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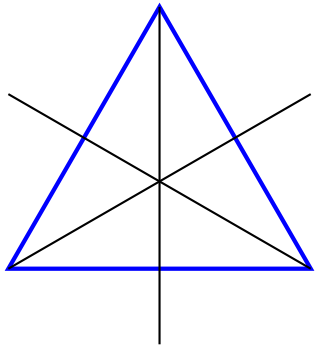

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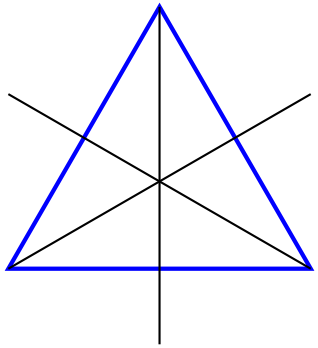

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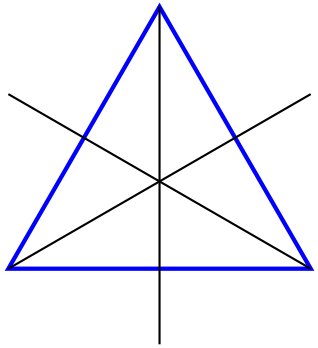

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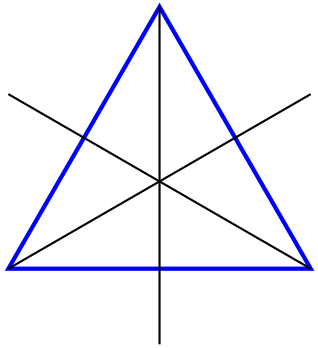

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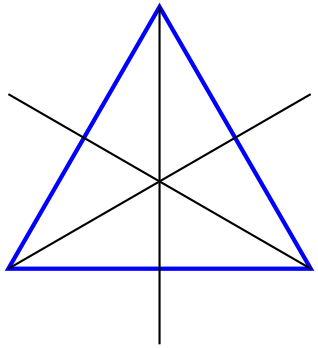

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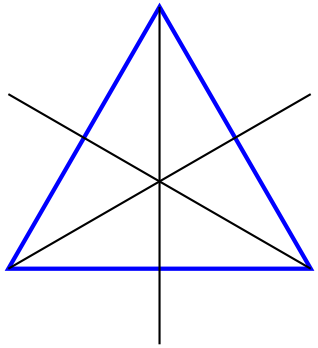

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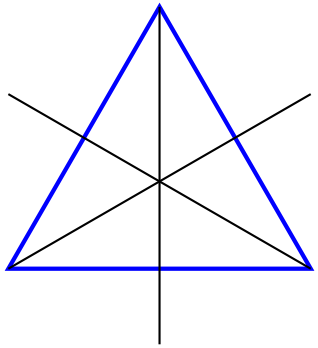

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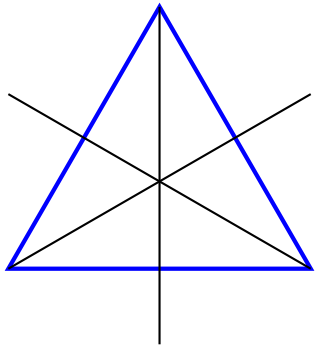

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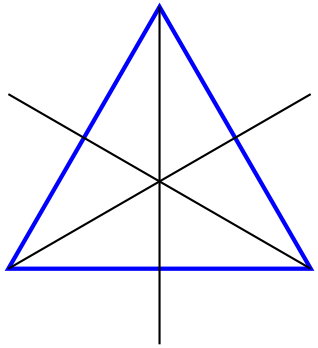

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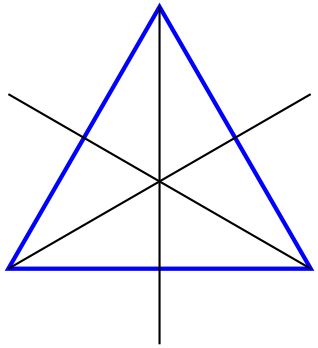

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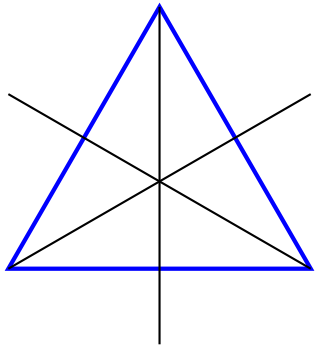

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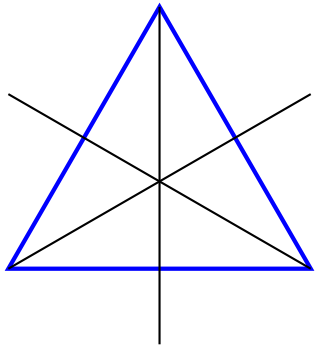

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This leads to the definition of a *group*.

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- If, additionally, $\forall\, a, b \in R \quad a \cdot b = b \cdot a$ ($\cdot$ is **commutative**),
  then R is called a *commutative* ring.
- If, additionally, $\exists\, 1 \in R \quad \forall\, a \in R \quad 1 \cdot a = a \cdot 1 = a$
  (there exists a **multiplicative identity**), then R is called a ring with *unity*.

Definition of ring

Definition. A *ring* R is a set with two operations, addition and multiplication, denoted by $+$ and $\cdot$, satisfying the following properties:

1. $\forall\, a, b \in R \quad a + b \in R$ (R is **closed** with respect to $+$)
2. $\forall\, a, b \in R \quad a \cdot b \in R$ (R is **closed** with respect to $\cdot$)
3. $\forall\, a, b, c \in R \quad (a + b) + c = a + (b + c)$ ($+$ is **associative**)
4. $\forall\, a, b \in R \quad a + b = b + a$ ($+$ is **commutative**)
5. $\exists\, 0 \in R \quad \forall\, a \in R \quad a + 0 = a$ (there exists an **additive identity** in R)
6. $\forall\, a \in R \;\; \exists\, -a \in R \;\; a + (-a) = 0$ (each element in R has an **additive inverse**)
7. $\forall\, a, b, c \in R \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$ ($\cdot$ is **associative**)
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The properties are called the *axioms* of a ring.

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Important: To prove that each of the listed above objects is a ring, we have to verify all ring axioms.

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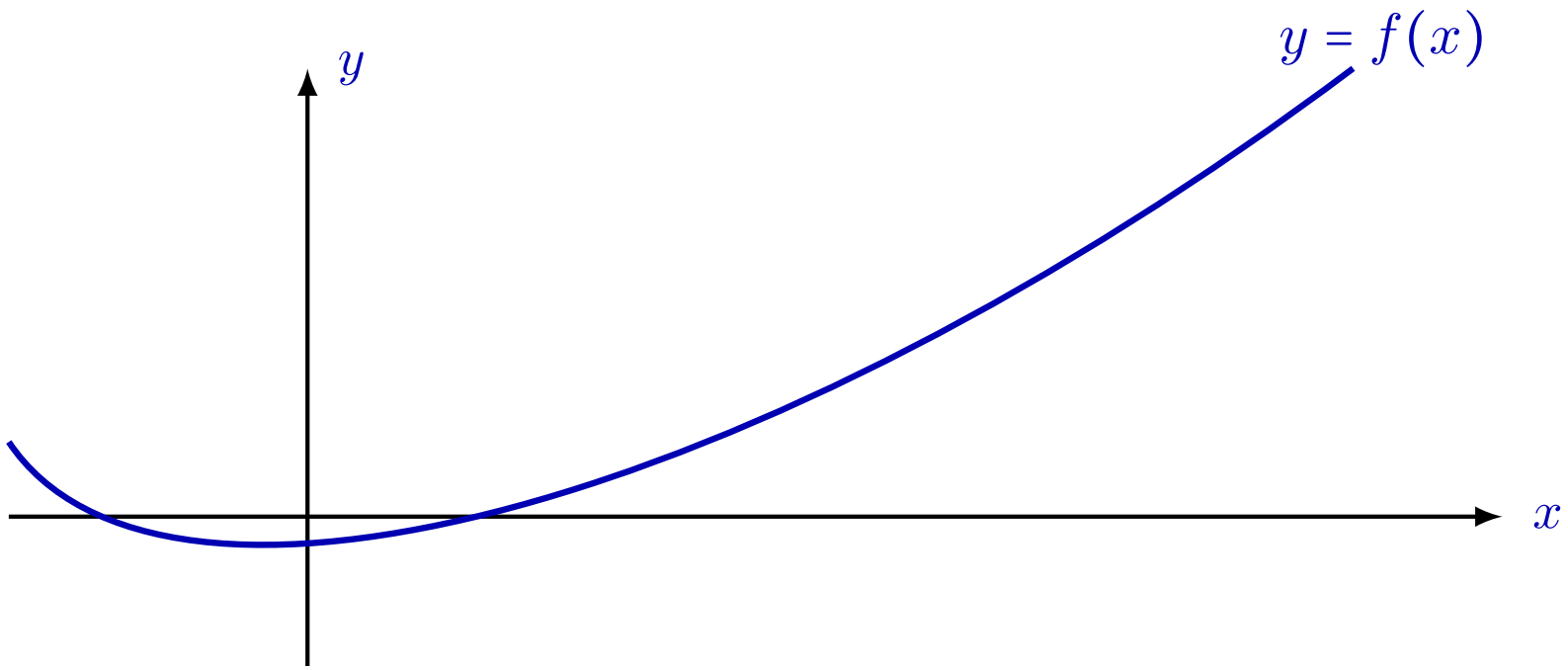

