Lecture 3

Welcome to MAT 250!

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Brightspace

Welcome to MAT 250!

Brightspace for MAT 250.01

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Activities:

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Activities: lectures

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Final (5/14)	35%
HW	15%
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The **final grade** is the **maximum** of the score for final exam and the total grade calculated according to the scheme discribed above.

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P	Q	$P\wedge Q$	$P\vee Q$	$P\Longrightarrow Q$	$P\Longleftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

P	$\neg P$
T	F
F	T

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In Calculus their counterparts are **polynomials**.

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Conjunction and disjunction are involved in a few simple relations,
which allow to simplify a disjunctive normal form.

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In order to answer, we need to study $\Longrightarrow$ and $\Longleftrightarrow$.

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□

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Conclusion. the connective $\Longrightarrow$ is needed for **growth** of theories.

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- How can we **work with** definitions?
- How should we **read** a mathematical text?

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Let us illustrate the nature of a mathematical definition
using the definition of a rational number.

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The definition is clear and unambiguous, but to fully understand it,
we must know what an integer is and what a quotient means.

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The definition is clear and unambiguous, but to fully understand it,
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Definition of a rational number

Definition. A number is called **rational number**
if it can be presented as a quotient of two integers.

This definition contains three essential parts:

– The **term** (word or phrase) to be defined - "rational number".

– The **class** it belongs to - "numbers".

– The **distinguishing characteristic** -
"can be presented as a quotient of two integers".

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Give a **definition**, not just an example!

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Consider the definition of an even function.

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Then $3 \mid (\underbrace{4 + 5}_{9})$, but it is not true that $3 \mid 4$ and it is not true that $3 \mid 5$.

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There is no special notation for a prime number, so there are words in the symbolic form of the definition.

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Answer: Yes, it is true that if a integer greater than 1 is not prime,
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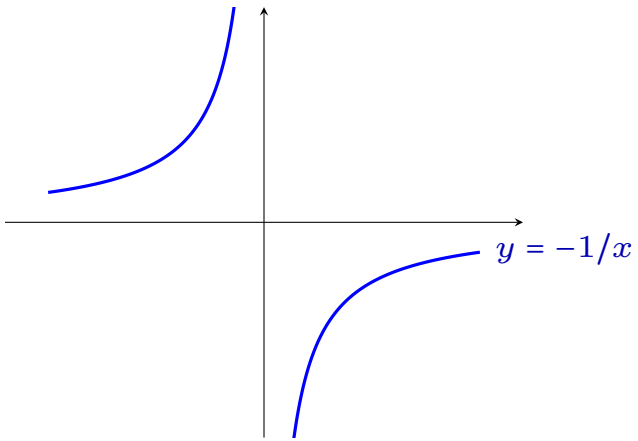

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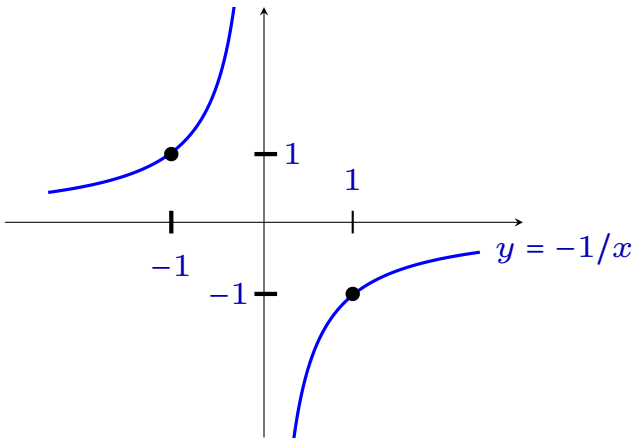

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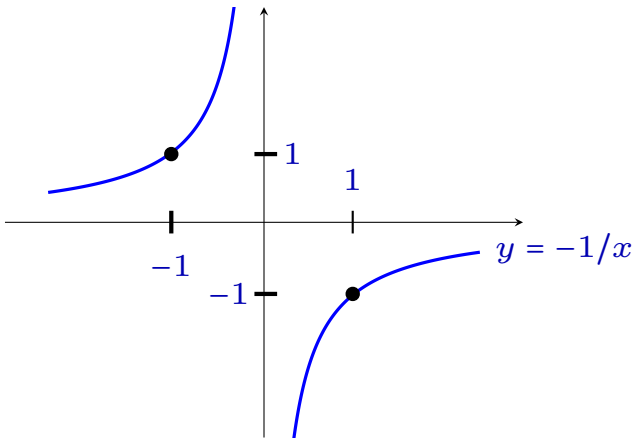

