Lecture 5

Proof techniques

Proofs

Basic schemes of proof

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In this lecture we will discuss basic proof techniques:

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- Direct proof

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- Direct proof
- Proof by contraposition
- Proof by contradiction
- Proof by exhaustion (proof by cases)

Direct proof (modus ponens)

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This is the **Extremal Property** of the inequality $AM \geq GM$.

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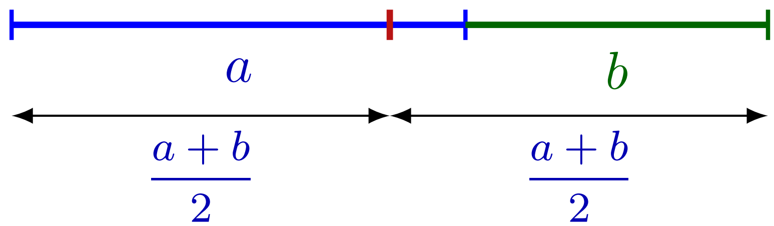

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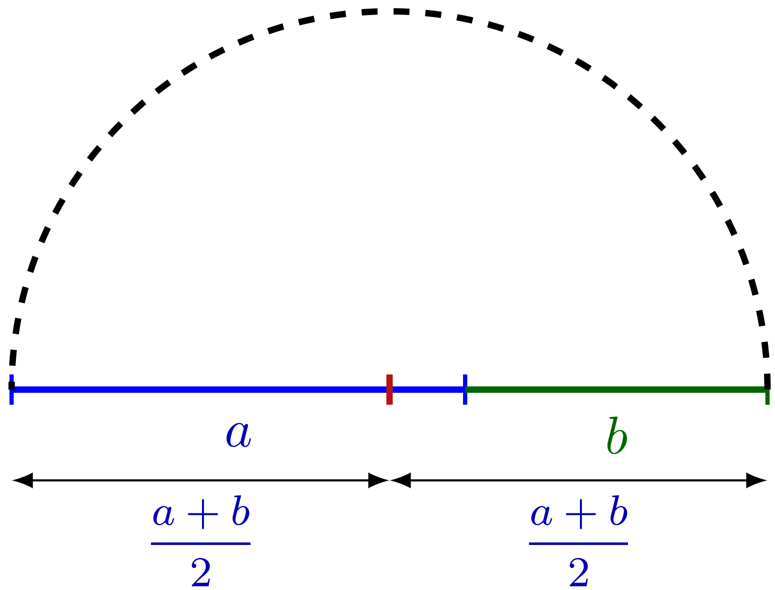

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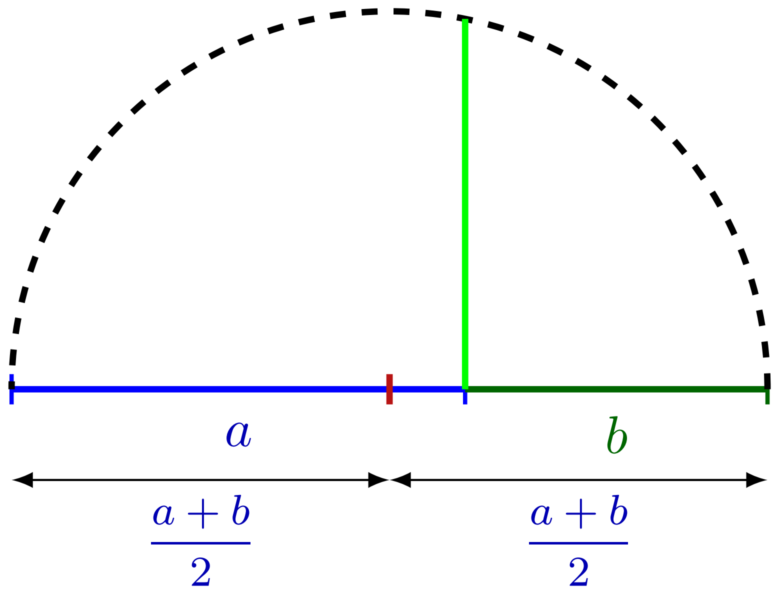

