## About definitions and quiz

**Definition.**

**Verbally.** Let $p > 1$ be an integer.
$p$ is called **prime**
if it has only two positive divisors: $1$ and $p$.

**Symbolically.** Let $p \in \mathbb{Z}$ and $p > 1$.
$p$ is **prime** $\iff \forall k \in \mathbb{Z}_+ \ (k \mid p \implies k = 1 \vee k = p)$.

Let $n > 1$ be an integer.
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# Proof by contradiction (indirect proof)



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Logical justification: $(\neg P \implies Q) \wedge (\neg P \implies \neg Q)) \implies P$ is a tautology.

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Logical justification: $(\neg P \implies Q) \land (\neg P \implies \neg Q)) \implies P$ is a tautology.

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**Proof.** The statement to prove: $\underbrace{\boxed{\sqrt{2} \text{ is irrational}}}_{P}$.

Assume, to the contrary, that $\underbrace{\boxed{\sqrt{2} \text{ is rational}}}_{\neg P}$.

Then $\sqrt{2} = \dfrac{p}{q}$ for some $p, q \in \mathbb{Z},\ q \neq 0$.

# $\sqrt{2}$ is irrational



Since any fraction $\dfrac{p}{q}$ can be reduced to lowest terms,

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For source and comments see

**Euclid's Elements**, Book IX, Proposition 20.

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If possible, let it be so. Now $A$, $B$, and $C$ measure $DE$, therefore $G$ also measures $DE$.

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Listen to the proof and try to write it down...

# The triangle inequality

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**Theorem (triangle inequality).** $|a+b| \leq |a| + |b|$ *for any real numbers* $a, b$.

# The triangle inequality



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**Proof**

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**Theorem (triangle inequality).** $|a+b| \leq |a|+|b|$ *for any real numbers* $a, b$.

**Proof** (by cases).

- Case 1. $a \geq 0$ and $b \geq 0$ .Then $|a|=a$, $|b|=b$, $|a+b|=a+b$, so $|a+b|=a+b=|a|+|b|$, and, by this $|a+b| \leq |a|+|b|$ .

- Case 2. $a \geq 0$ and $b<0$ .Then $|a|=a$, $|b|=-b$, $|a+b|=?$

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Therefore, $|a+b| \leq |a|+|b|$ for all real numbers $a$ and $b$,

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Therefore, $|a+b| \leq |a| + |b|$ for all real numbers $a$ and $b$, as required.

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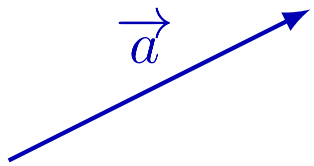

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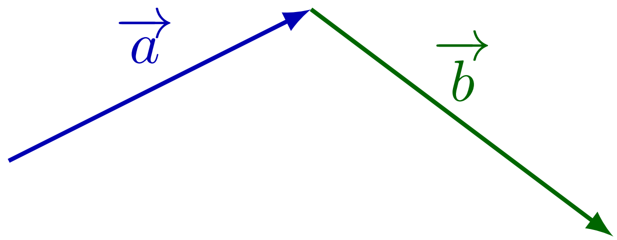

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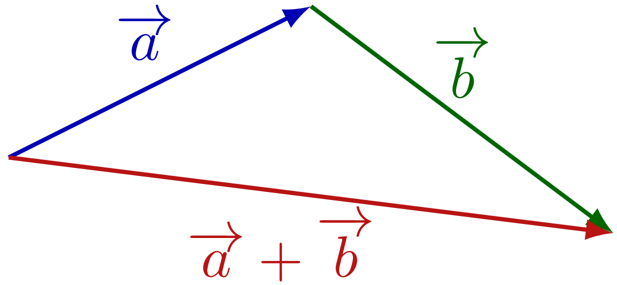

