "**All integers are even**",
construct a denial (both in words and symbols).

Solution. We start with writing the sentence in a symbolic form.
$\forall n(n \in \mathbb{Z} \implies 2 \,|\, n)$

Let us construct a useful denial of this sentence.
☞ remember that that implication $P \implies Q$ is negated as $P \wedge \neg Q$:

$\neg\,(\forall\, n(n \in \mathbb{Z} \implies 2 \,|\, n)) \iff \exists\, n\,(n \in \mathbb{Z} \wedge 2 \nmid n)$

So the denial in symbolic form is $\exists\, n\,(n \in \mathbb{Z} \wedge 2 \nmid n)$. Wording?

$\exists\, n\,(n \in \mathbb{Z} \wedge 2 \nmid n)$ says "**Some integers are not even.**"

Passing negation through quantifies

Example. Let $P(x,z)$ and $Q(y,w)$ be predicates.
Construct a useful denial of the following sentence:

$$\forall\, x\;\exists\, y\;\exists\, z\;\forall\, w\,(P(x,z) \implies Q(y,w))\,.$$

Solution.

$$\neg\,(\forall\, x\;\exists\, y\;\exists\, z\;\forall\, w\;\;(P(x,z) \implies Q(y,w))) \iff$$

$$\exists\, x\;\forall\, y\;\forall\, z\;\exists\, w\;\neg\,(P(x,z) \implies Q(y,w)) \iff$$

$$\exists\, x\;\forall\, y\;\forall\, z\;\exists\, w\;\;(P(x,z) \wedge \neg\, Q(y,w))\,.$$

Quantified sentences with two quantifiers

Let $P(x,y)$ be a predicate in two variables x and y.

Let us construct all possible quantified sentences with $P(x,y)$:

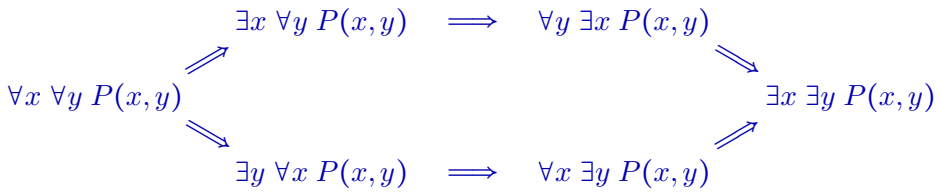
$\exists x\ \exists y\ P(x,y) \iff \exists y\ \exists x\ P(x,y)$

$\exists x\ \forall y\ P(x,y) \implies \forall y\ \exists x\ P(x,y)$

$\forall x\ \exists y\ P(x,y) \impliedby \exists y\ \forall x\ P(x,y)$

$\forall x\ \forall y\ P(x,y) \iff \forall y\ \forall x\ P(x,y)$

Only quantifiers of the same type **commute**.
The existential quantifier and the universal quantifier do **not** commute.



Different quantifiers don't commute

Example 1. Let C be the set of all children and F be the set of all fathers.
Let $P(x,y)$ be the predicate "y **is the father of** x**.**"

Determine the truth value of the following sentences:

$\forall x \in C\ \exists y \in F\ P(x,y)$ Each child has a father. T

$\exists y \in F\ \forall x \in C\ P(x,y)$ There exists a person who is the father of all children.

F

Example 2. Let $x, y \in \mathbb{R}$. Determine the truth value of the following sentences:

$\forall x\ \exists y\ \ x = y$ T

$\exists y\ \forall x\ \ x = y$ F

Example 3. True of false?

$\forall a \in \mathbb{R}\ \exists x \in \mathbb{C}\ \ x^2 = a$ T

$\exists x \in \mathbb{C}\ \forall a \in \mathbb{R}\ \ x^2 = a$ F

All quantified sentences

Problem. Construct all possible quantified sentences using
real variables x, y,
the universal and existential quantifiers, and
the predicate $x + y = 0$.
Determine the truth value of the sentences. Justify your answer.
Solution. There are eight quantified sentences:

1) $\exists x \ \exists y \ \ x + y = 0$
2) $\exists x \ \forall y \ \ x + y = 0$
3) $\forall x \ \exists y \ \ x + y = 0$
4) $\forall x \ \forall y \ \ x + y = 0$
5) $\exists y \ \exists x \ \ x + y = 0$
6) $\exists y \ \forall x \ \ x + y = 0$
7) $\forall y \ \exists x \ \ x + y = 0$
8) $\forall y \ \forall x \ \ x + y = 0$

Let us determine the truth value of each quantified sentence.