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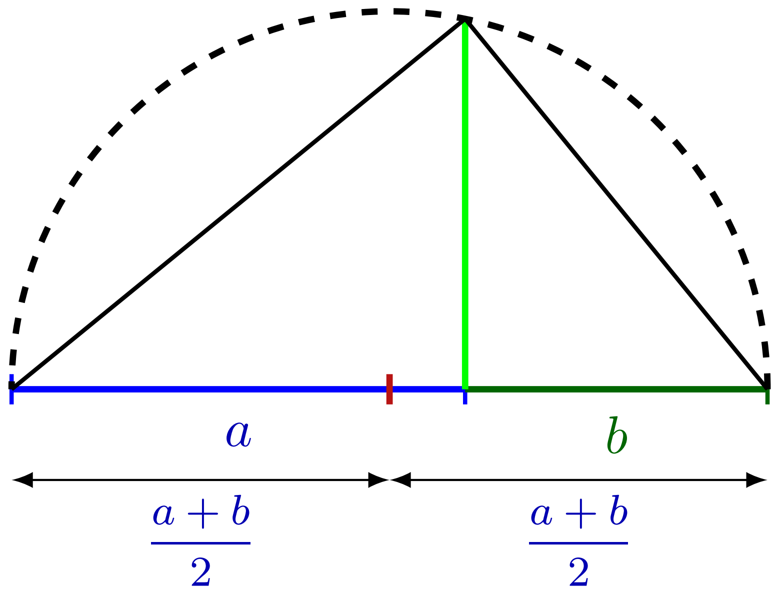

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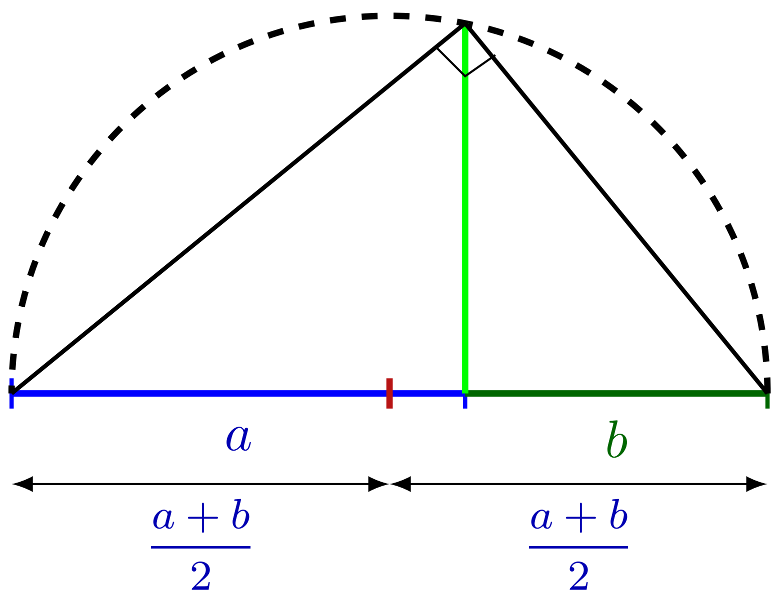

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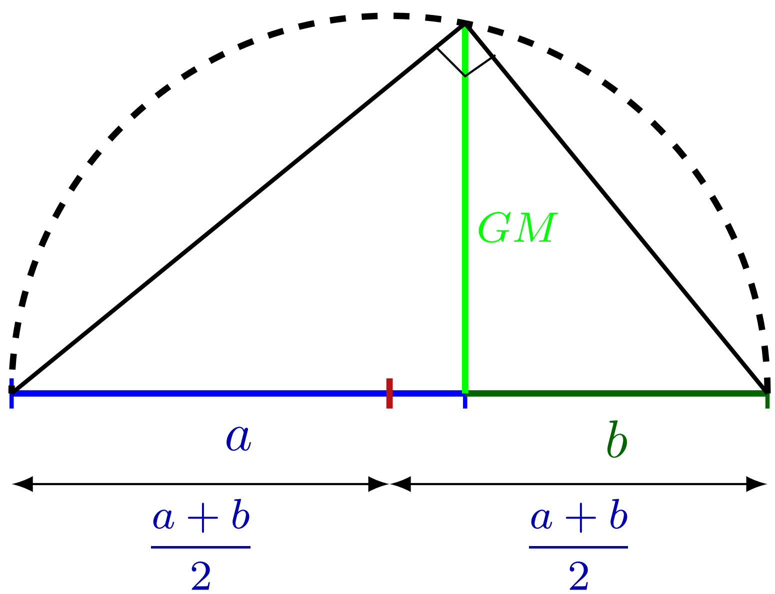