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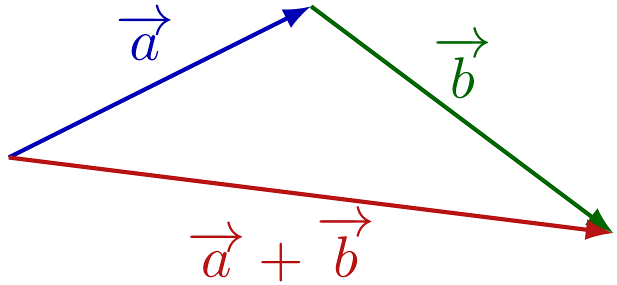


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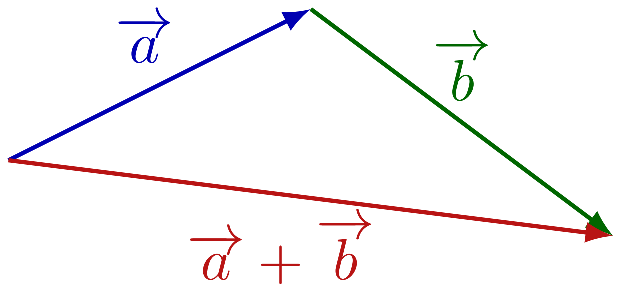


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**Corollary 1.** $|a-b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

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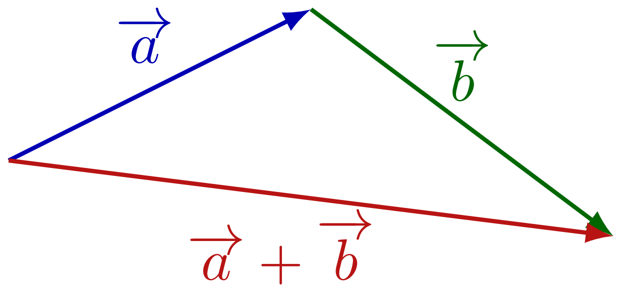


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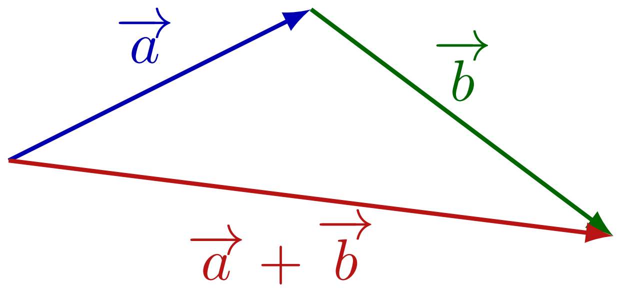


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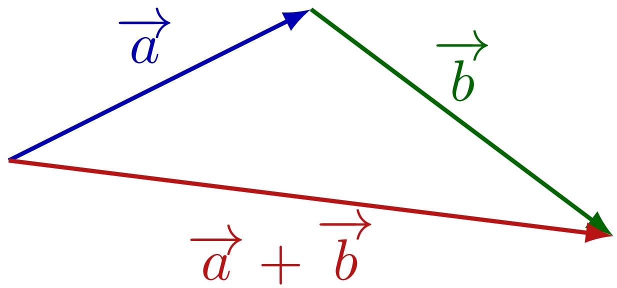


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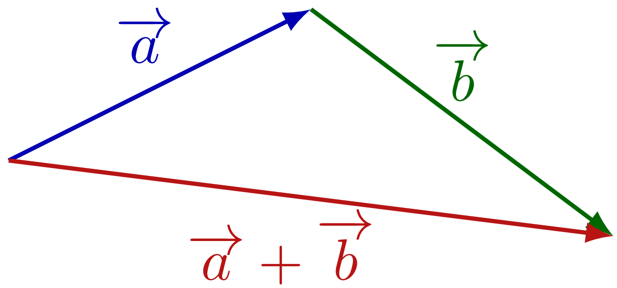