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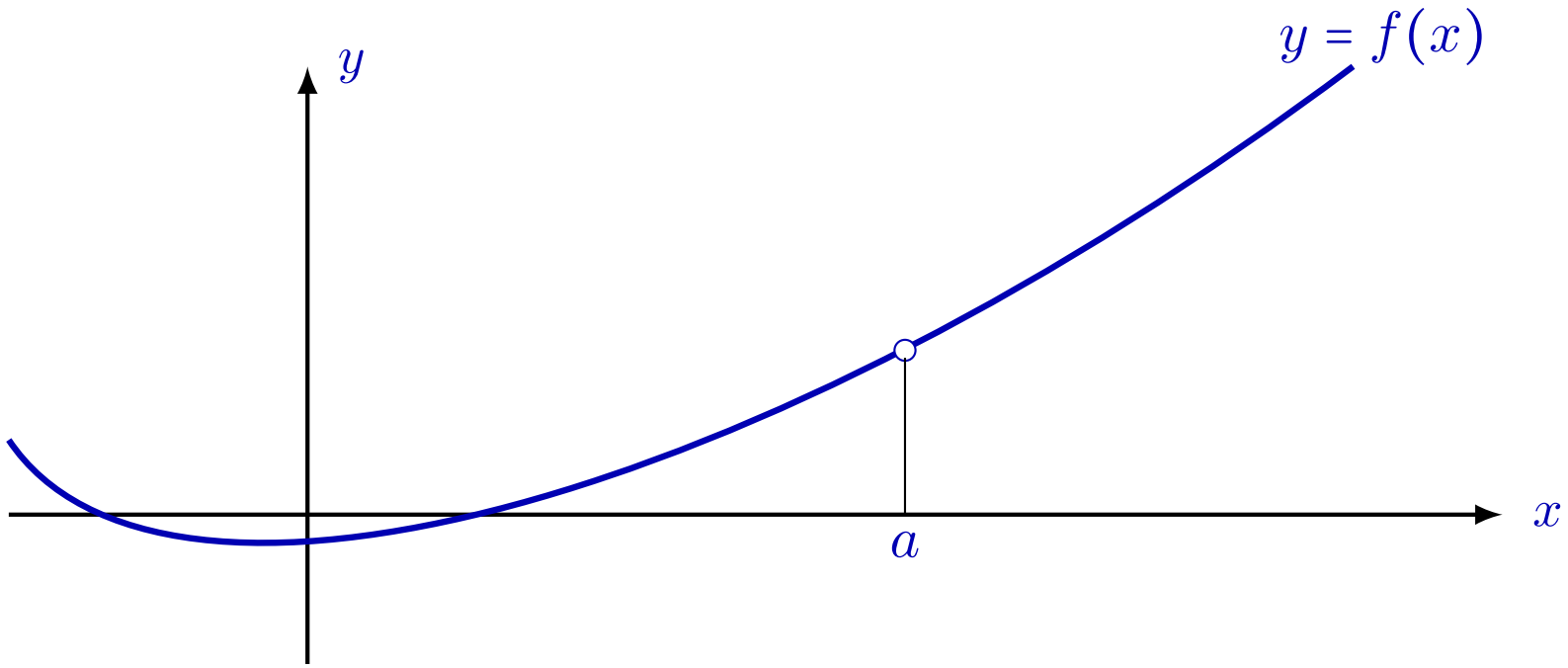

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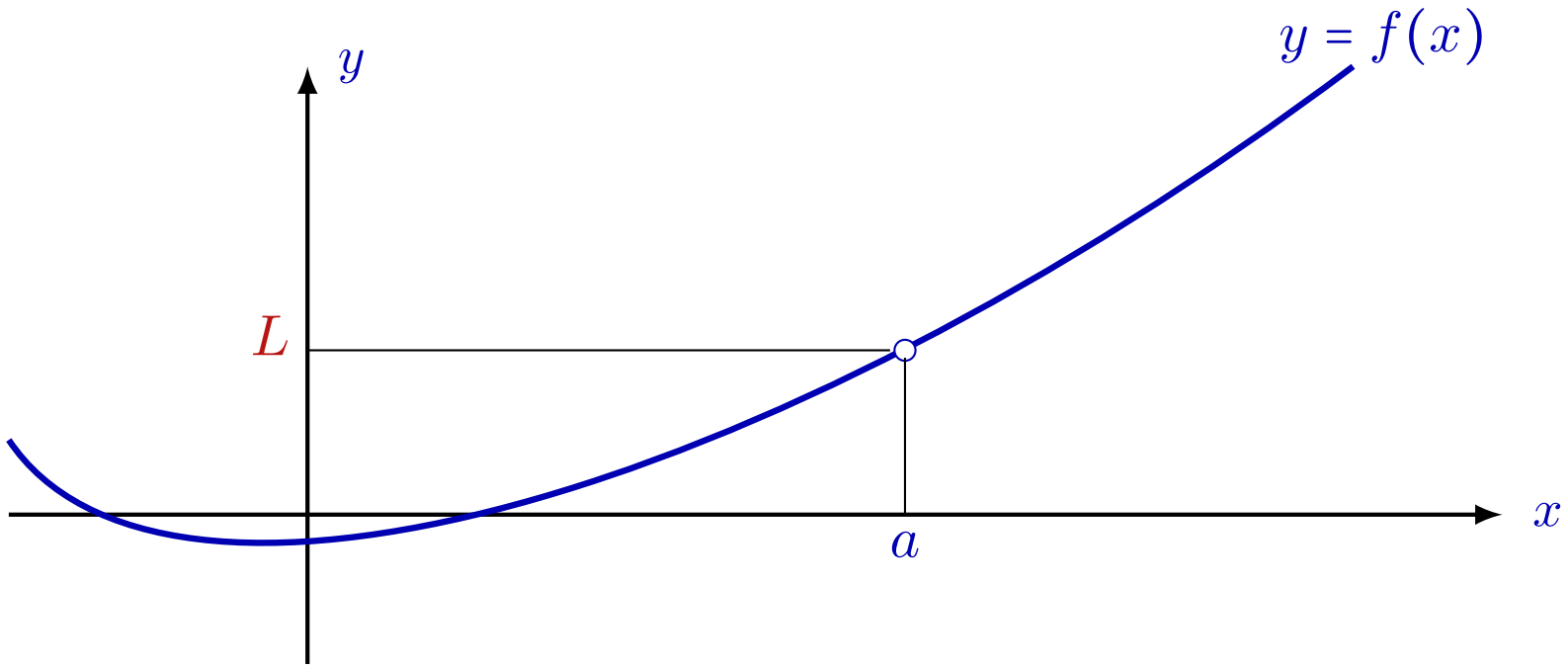

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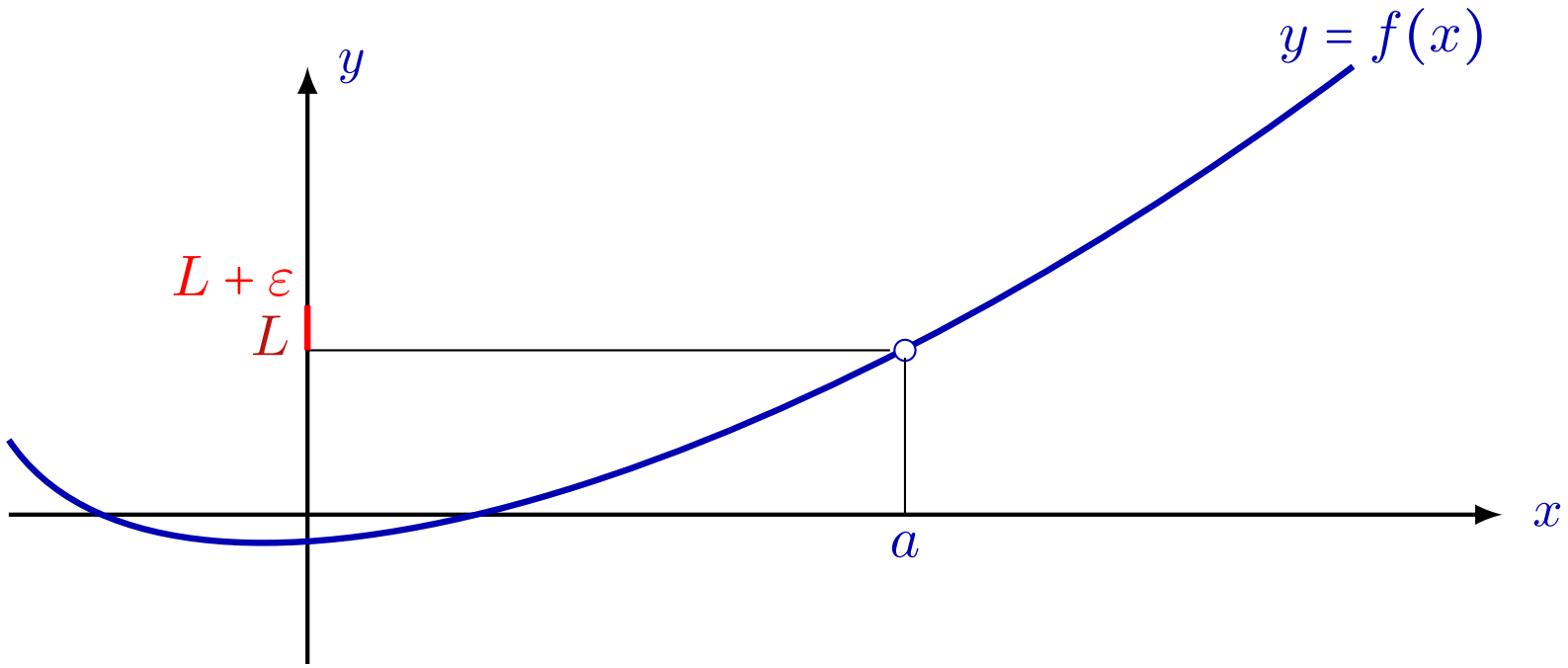

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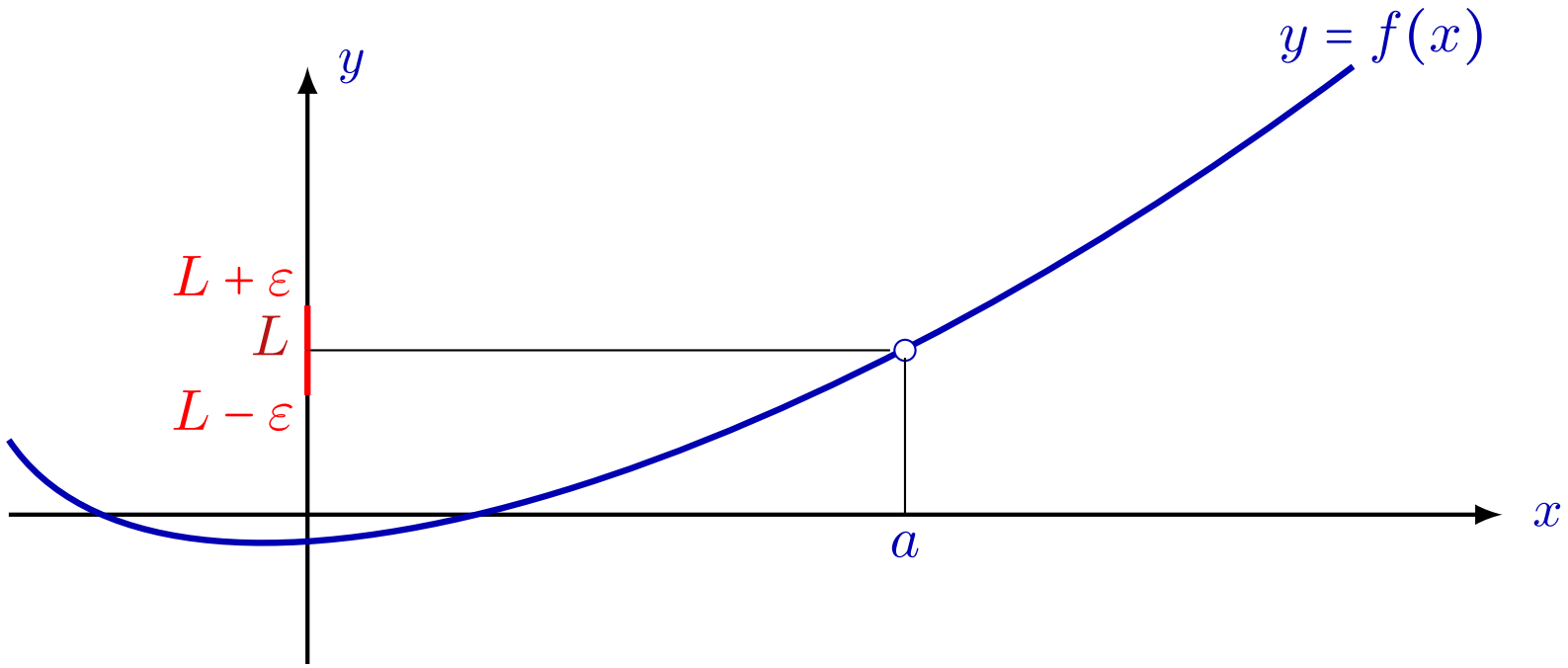