1) $\exists x \ \exists y \ \ x + y = 0$ T Why?

Indeed, there exists x, for example, $x = 2$, and there exists y, namely $y = -2$, such that $\underbrace{x}_{2} + \underbrace{y}_{-2} = 0$.

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All quantified sentences

2) $\exists x \ \forall y \ \ x + y = 0$ F Why?
The statement says that there exists **some** special number x,
to which we can add **any** number y and get 0 as the result.
This can't be true, such special number doesn't exist.

The justification above requires many words. Let us see how to reason more efficiently. Consider the **denial** of the statement:

$\forall x \ \exists y \ \ x + y \neq 0$. This is true, since for any x there exists y, namely, $y = 1 - x$, such that $x + \underbrace{y}_{1-x} \neq 0$.

Therefore, the denial is true, and the original statement is **false**.

Warning. Here is a typical example of **incorrect** reasoning.

The statement $\exists x \ \forall y \ \ x + y = 0$ is true,
since there is x, namely $x = -y$, such that for all y we have $\underbrace{x}_{-y} + y = 0$.

Why this reasoning is incorrect? The statement claims existence of some x that serves all y. So x should not depend on y.

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All quantified sentences

3) $\forall x \ \exists y \ \ x + y = 0$ T Why?

No matter which x is chosen, we may always find y, namely, $y = -x$,

such that $x + \underbrace{y}_{-x} = 0$

Warning. Here is a typical example of **incorrect** reasoning.

The statement $\forall x \ \exists y \ \ x + y = 0$ is true.

Indeed, for $x = 2$, there exists y, namely $y = -2$, such that $\underbrace{x}_{2} + \underbrace{y}_{-2} = 0$.

Why this reasoning is incorrect? Since x stands under universal quantifier,
one can not assign any value for it.

All quantified sentences

4) $\forall x \ \forall y \ \ x + y = 0$ F Why?

It is not true that the sum of any two numbers is always 0.
An effective way to prove that a statement is false is to construct its useful denial and show that the denial it true.

The **denial** of $\forall x \ \forall y \ \ x + y = 0$ is $\exists x \ \exists y \ \ x + y \neq 0$.
The denial is true: there exists x, namely $x = 1$, and there exists y, namely $y = 2$, such that $\underbrace{x}_{1} + \underbrace{y}_{2} \neq 0$.
Therefore, the denial is true, and, by this, the original statement is false.

5) $\exists y \ \exists x \ \ x + y = 0$ is equivalent to $\exists x \ \exists y \ \ x + y = 0$, so it is true, see **1)**.

6) $\exists y \ \forall x \ \ x + y = 0$ F Why?

The denial is $\forall y \ \exists x \ \ x + y \neq 0.$ This is true, since for any y one can find x, namely, $x = 3 - y$, such that $x + \underbrace{y}_{3-x} \neq 0$.
Therefore, the denial is true, and the original statement is **false**.

All quantified sentences

7) $\forall\, y\ \exists\, x\quad x + y = 0$ T Why?

No matter which y is chosen, we may always find x, namely, $y = -x$,

such that $x + \underbrace{y}_{-x} = 0$

8) $\forall\, y\ \forall\, x\quad x + y = 0$ is equivalent to $\forall\, x\ \forall\, y\quad x + y = 0$, so it is false, see **4)**.

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Hidden quantifiers

Consider the following problem:

Let a be a real number. Prove that the equation $ax^2 + x - a = 0$ has a real solution.

Although the problem contains no explicit quantifiers,
expressing the statement symbolically reveals the presence of two:

$$\forall a \in \mathbb{R}\ \exists x \in \mathbb{R}\quad ax^2 + x - a = 0\,.$$

Here's a rewording that shows the quantifiers explicitly:

For any real number a, there exists a real number x

that satisfies the equation $ax^2 + x - a = 0$.

To solve the problem means to show that the statement above is true, that is,
any choice of a will guarantee some solution x to the equation $ax^2 + x - a = 0$.

This is a quadratic equation (if $a \neq 0$) whose discriminant $1 + 4a^2$ is positive.
So there is a solution (even two solutions).

If $a = 0$, then the equation turns into $x = 0$. So there is a solution in this case also.

Overall, for arbitrary real a, there is a solution x, as required.

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Are we in a good standing?

Our next two exercises will test how well we:

– handle symbolic writing

– understand sentences with several quantifiers

– construct useful denials of quantified sentences.

Constructing denials

Example. Consider the following statement:
For every real number x, one can find a real number y such that $x > 2$ whenever $x + y \leq 3$.

a) Rewrite the statement in symbolic form (without words).
b) Construct a useful denial of the statement.
c) Determine the truth values of the statement and its denial.
d) Give a geometric interpretation of the statement.