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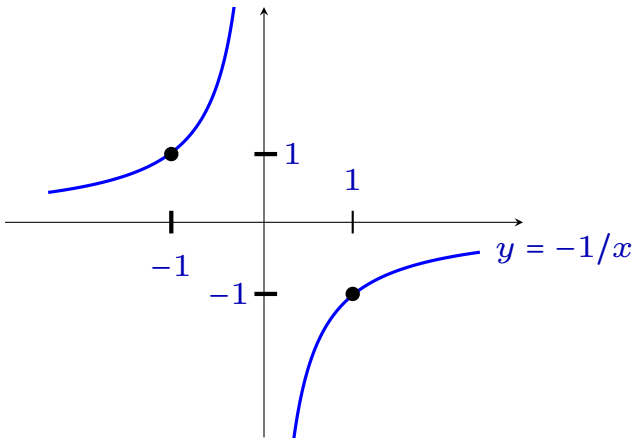

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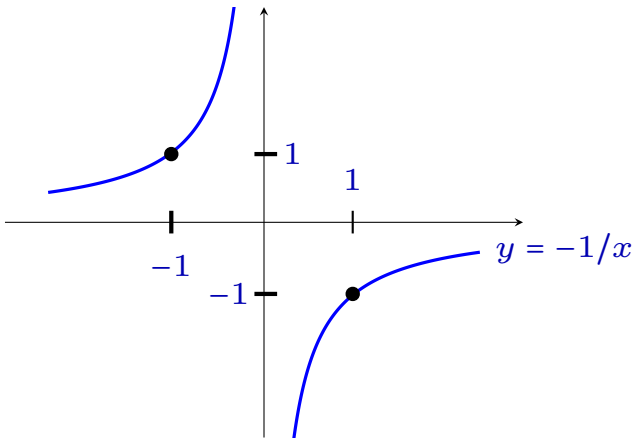

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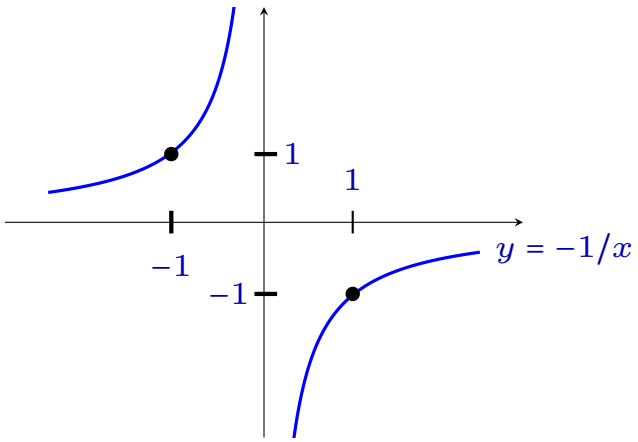

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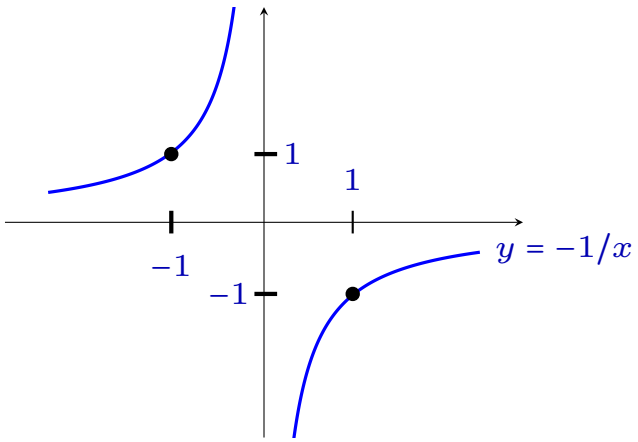