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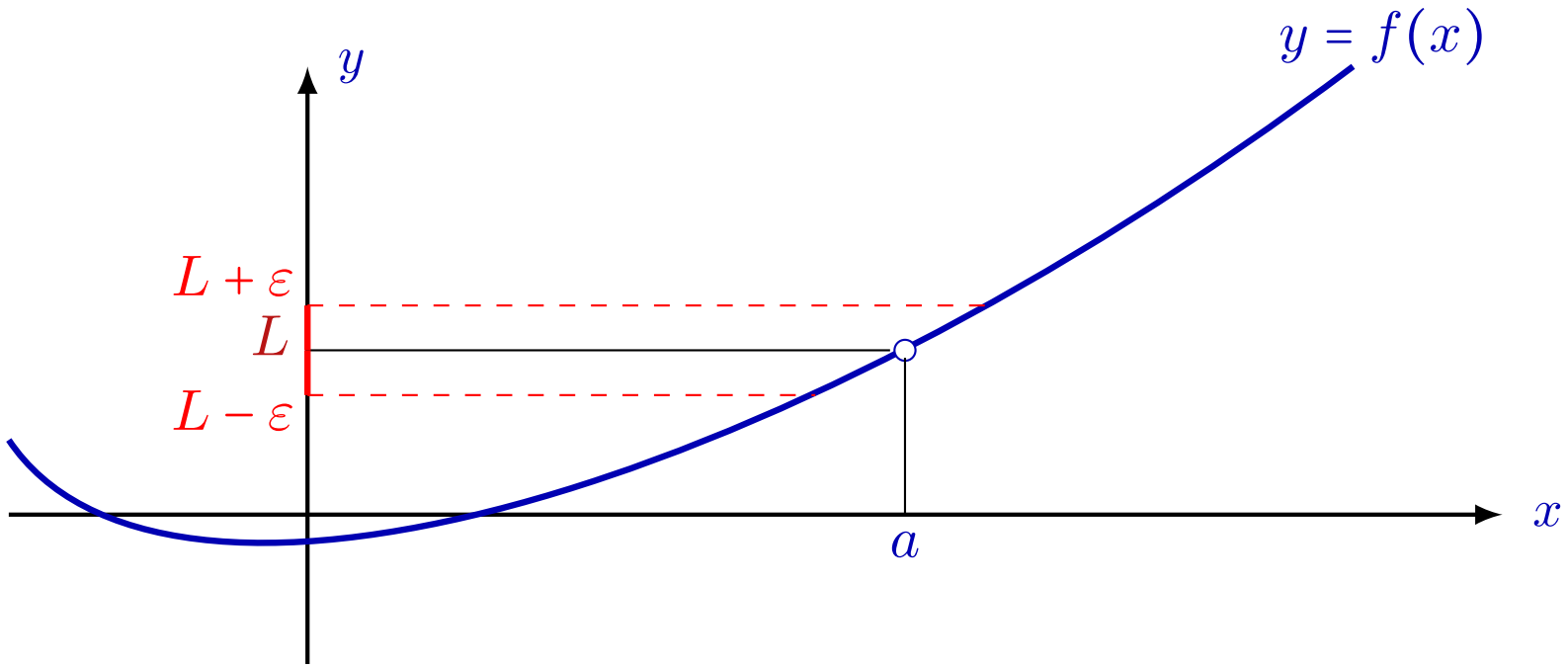

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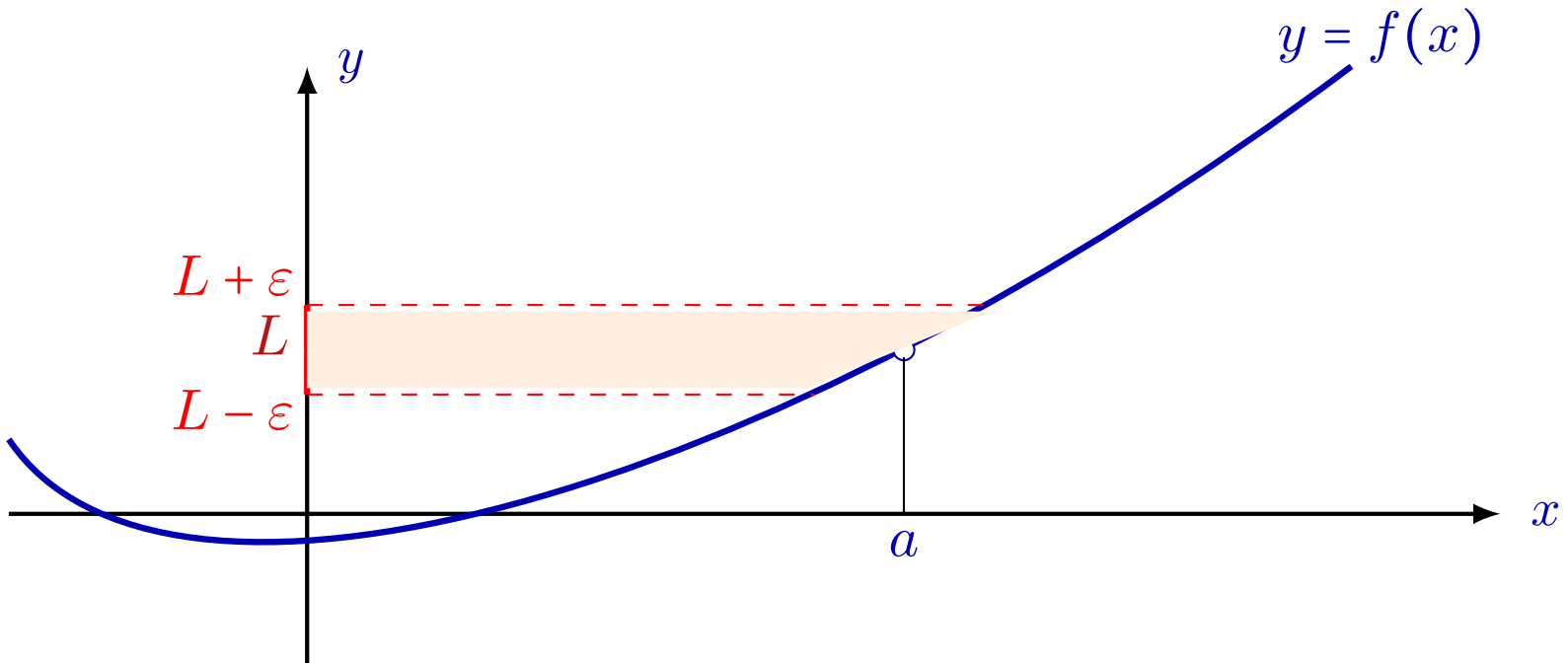

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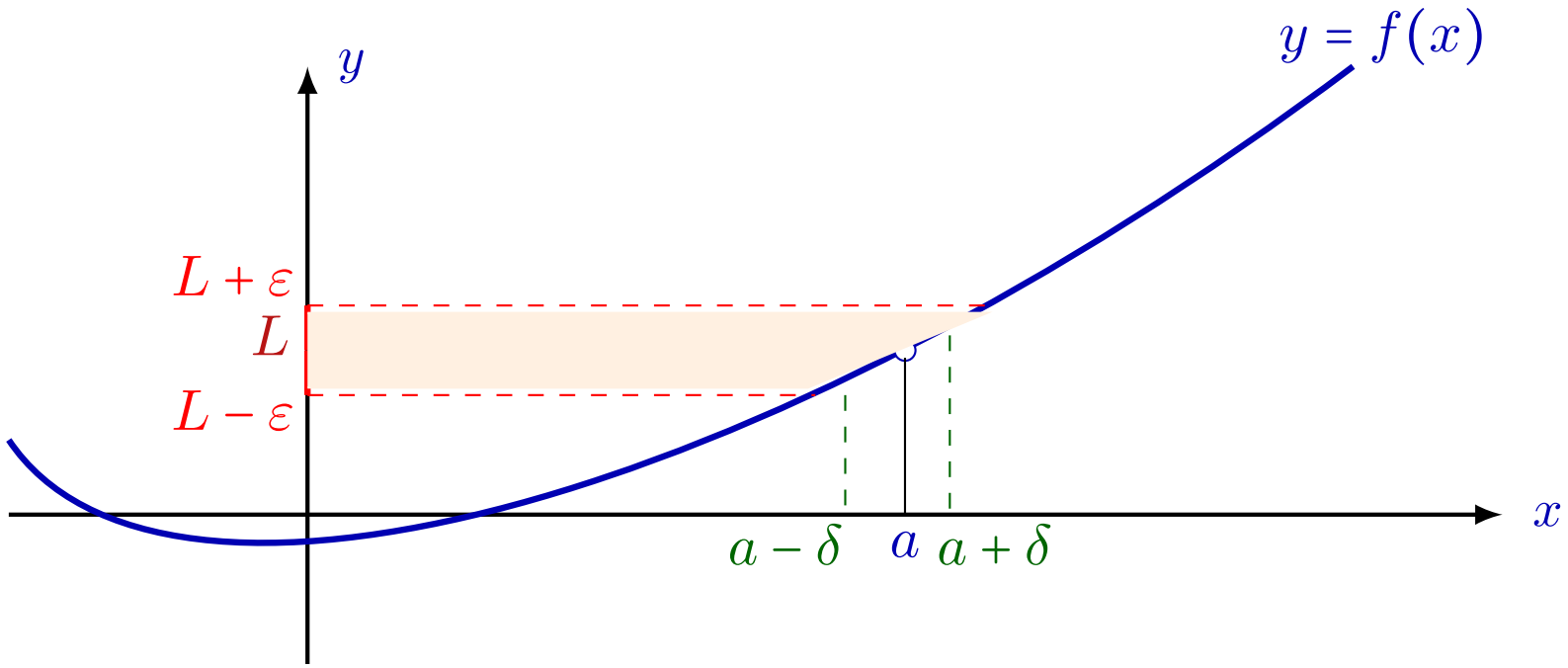