Solution. a) $\forall x\ \exists y\ \ x + y \leq 3 \implies x > 2$.
b) For a **denial**, we need to switch the quantifiers and negate the implication.
We remember that $\neg(P \implies Q) \iff P \wedge \neg Q$. Therefore,
$\neg(\forall x\ \exists y\ \ x + y \leq 3 \implies x > 2) \iff (\exists x\ \forall y\ \ x + y \leq 3\ \wedge x \leq 2)$

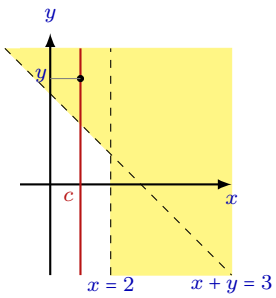
c) To determine the truth value of the statement, we rewrite it as a **disjunction**.
Remember: $(P \implies Q) \iff (\neg P \vee Q)$. Therefore,
$(\forall x\ \exists y\ \ x + y \leq 3 \implies x > 2) \iff (\forall x\ \exists y\ \ x + y > 3 \vee x > 2)$, which is true.
Indeed, for any x, there exists y, namely $y = 4 - x$, such that $x + y = x + (4 - x) = 4 > 3$, which makes the part $x > 2$ irrelevant.
The negation of the statement is therefore false.

Geometric interpretation

d) For a geometric interpretation of the original statement $\forall x\ \exists y\ \ x+y \le 3 \implies x > 2,$ we use its disjunctive form: $(\forall x\ \exists y\ \ x+y \le 3 \implies x > 2) \iff (\forall x\ \exists y\ \ x+y > 3 \vee x > 2)$.

Let us draw on the coordinate system the set of all points (x, y) for which $x+y > 3$ **or** $x > 2$.

It's a union of two half-planes determined by the inequalities $x+y > 3$ and $x > 2$:



The statement says that for **any** value of x (draw vertical line $x = c$, where c is an arbitrary number) there exists y such that the point (c, y) belongs to the shaded region.

For example, we can take $y = 4 - x$ to ensure that $x+y$ will be greater than 3 and so the statement $x+y > 3 \vee x > 2$ will hold true.

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A problem with parameter

Problem. Let a and x be real numbers.
Find **all** values of a for which each of the propositional forms below holds true.
(a) $\exists x\ \ a^2x - x + 1 = 0$ **(b)** $\forall x\ \ a^2x - x + 1 = 0$ **(c)** $\exists x\ \ a^2x - x + 1 > 0$

Solution. (a) What does it mean that $\exists x\ \ a^2x - x + 1 = 0$? It means that there is x which is a solution of the equation $a^2x - x + 1 = 0$.
Can we solve this equation? Sure! This is a **linear** equation
$a^2x - x + 1 = 0 \iff (a^2 - 1)x + 1 = 0$.
It has a solution $x = \dfrac{1}{1-a^2}$ provided $a \neq \pm 1$.

Therefore, for **any** $a \neq \pm 1$, there is x such that $a^2x - x + 1 = 0$.
So we have found all values of a such that $\exists x\ \ a^2x - x + 1 = 0$.

Answer: $a \neq \pm 1$.

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A problem with parameter

In **(b)**, we need to find **all** values of a for which the sentence

$$\forall\, x \;\; a^2x - x + 1 = 0$$ holds true.

This sentence means that **any** x is a solution of the linear equation

$$a^2x - x + 1 = 0\,.$$

Can this happen for whatever a? No, since

$a^2x - x + 1 = 0 \iff (a^2 - 1)x + 1 = 0\,.$

To make our reasoning clear, let us consider a **denial** of the original statement.

$\neg(\forall\, x \;\; a^2x - x + 1 = 0) \iff \exists\, x \;\; a^2x - x + 1 \neq 0\,.$

The denial is true: no matter what a is, there is x, namely $x = 0$,

such that $a^2x - x + 1\Big|_{x=0} = 1 \neq 0\,.$

Therefore, the denial is true for any a,

so the original statement is false for any a.

Answer: There no such values of a.

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A problem with parameter

In **(c)**, we need to find **all** values of a for which the sentence

$$\exists\, x \;\; a^2x - x + 1 > 0$$

holds true.

Simple observation gives us $x = 0$ which guarantees that $a^2x - x + 1\Big|_{x=0} = 1 > 0$

no matter what a is.

Answer: For any $a \in \mathbb{R}$, the statement hols true.

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