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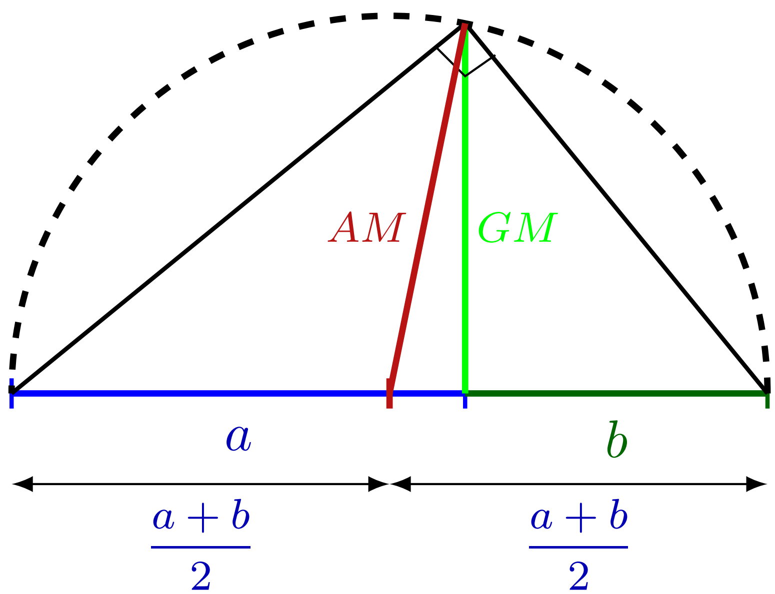

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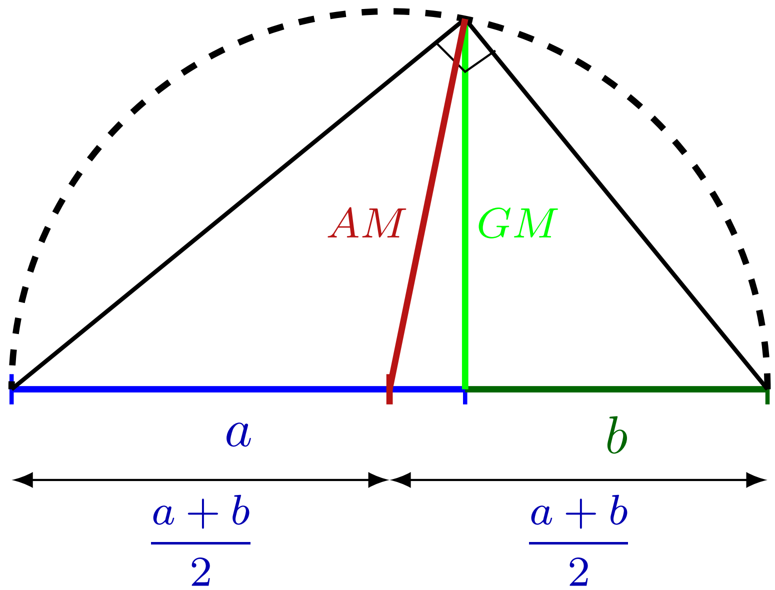