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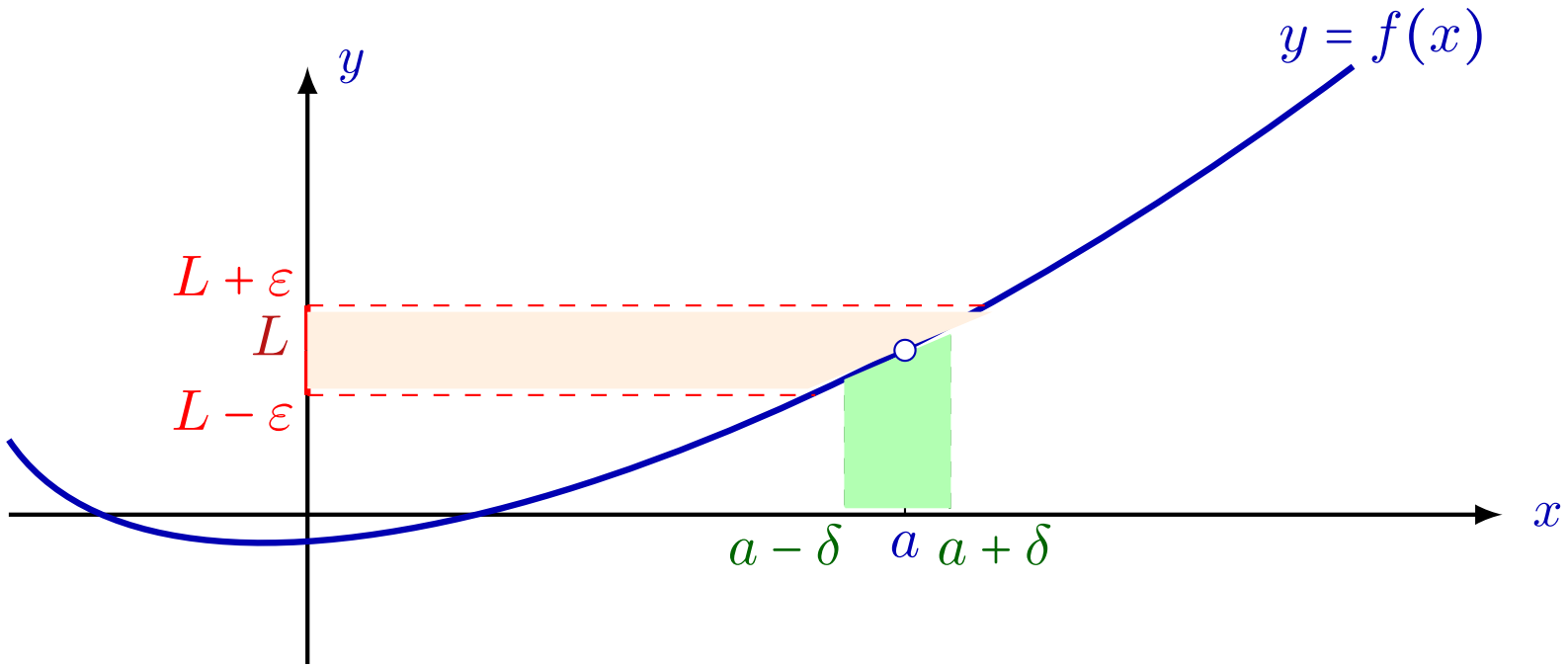

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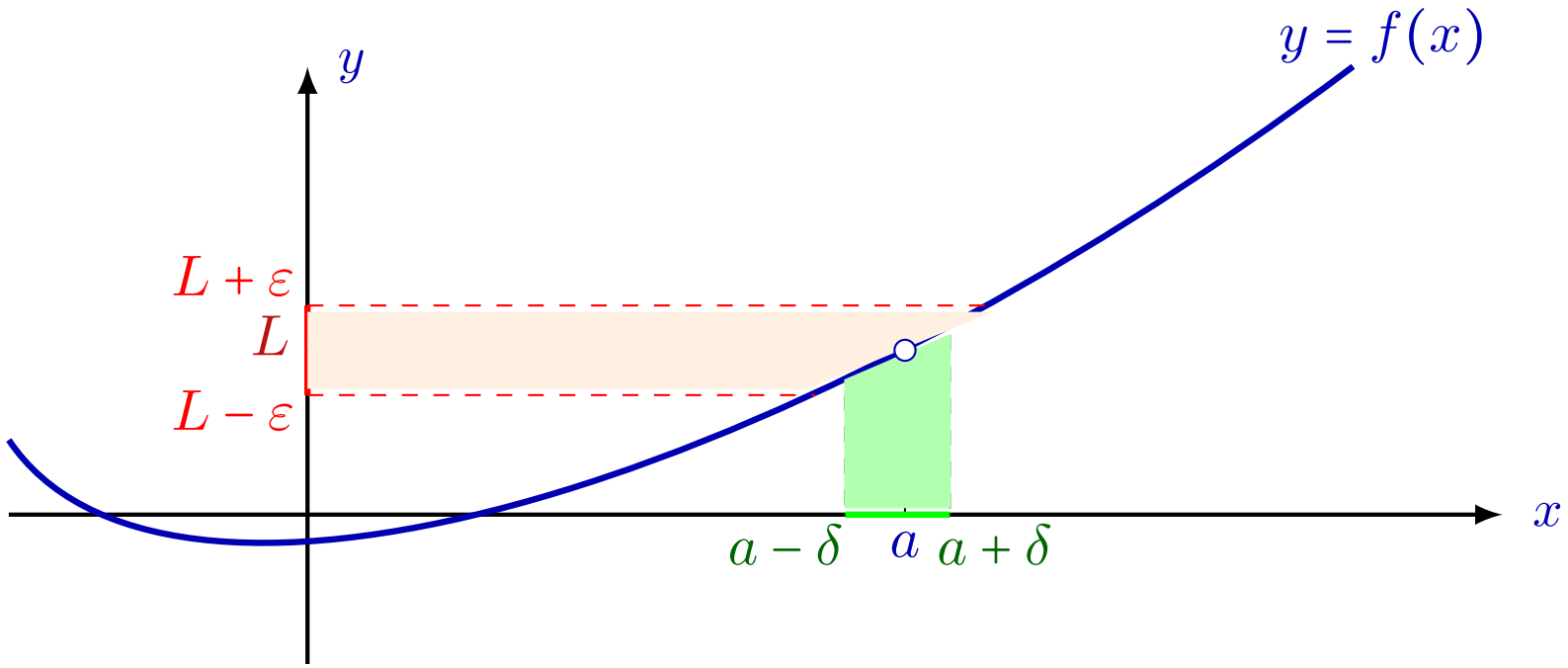

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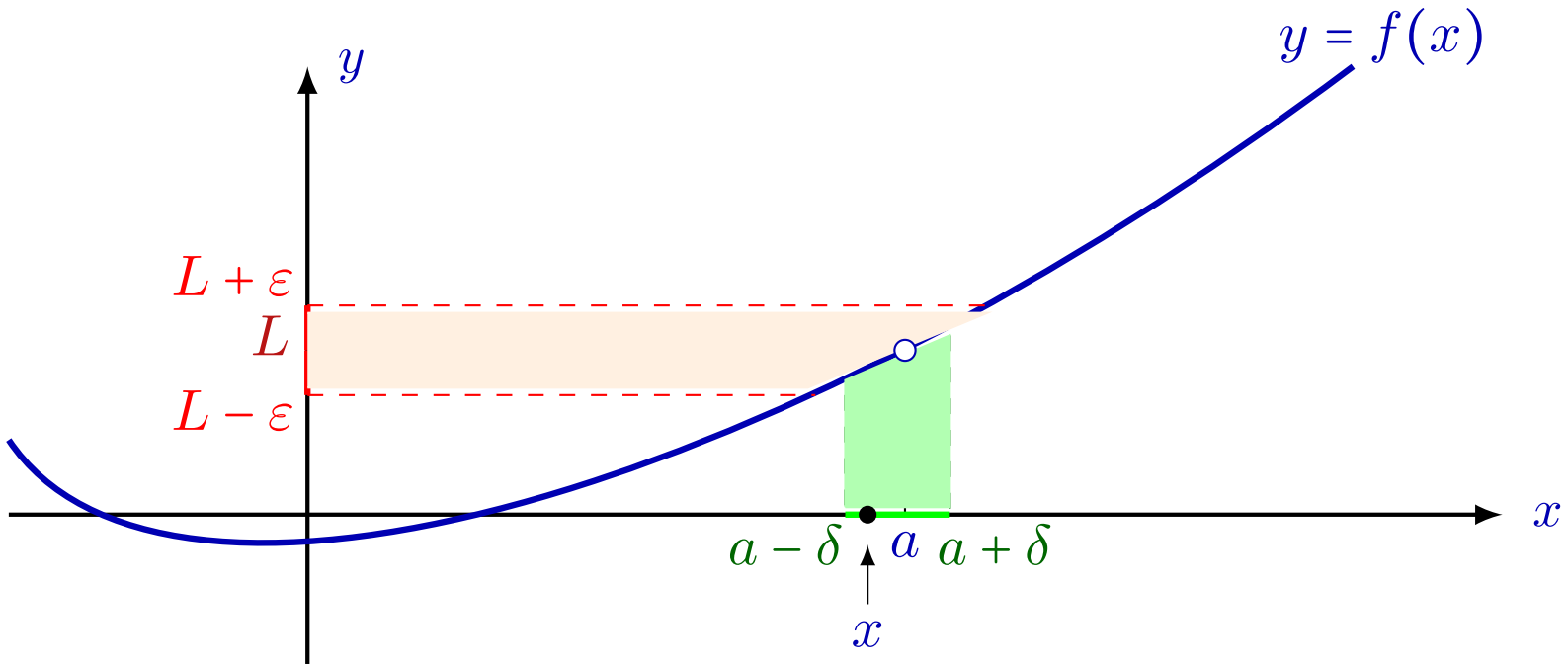