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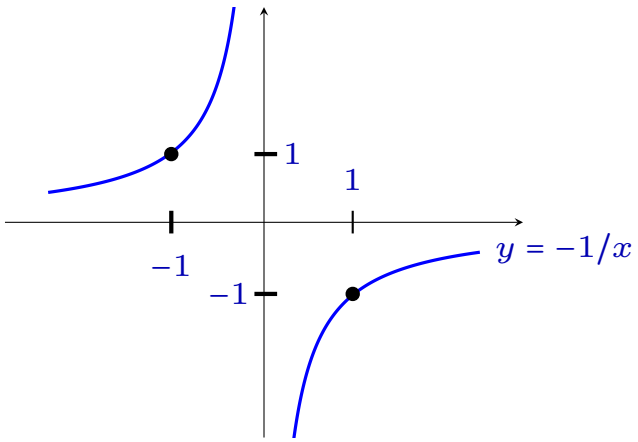

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Then $x_1 < x_2$, but $\underbrace{f(x_1)}_{1} \geq \underbrace{f(x_2)}_{-1}$.
So $\exists x_1, x_2 \ \big(x_1 < x_2 \wedge f(x_1) \geq f(x_2)\big)$.
Therefore, f is **not** increasing on its domain.

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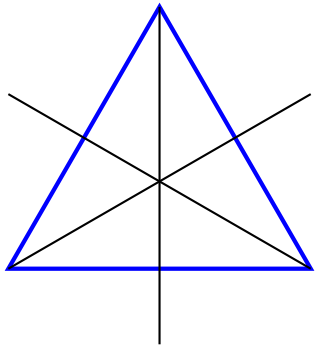

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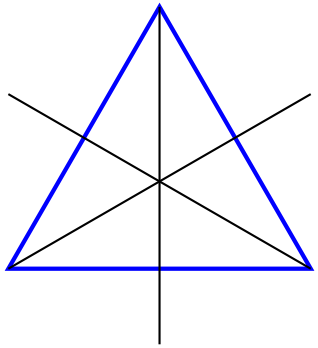

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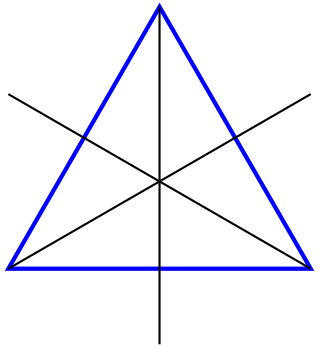

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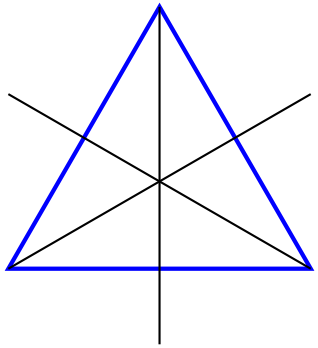

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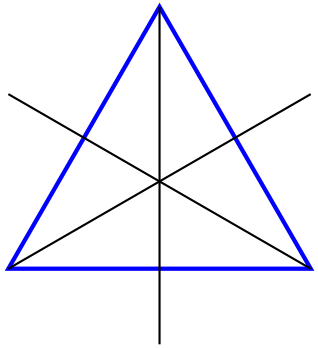

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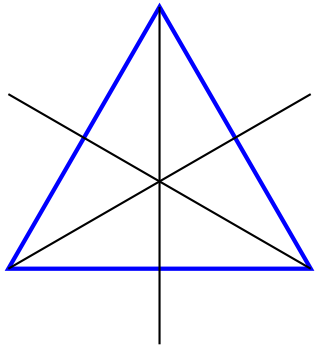

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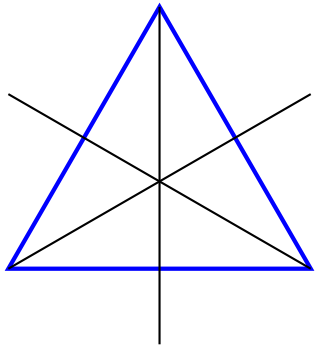

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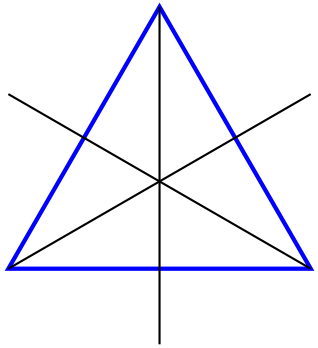

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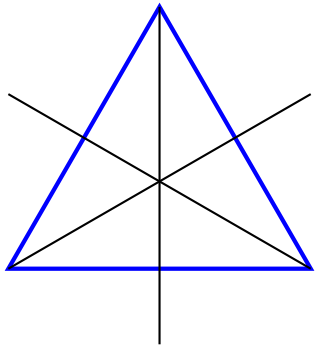