$$AM = \frac{a+b}{2}$$

$$GM = \sqrt{ab}$$

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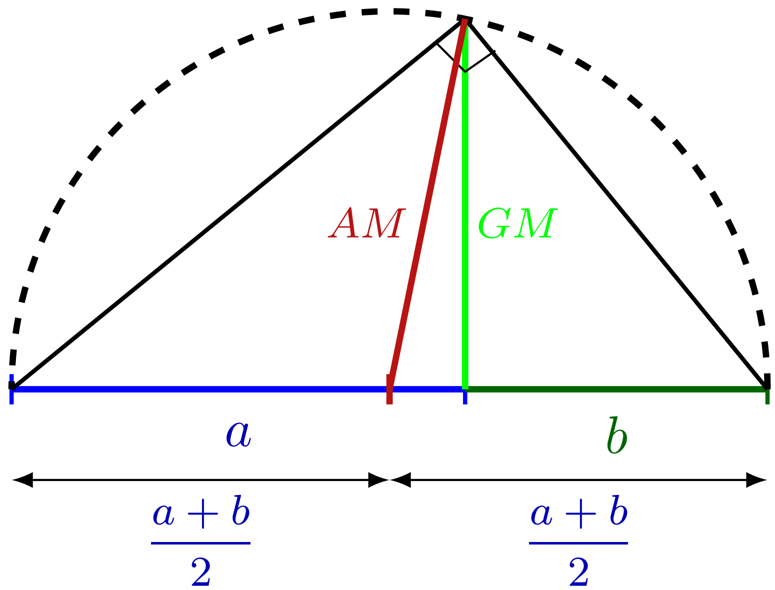

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$$AM = GM \iff a = b$$

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$= f'(a)\cdot 0 \ = 0$, as required.

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Therefore, $\lim_{x\to a} f(x) = f(a)$, and, by this, f is continuous at a, as required.

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Example 1. Let n be an integer.

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Example 1. Let n be an integer. Prove that if n^2 is odd then n is odd.

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This rule of logical deduction
$((P \implies Q) \wedge \neg Q) \implies \neg P$ is called

modus tollens.

Method: Assume (let) $\neg Q$. Then ... Then ... Therefore, $\neg P$.

So $\neg Q \implies \neg P$. By contraposition, $P \implies Q$.

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Why not to prove like this: n^2 is odd $\implies$ $\sqrt{n^2} = n$ is odd?

What to choose: direct proof or proof by contraposition?

For a **direct** proof of

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Cast off crutches:

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Proposition. *For any integer n, if n^2 is odd then n is odd.*

Proof. Let n be even. Then $n = 2k$ for some integer k. So $n^2 = 4k^2$, which is even. Therefore, by contraposition, if n^2 is odd then n is odd, as required. □

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qed

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Assume that $2 \nmid n$ $(\neg Q)$. Then $n = 2k + 1$ for some integer k.

Calculate $n^2 - 1$:

$n^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k = 4 \underbrace{k(k+1)}_{\text{divisible by } 2}$ is divisible by 8 ($\neg P$).

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By contraposition, $8 \nmid (n^2-1) \implies 2 \mid n$, as required.

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Proof by contradiction (indirect proof)

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Then $\sqrt{2} = \dfrac{p}{q}$ for some $p, q \in \mathbb{Z},\ q \neq 0$.

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But p is also even, that is $2\,|\,p$. We have got that $2\,|\,p$ and $2\,|\,q$.

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For source and comments see
Euclid's Elements, Book IX, Proposition 20.
http://aleph0.clarku.edu/ djoyce/java/elements/bookIX/propIX20.html

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Listen to the proof and try to write it down...

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- Case 1. $a \geq 0$ and $b \geq 0$.

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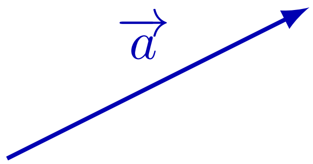

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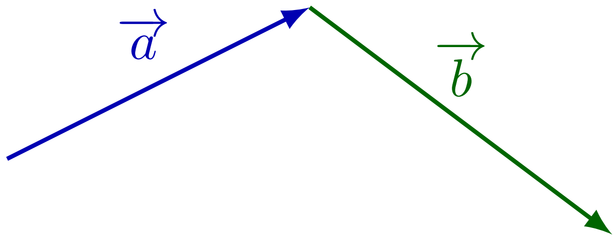