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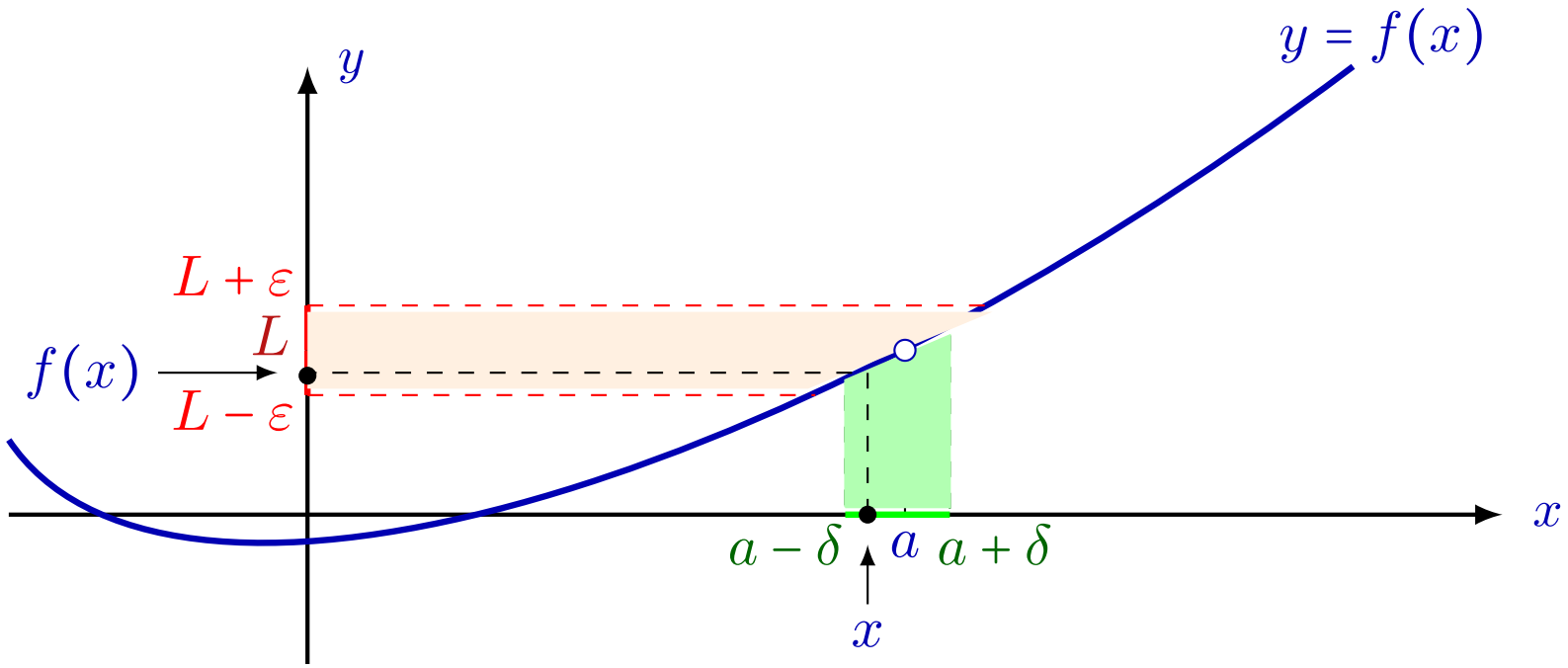

For any x such that $x \in (a - \delta, a + \delta)$, we have $f(x) \in (L - \varepsilon, L + \varepsilon)$.

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A number δ which we are looking for, is, therefore, $\delta=\frac{\varepsilon}{2}$.

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$$\forall\,\varepsilon > 0 \quad \exists\,\delta = \frac{\varepsilon}{2} \quad \forall\,x \quad |x-3| < \delta \implies |(2x+1)-7| < \varepsilon\,.$$

Therefore, by definition of limit,

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Let's write our reasoning in a form suitable to be a proof of the fact

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For **any** $\varepsilon > 0$, there exists δ, namely $\delta = \dfrac{\varepsilon}{2}$, such that if $|x-3| < \delta$,

then $|(2x+1)-7| = |2x-6| = 2|x-3| < 2\delta = 2\,\dfrac{\varepsilon}{2} = \varepsilon$.

In symbols only, this is written as follows:

$$\forall\,\varepsilon > 0 \quad \exists\,\delta = \frac{\varepsilon}{2} \quad \forall\,x \quad |x-3| < \delta \implies |(2x+1)-7| < \varepsilon\,.$$

Therefore, by definition of limit, $\lim_{x\to 3}(2x+1) = 7$.

Geometric interpretation

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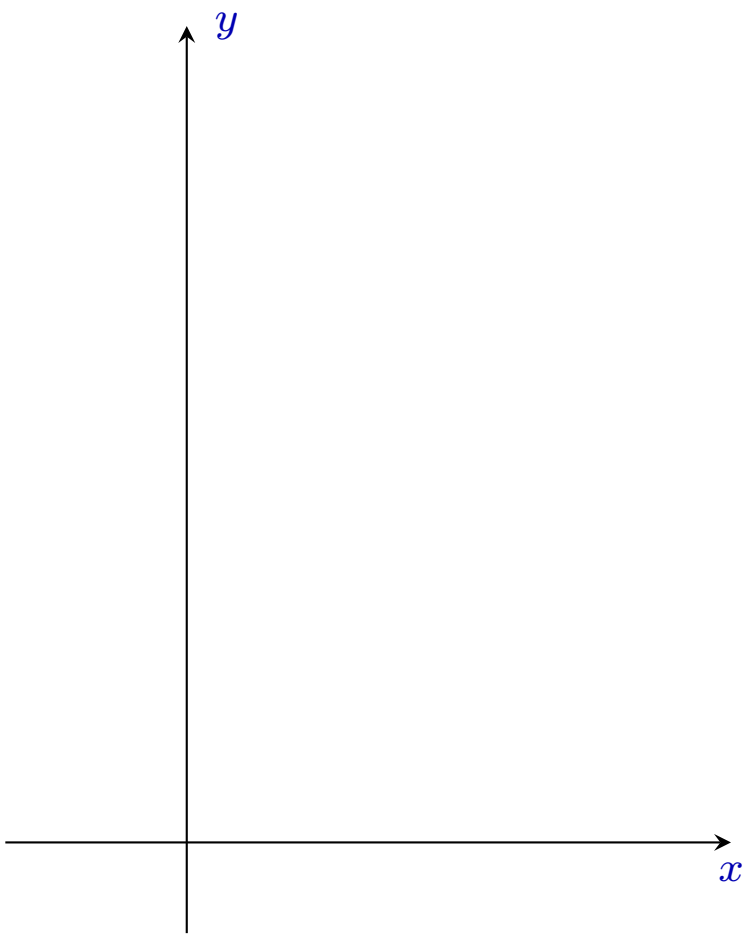

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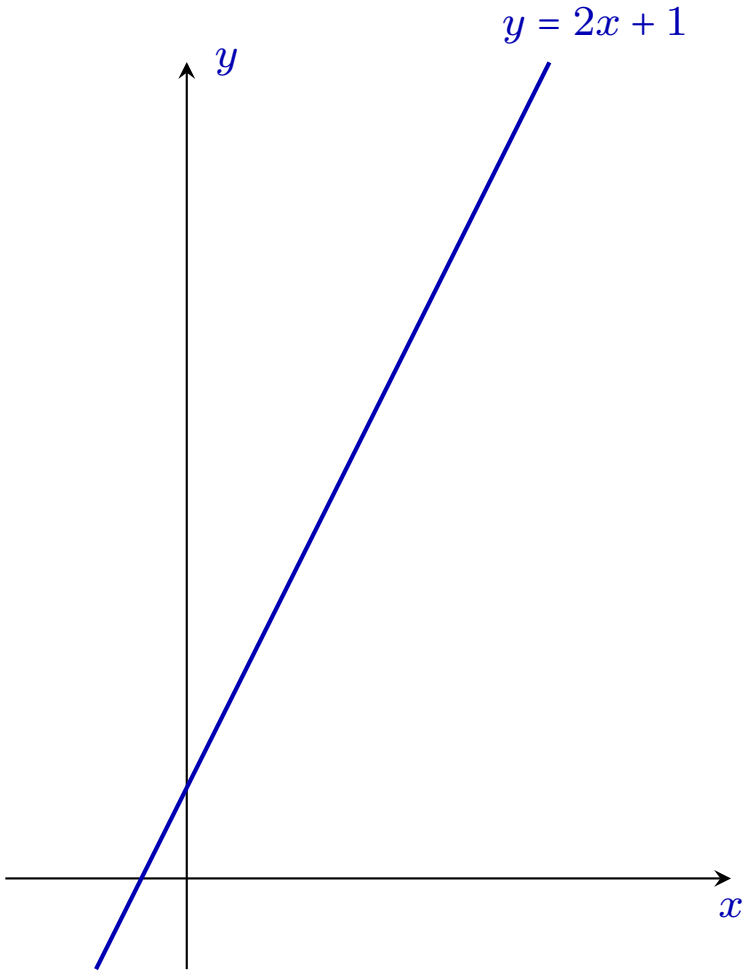

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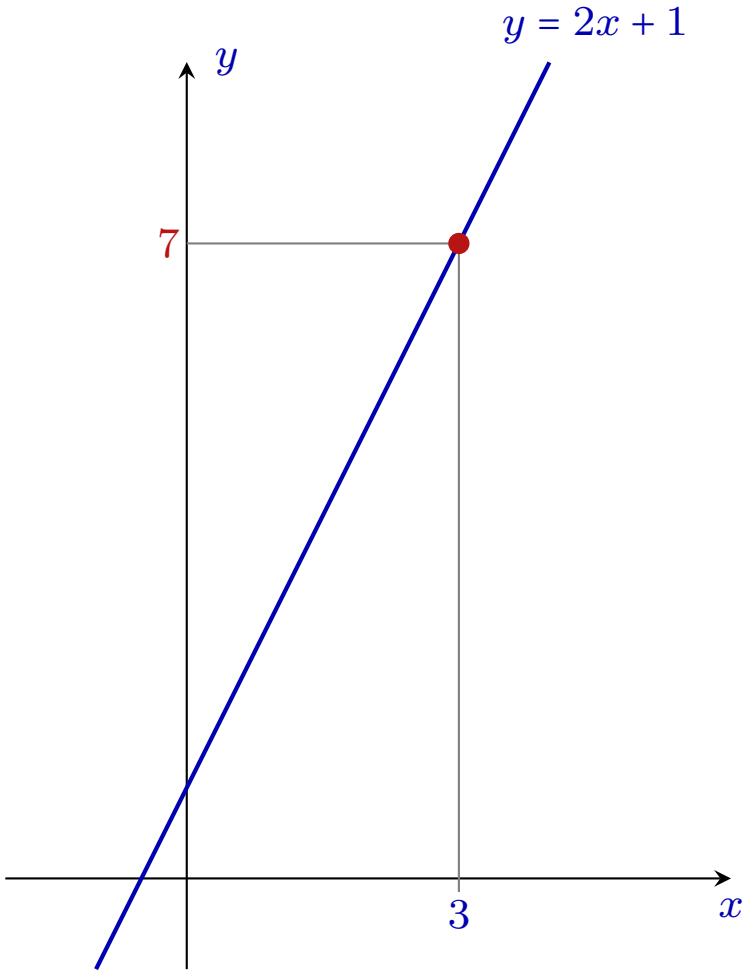