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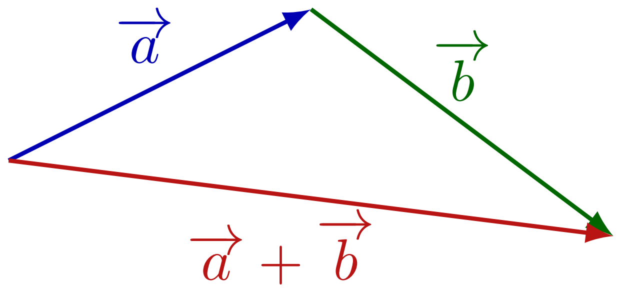


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**Corollary 2.** $||a| - |b|| \leq |a-b|$ for all $a, b \in \mathbb{R}$.

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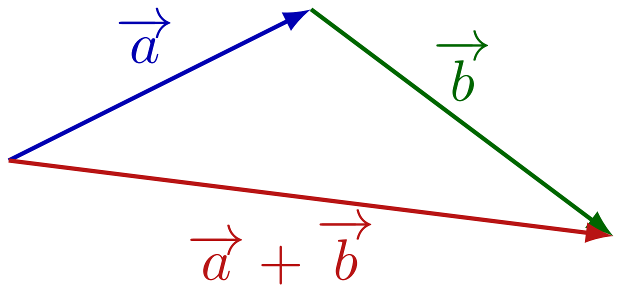


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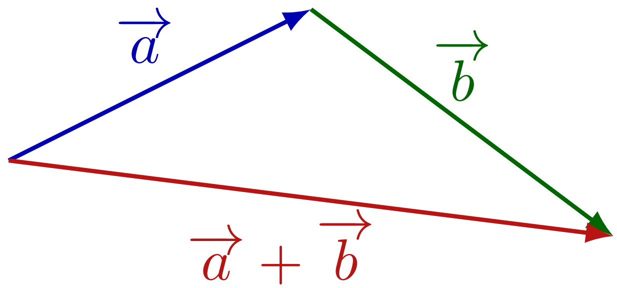


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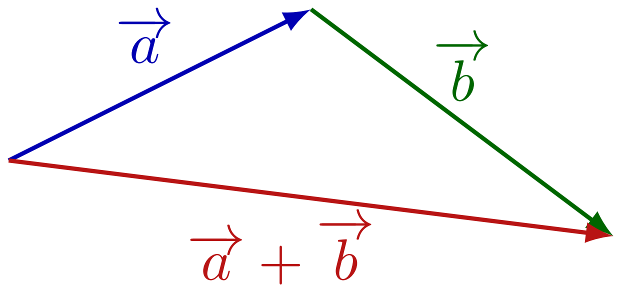


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**Corollary 1.** $|a-b| \le |a| + |b|$ for all $a, b \in \mathbb{R}$.

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Since $a+(-b) = a-b$ and $|-b| = |b|$,
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**Corollary 2.** $||a| - |b|| \le |a-b|$ for all $a, b \in \mathbb{R}$.

**Proof.** $|a| = |(a-b)+b| \le |a-b| + |b| \implies |a| - |b| \le |a-b|$.

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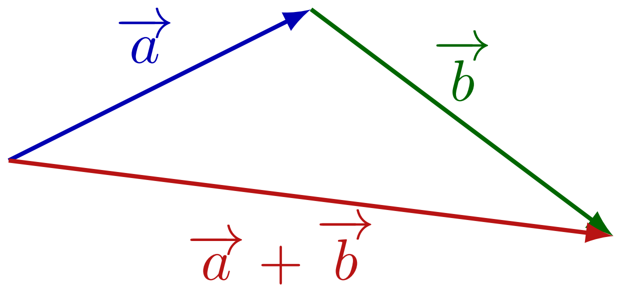