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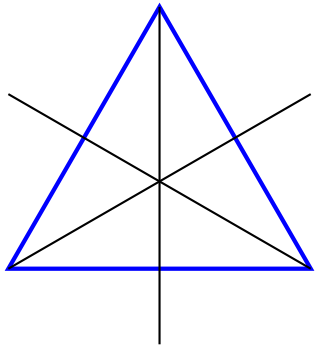

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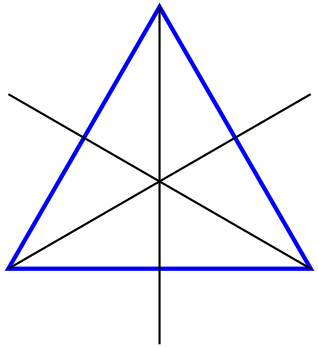

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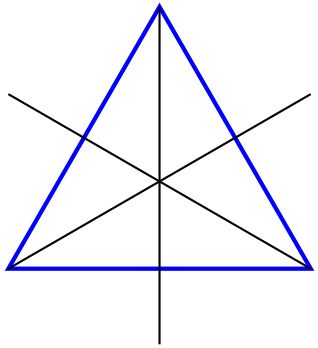

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This leads to the definition of a **group**.

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In the next rounds, you explore examples, and then engage in a careful, detailed reading of the proofs.

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As the first step towards classifying the lengths which can be constructed by straightedge and compass, this chapter introduces the concept of an algebraic number. Each such number will satisfy many polynomial equations and our immediate goal is to choose the simplest one.

A number $\alpha \in \mathbb{C}$ is said to be *algebraic over a field* $\mathbb{F} \subseteq \mathbb{C}$ if there exists a nonzero polynomial $f(x) \in \mathbb{F}[x]$ such that α is a zero of $f(x)$.

For each field $\mathbb{F}$, every number α in $\mathbb{F}$ is algebraic over $\mathbb{F}$ because α is a zero of the polynomial $f(x) = x - \alpha \in \mathbb{F}[x]$. This implies that e and π are algebraic over $\mathbb{R}$, though they are not algebraic over $\mathbb{Q}$ as we will prove later.

Let us read!

The number $\sqrt{2}$ is algebraic over $\mathbb{Q}$ because it is zero of the polynomial $f(x) = x^2 - 2$, which is nonzero and has coefficients in $\mathbb{Q}$.

In order to show that a number is algebraic, we look for a suitable polynomial having that number as zero. Try to prove that $1 + \sqrt{3}$ is algebraic over $\mathbb{Q}$.

It is useful to be able to recognize the definition of "algebraic over a field $\mathbb{F}$" when it appears in different guises: a number $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{F} \subseteq \mathbb{C}$ if and only if there is a positive integer n such that $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}, \alpha^n\}$ are linearly dependent over $\mathbb{F}$.

Indeed, if $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{F} \subseteq \mathbb{C}$ then there exists a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, whose coefficients $a_0, a_1, \ldots, a_n$ all belong to $\mathbb{F}$, at least one of these coefficients is nonzero, and $f(\alpha) = 0$, that is

$$a_0 + a_1\alpha + a_2\alpha^2 \cdots + a_{n-1}\alpha^{n-1} + a_n\alpha^n = 0. \qquad (*)$$

Let us read!

Since $\mathbb{F}$ is a subfield of $\mathbb{C}$, we can regard $\mathbb{C}$ as a vector space over $\mathbb{F}$. The numbers $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}, \alpha^n$ are all elements in $\mathbb{C}$, and hence can be regarded as vectors in the vector space $\mathbb{C}$ over $\mathbb{F}$.

The coefficients $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$, on the other hand, are all in $\mathbb{F}$ so we can regard them as scalars. Thus, the equality $(*)$ can be interpreted as a linear dependence of vectors $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}, \alpha^n$ in $\mathbb{C}$.

You will often meet the terms "algebraic number" and "transcendental number" where no field is specified. In such cases the field is taken to be $\mathbb{Q}$. We formalize this as follows.

A complex number is said to be an *algebraic number* if it is algebraic over $\mathbb{Q}$; a *transcendental number* if it is not algebraic over $\mathbb{Q}$.