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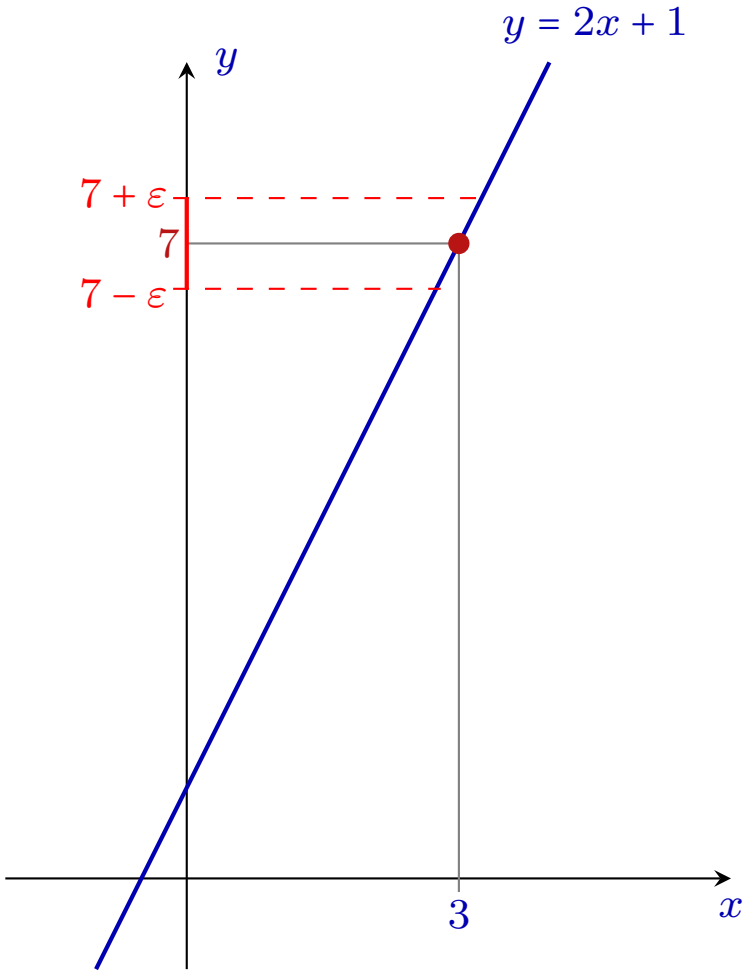

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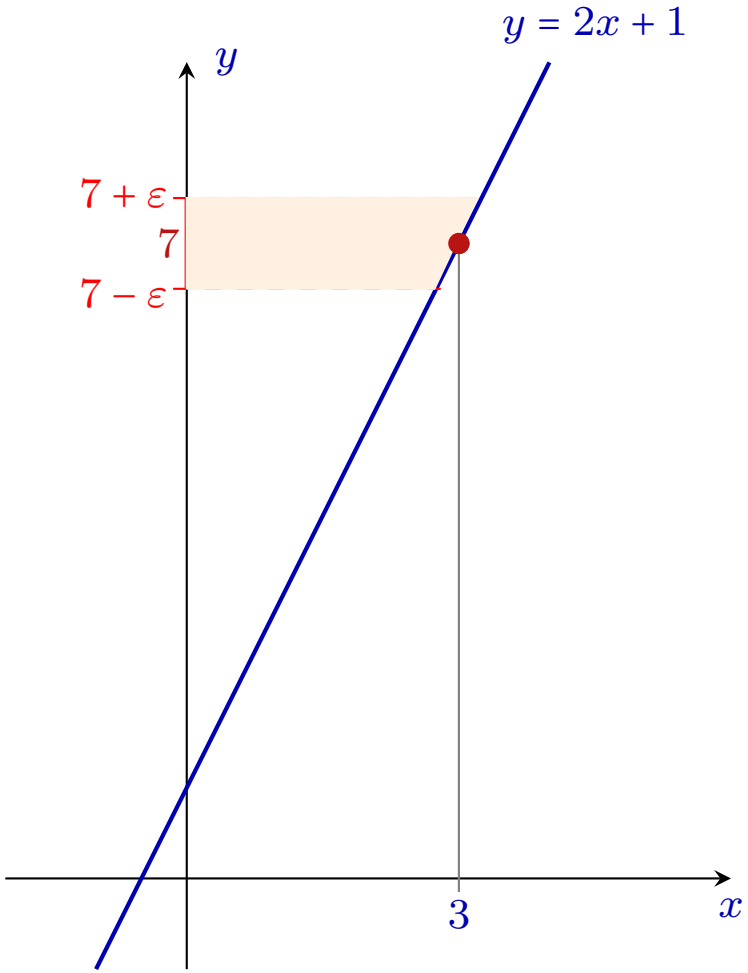

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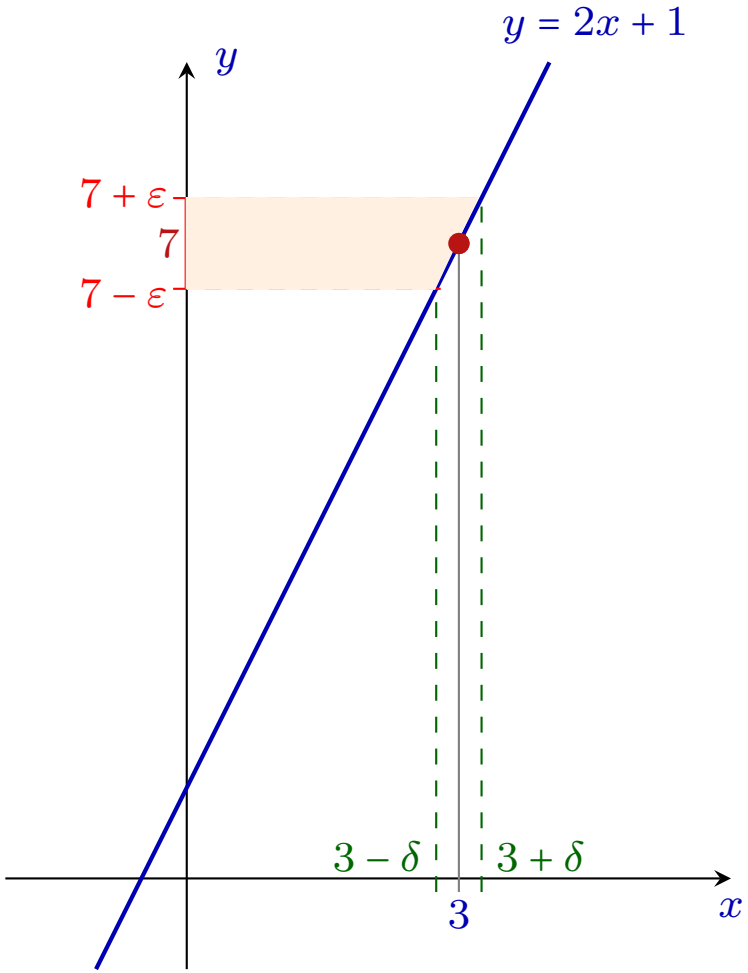

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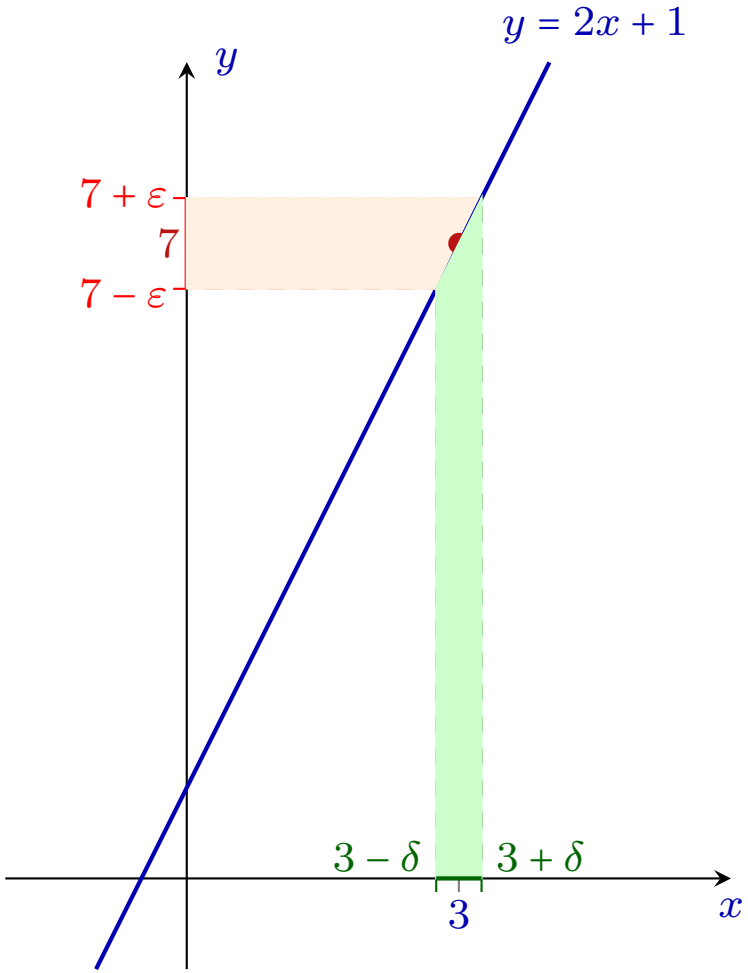

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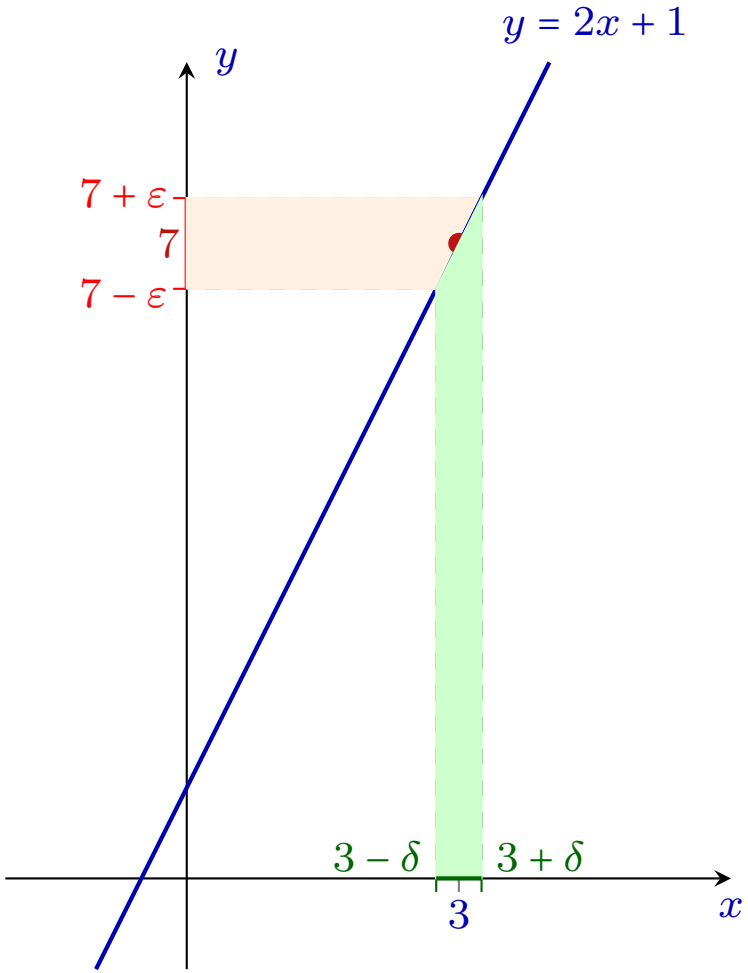