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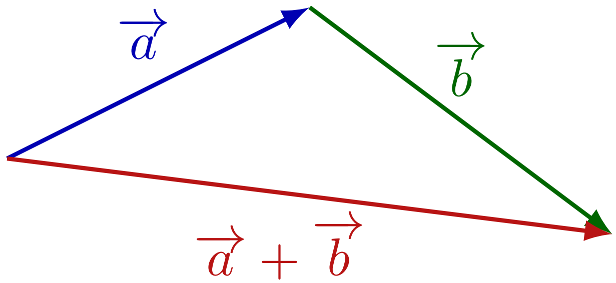

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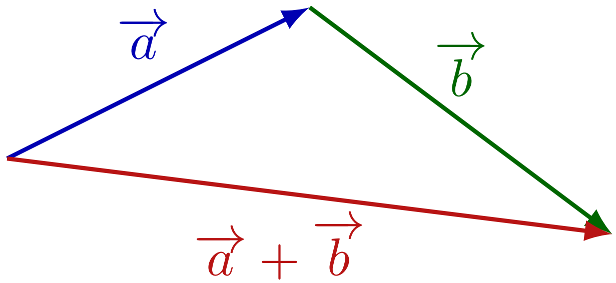

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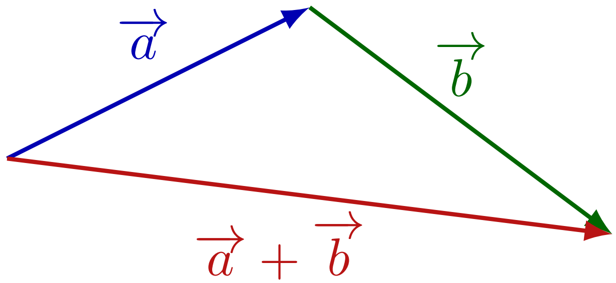

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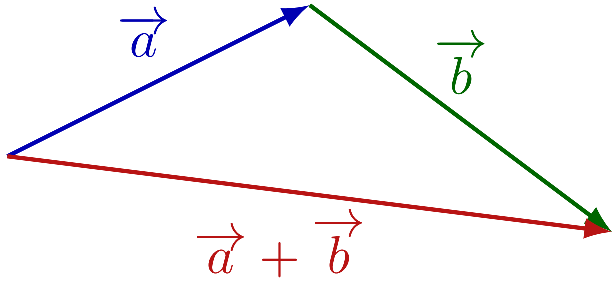

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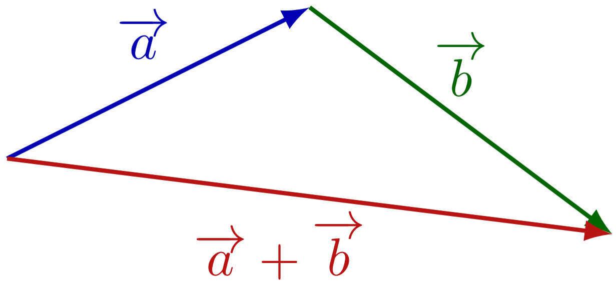

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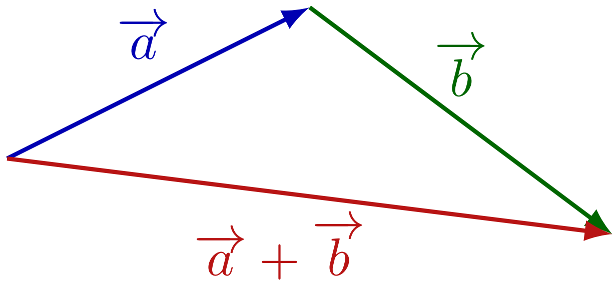

