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1. Let $P(x)$, $Q(x)$, and $R(x)$ be predicates defined on the universe U and $A = \{x \in U \mid P(x)\}$, $B = \{x \in U \mid Q(x)\}$, and $C = \{x \in U \mid R(x)\}$ be sets. Reformulate the propositions

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1.1 $\forall x \in U(P(x) \implies \neg(Q(x) \implies \neg R(x)))$

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as statements about the sets A, B and C.

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The latter can be rewritten as $A \subset (B \smallsetminus C)$.

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The latter can be rewritten as $A \subset (B \setminus C)$.
The answer for 1.1: $A \subset (B \setminus C)$

Solution for 1.2

We proved that the proposition $\exists x \in U(P(x) \implies \neg(Q(x) \implies \neg R(x)))$ is equivalent to $\exists x \in U(P(x) \implies (Q(x) \wedge \neg R(x)))$.

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Problem 2

Let $A_n = \left(-\frac{1}{n}, \frac{2n-1}{n}\right]$. Find $\bigcap_{n=1}^{\infty} A_n$ and $\bigcup_{n=1}^{\infty} A_n$. Justify your result.

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Preliminary considerations.

Observe that $A_1 = (-1, 1]$, $A_2 = (-\frac{1}{2}, \frac{3}{2}]$, $A_3 = (-\frac{1}{3}, \frac{5}{3}]$, ...

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The left end points converge to 0: $-\frac{1}{n} \;\to 0$,

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We have to prove these inclusions, and check if the end points are contained in X and Y .

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And also we have to prove that no other points are contained.

Solution of Problem 2

Intervals and inequalities.

$$A_n = \left(-\frac{1}{n}, \frac{2n-1}{n}\right] = \left\{x \in \mathbb{R} \mid -\frac{1}{n} < x \leq \frac{2n-1}{n}\right\}$$

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Intervals and inequalities.

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Indeed, if $0 < x$, then $\forall n \in \mathbb{N}\ (-\frac{1}{n} < x)$
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Moreover, for $x = 0$ we have $\forall n \in \mathbb{N}\ (-\frac{1}{n} < x)$ and for $x = 1$ we have $\forall n \in \mathbb{N}\ (x < \frac{2n-1}{n})$.

Intervals and inequalities.

$$A_n = \left(-\frac{1}{n}, \frac{2n-1}{n}\right] = \left\{x \in \mathbb{R} \mid -\frac{1}{n} < x \leq \frac{2n-1}{n}\right\}$$

$$X = \bigcap_{n=1}^{\infty} A_n = \left\{x \in \mathbb{R} \mid \forall n \in \mathbb{N} \left(-\frac{1}{n} < x \leq \frac{2n-1}{n}\right)\right\}$$

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Proof $(0,1) \subset X$ means

$$\forall x \left((0 < x < 1) \implies \forall n \in \mathbb{N} \left(-\frac{1}{n} < x \leq \frac{2n-1}{n}\right)\right)$$

Indeed, if $0 < x$, then $\forall n \in \mathbb{N}\ (-\frac{1}{n} < x)$
and if $x < 1$, then $\forall n \in \mathbb{N}\ (x < \frac{2n-1}{n} = 2 - \frac{1}{n})$.
Moreover, for $x = 0$ we have $\forall n \in \mathbb{N}\ (-\frac{1}{n} < x)$ and for $x = 1$ we have $\forall n \in \mathbb{N}\ (x < \frac{2n-1}{n})$. Thus $0, 1 \in X$ (and $[0,1] \subset X$).

Solution of Problem 2

Intervals and inequalities.

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In order to prove equality $[0,1] = X$, we still have to prove $[0,1] \supset X$.
The latter is equivalent to

$$\forall x \left(\forall n \in \mathbb{N} \left(-\frac{1}{n} < x \leq \frac{2n-1}{n}\right) \implies (0 \leq x \leq 1)\right).$$

Solution of Problem 2

Thus we have to prove

$$\forall x \left(\forall n \in \mathbb{N} \left(-\frac{1}{n} < x \leq \frac{2n-1}{n} \right) \implies (0 \leq x \leq 1) \right).$$

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Thus we have to prove

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Assume to the contrary that

$$\exists x \left(\forall n \in \mathbb{N} \left(-\frac{1}{n} < x \leq \frac{2n-1}{n} \right) \wedge ((x < 0) \vee (1 < x)) \right)$$

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The sequence $-\dfrac{1}{n}$ converges to 0.

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The sequence $\frac{2n-1}{n}$ tends to 1. Therefore if $1 > x$, then $\exists n \left(\frac{2n-1}{n} < x \right)$.

Solution of Problem 2

Thus we have to prove

$$\forall x \left(\forall n \in \mathbb{N} \left(-\frac{1}{n} < x \leq \frac{2n-1}{n} \right) \implies (0 \leq x \leq 1) \right).$$

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The sequence $\dfrac{2n-1}{n}$ tends to 1. Therefore if $1 > x$, then $\exists n \left(\dfrac{2n-1}{n} < x \right)$.

These contradicts our assumption. □

Problem 3

Prove that if the lengths of all sides of a right triangle are integers, then one of them is divisible by 5.

Problem 4

Let $f : X \to Y$, $g : Y \to Z$, $h : Z \to X$ be maps. Prove that if $f \circ h \circ g$ and $g \circ f \circ h$ are injections and $h \circ g \circ f$ is surjection, then f, g and h are bijections.

Problem 5

In the following two situations verify if $\sim$ is an equivalence relation in X and,
if it is, then list the equivalence classes,
if it is not, then explain why.

a) $X = \{2, 4, 7, 8, 9, 15\}$ and $x \sim y \iff \text{gcd}(x, y) \neq 1$.

b) $X = \{2, 4, 6, 7, 8, 9, 15\}$ and $x \sim y \iff \text{gcd}(x, y) \neq 1$.

Problem 5

In the following two situations verify if $\sim$ is an equivalence relation in X and,
if it is, then list the equivalence classes,
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a) $X = \{2, 4, 7, 8, 9, 15\}$ and $x \sim y \iff \text{gcd}(x, y) \neq 1$.

b) $X = \{2, 4, 6, 7, 8, 9, 15\}$ and $x \sim y \iff \text{gcd}(x, y) \neq 1$.

Partition or relation?

Problem 6

6. A number $z \in \mathbb{C}$ is called *algebraic* if it is a root of equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, where $n \in \mathbb{N}$ and $a_i \in \mathbb{Z}$ for $i = 0, \ldots, n$. Prove that the set of all algebraic numbers is countable.