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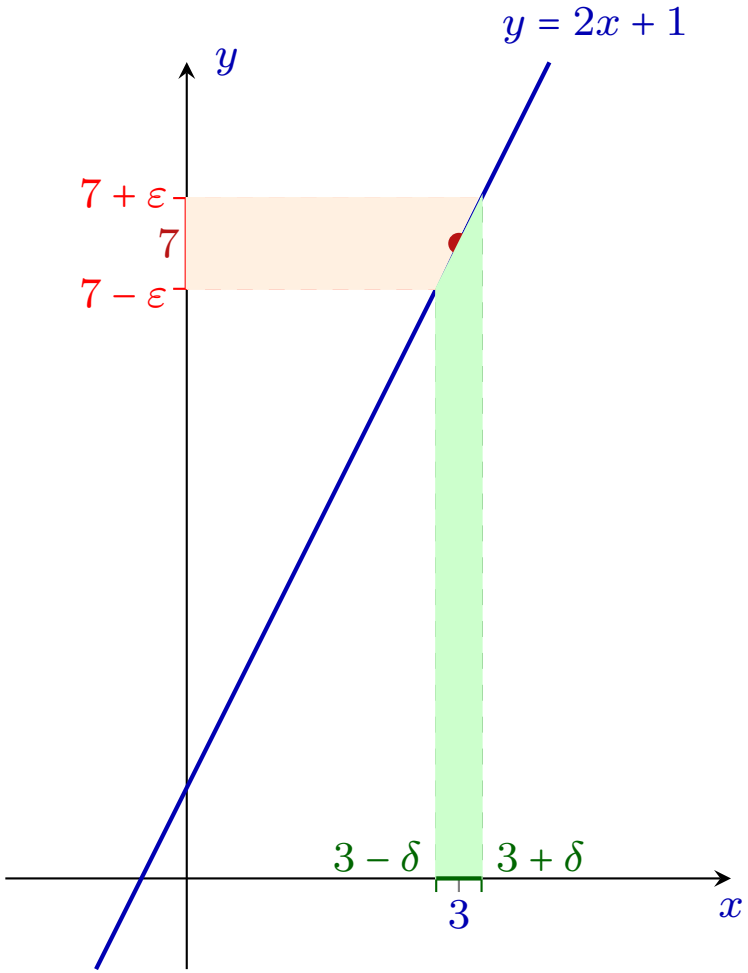

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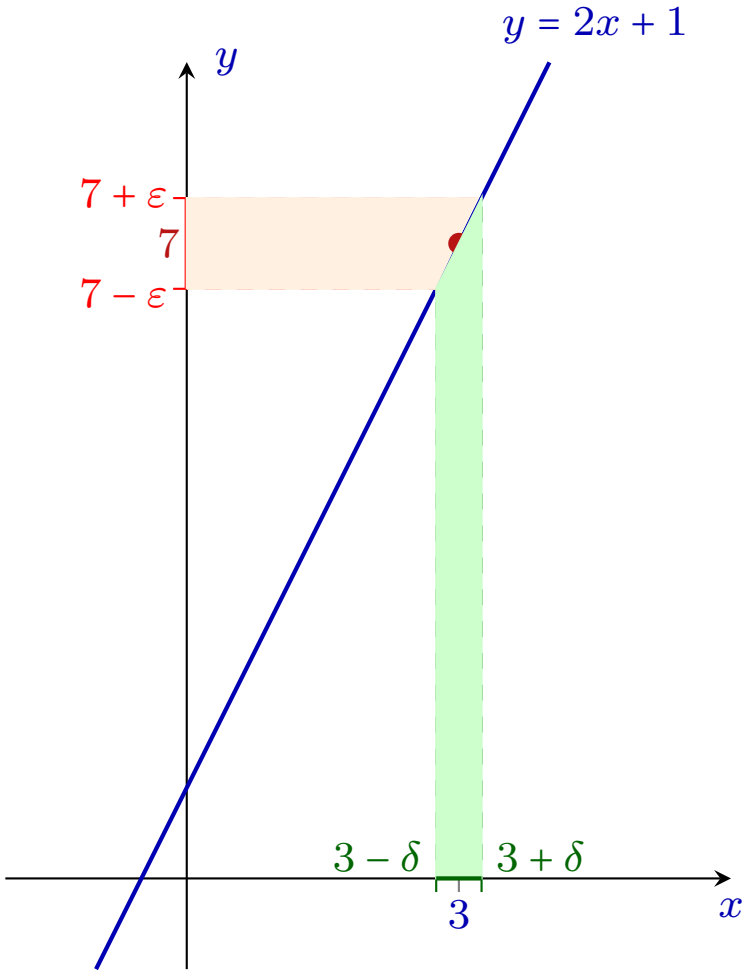

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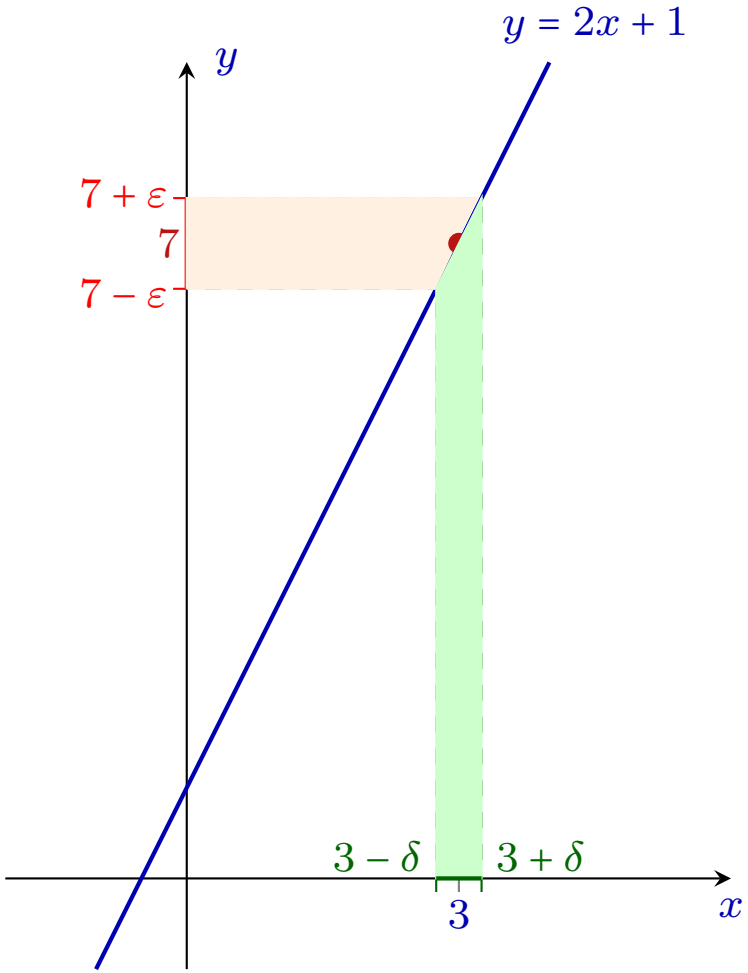

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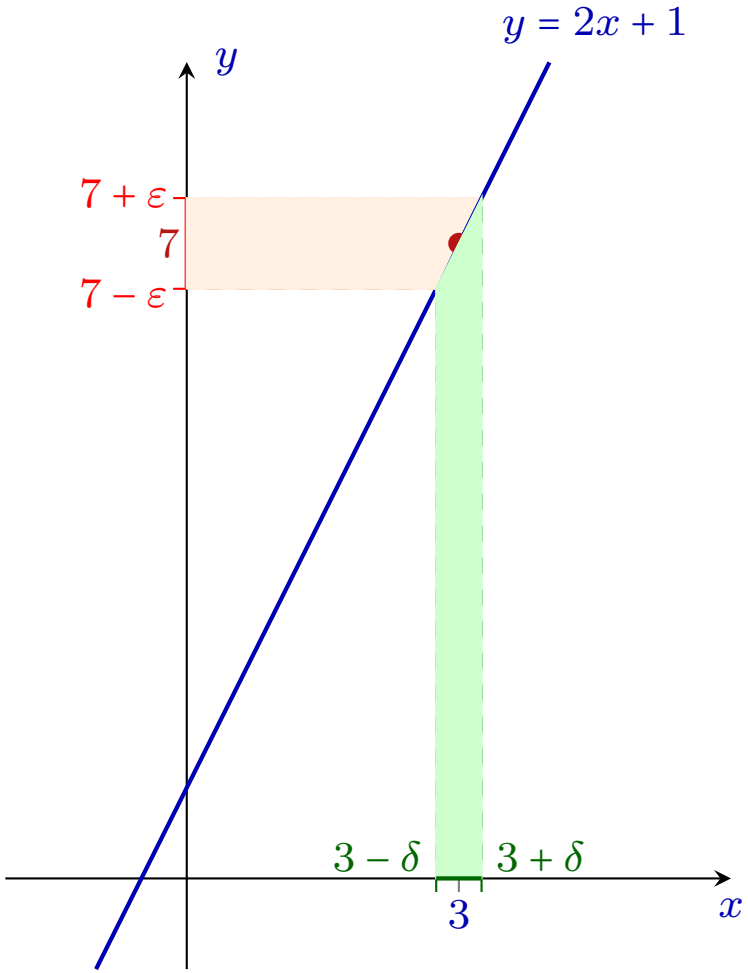

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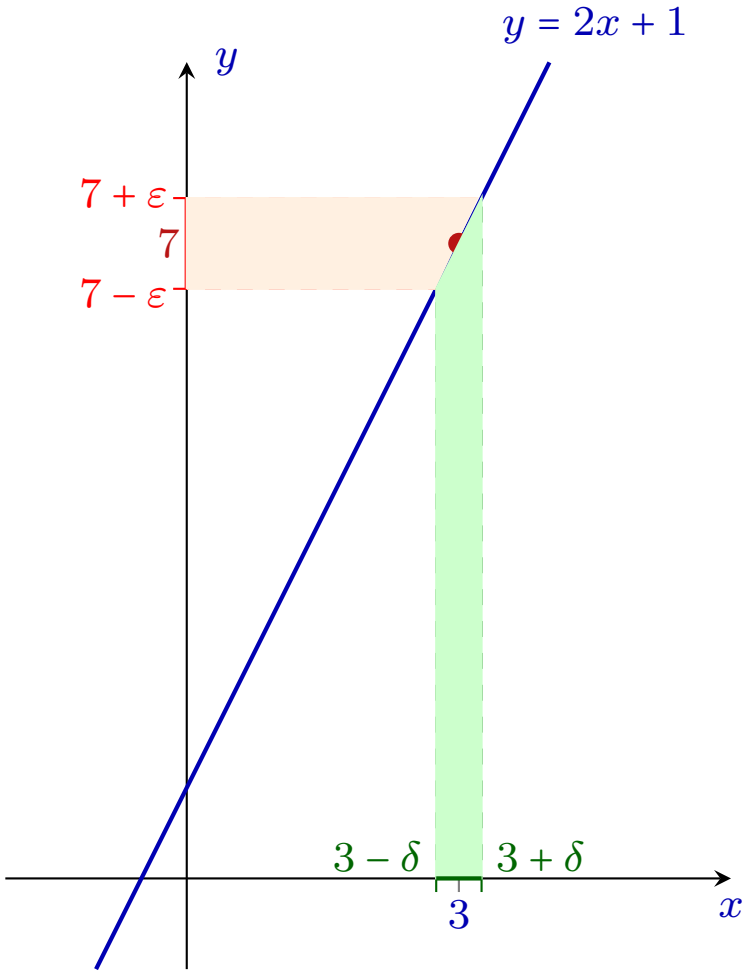