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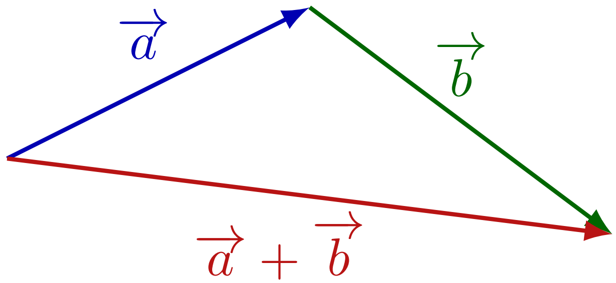

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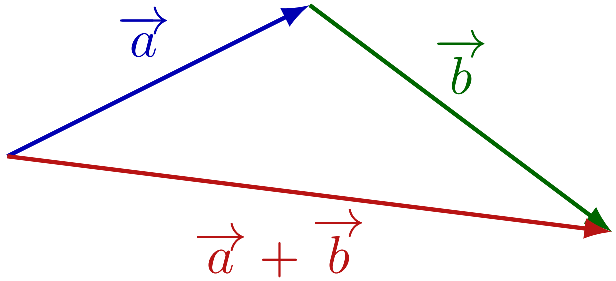

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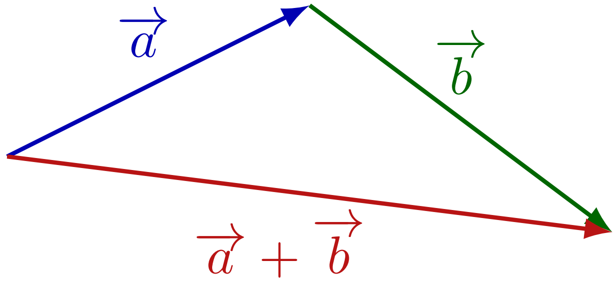

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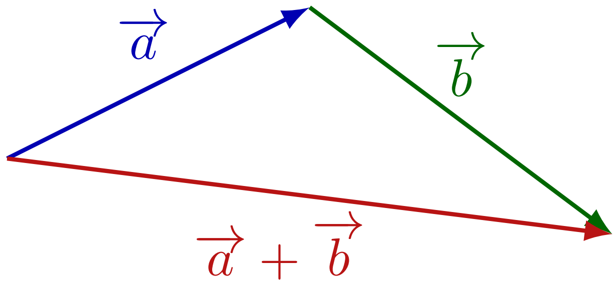

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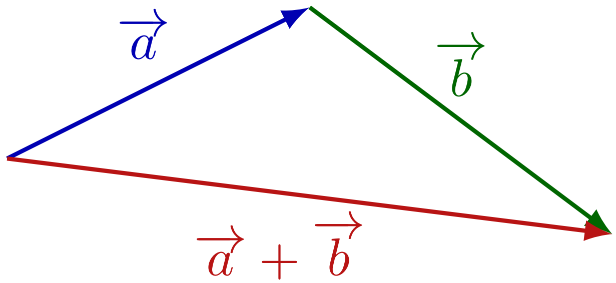

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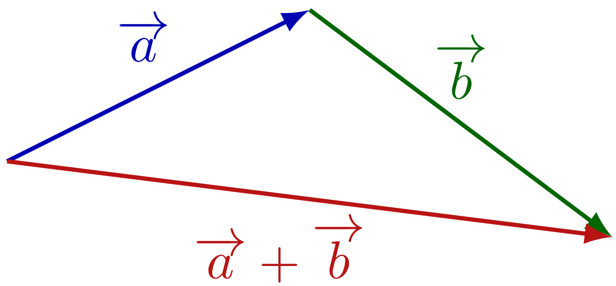

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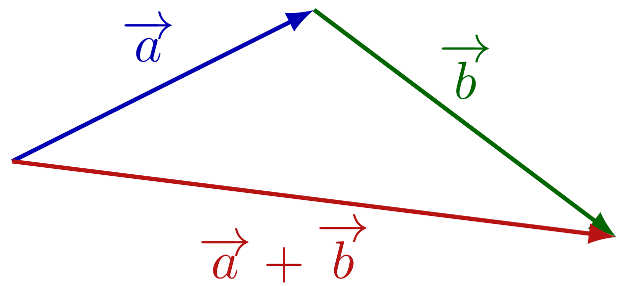