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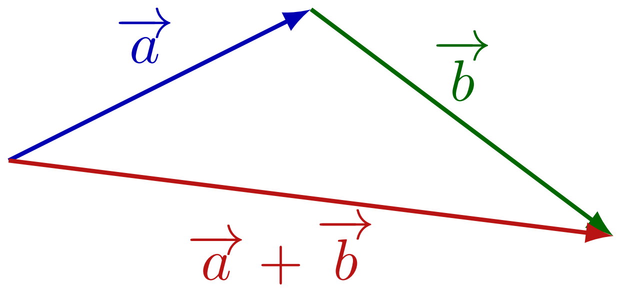


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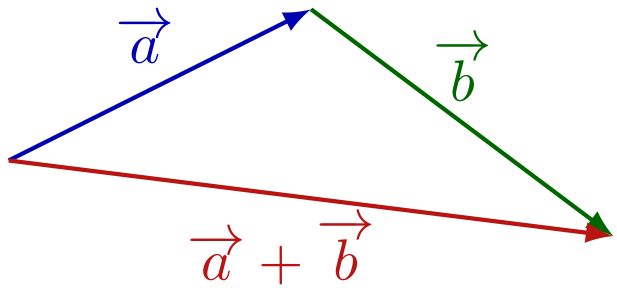


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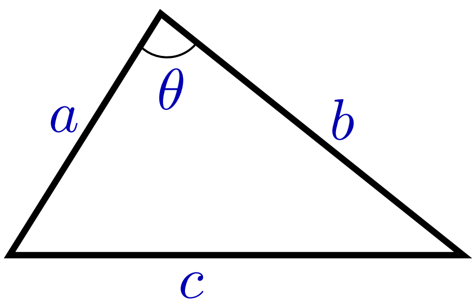

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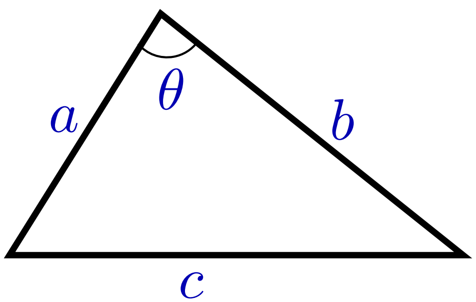


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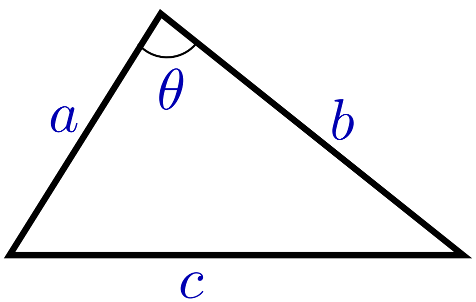


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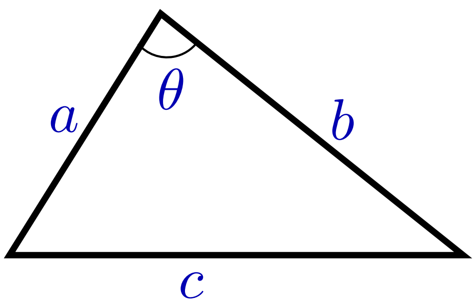


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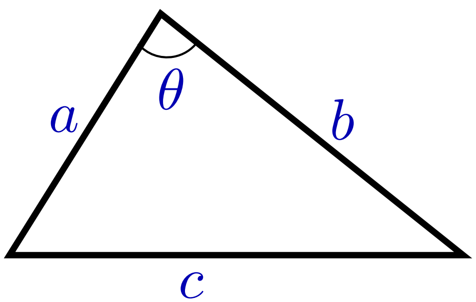


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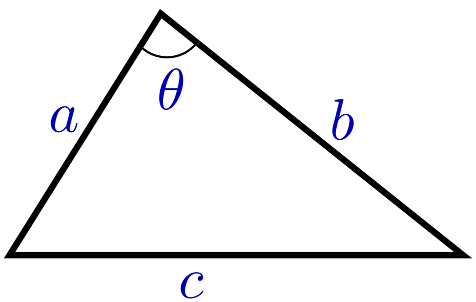


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