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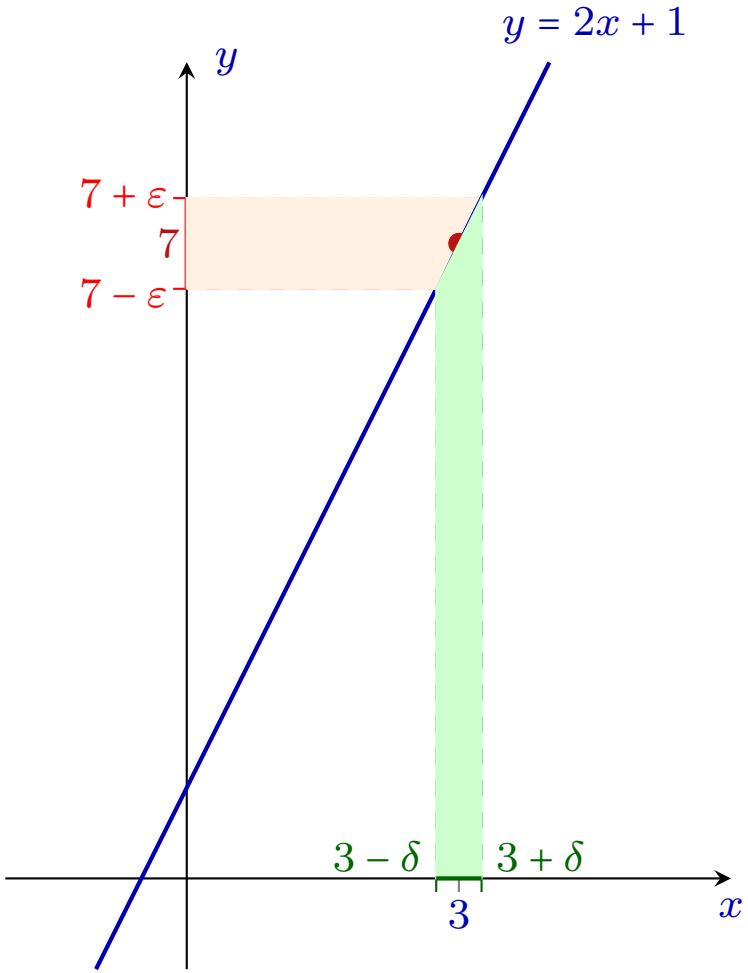

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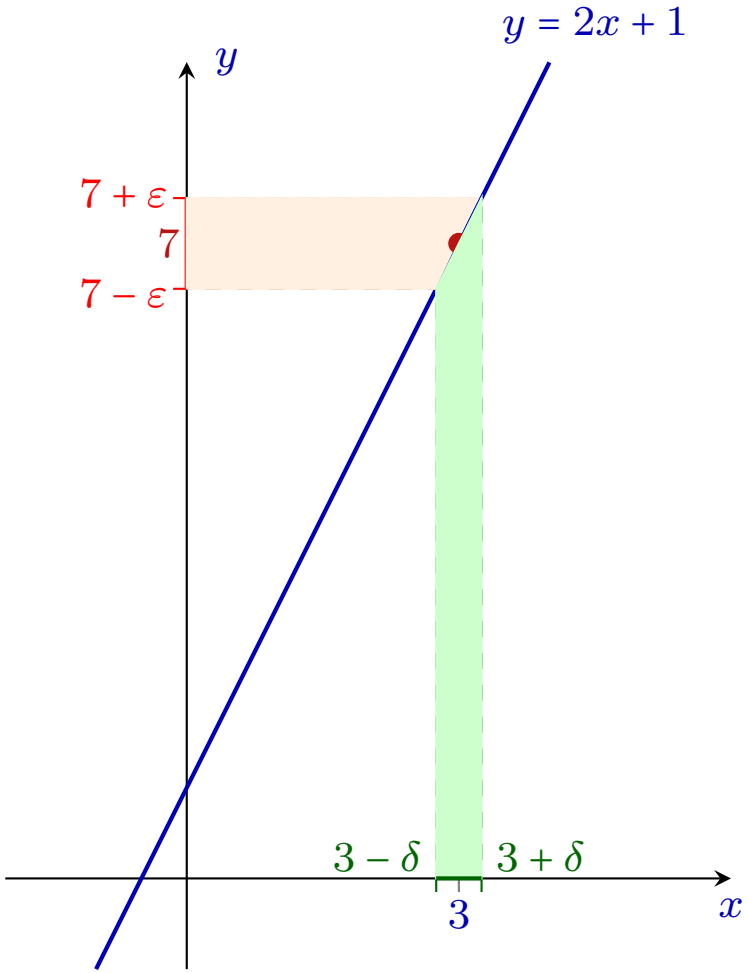

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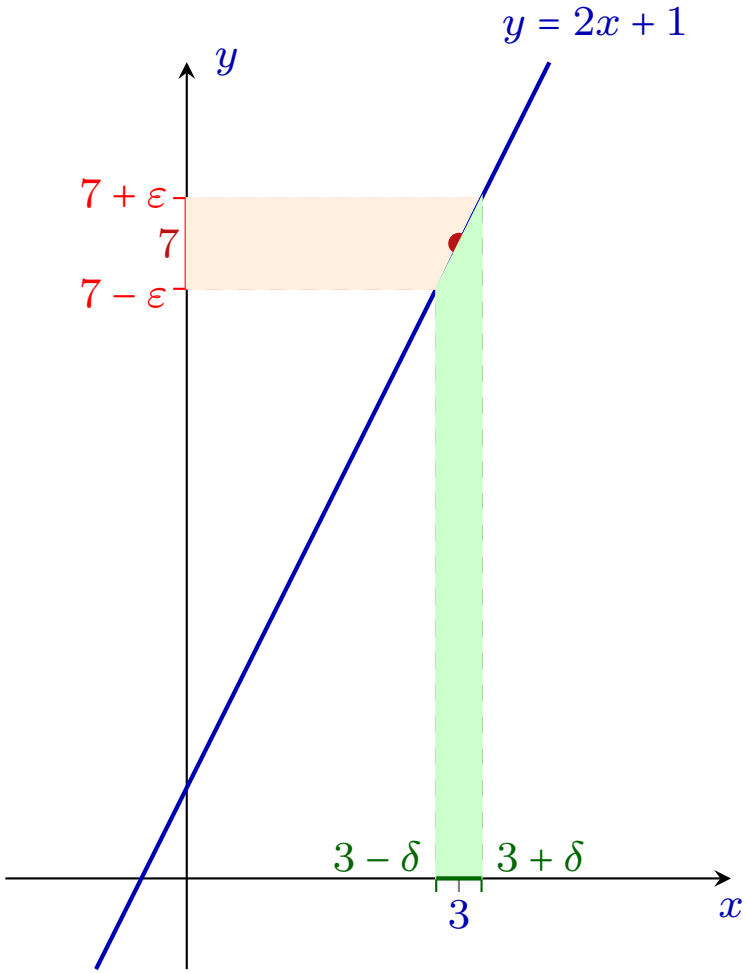

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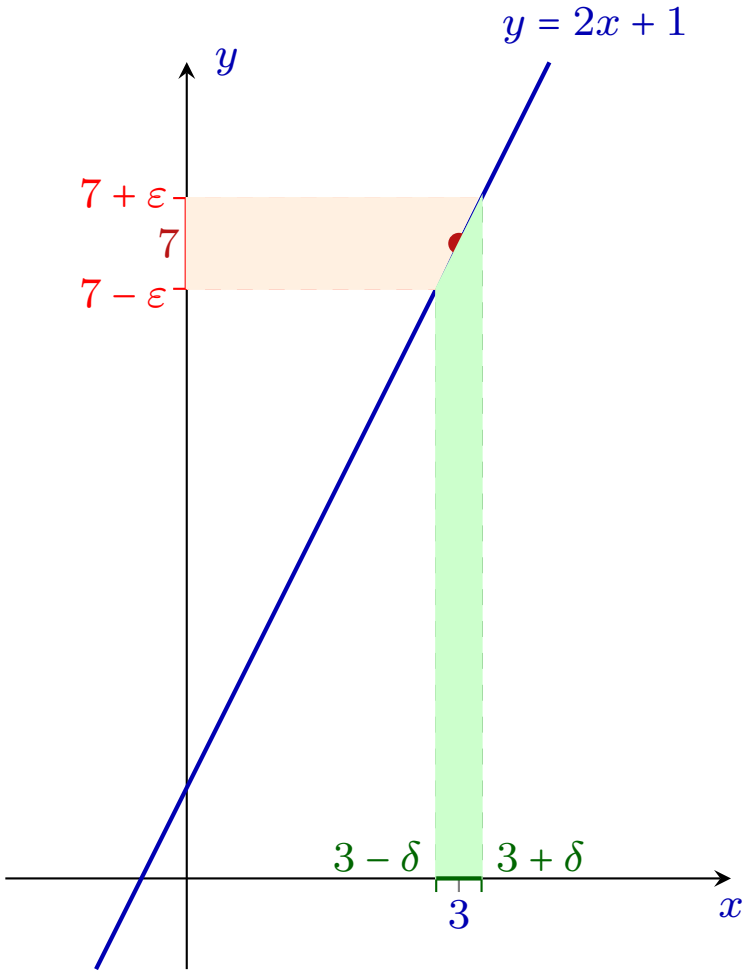

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It means, by definition of limit,

$7 = \lim\limits_{x\to 3} f(x)$, where $f(x) = 2x+1$.

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Exercise. Use the definition of limit to prove that $\lim\limits_{x \to 0} \left(\sin \dfrac{1}{x} \right)$ **does not exist**.

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A function f is said to be **continuous** at a if
the preimage $f^{-1}(U)$ of any neighborhood U of $f(a)$ is a neighborhood of a.