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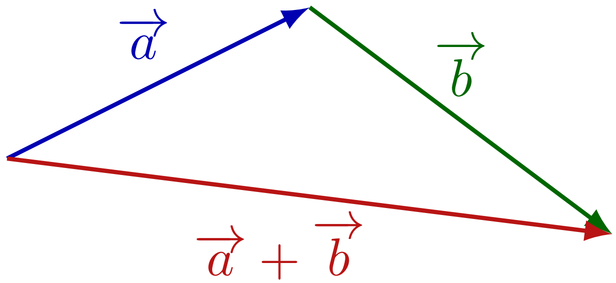

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Since $||a| + |b||| = |a| + |b|$, we get $|a + b| \leq |a| + |b|$.

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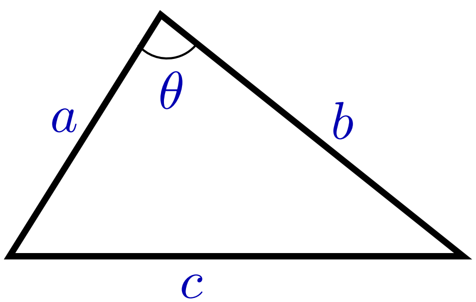

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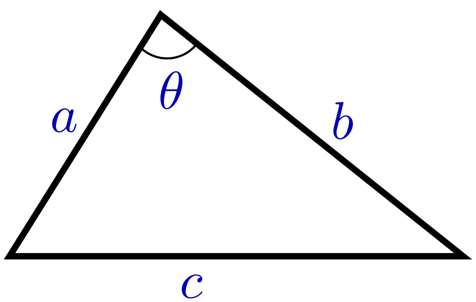

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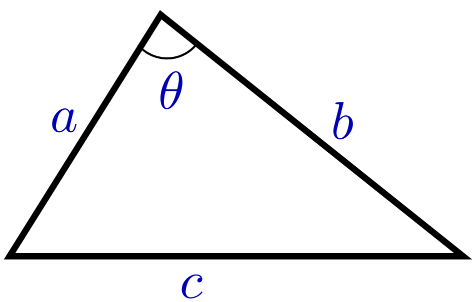

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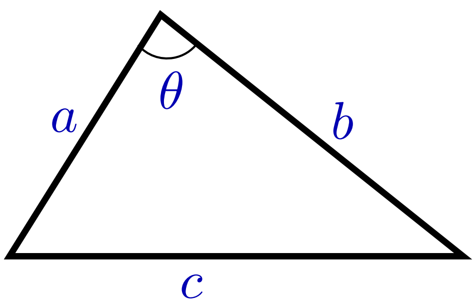

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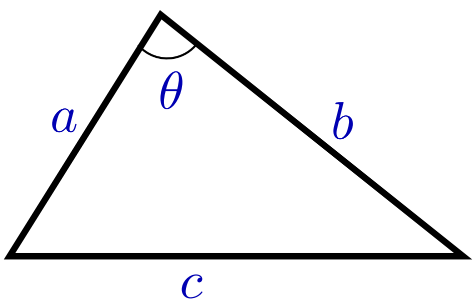

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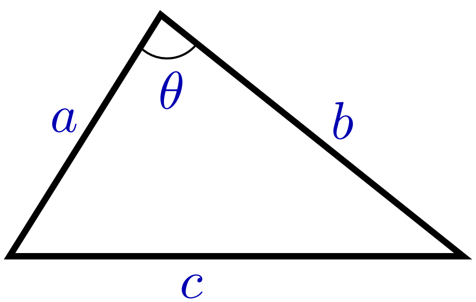

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  and conclusion (what should be proven).
- Recall all relevant definitions and theorems in their **precise** form.
- Do math. Logic can't replace missing mathematics.
- Put math in a correct logical form.
- Avoid **typical logical mistakes**:
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     Prove $P \implies Q$.
     "Proof." Let $Q \ldots$
  2. **Denying the antecedent**
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Strategies for constructing proofs

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     $\exists x\, P(x) \implies \forall x\, P(x)$