Lecture 13

Cardinalities

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However,
cardinalities of infinite sets and the **numbers of elements** of finite sets have quite different properties.

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$|\mathbb{N}| = |A|$, because of bijection $\mathbb{N} \to A : x \mapsto 2x$. □

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The cardinality of $\mathbb{N}$ is denoted by $\aleph_0$, that is $|\mathbb{N}| = \aleph_0$.

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$\mathbb{N}$ vs. $\mathbb{Z}$

Which set has more elements, $\mathbb{N}$ or $\mathbb{Z}$?
$\mathbb{N} \subset \mathbb{Z}$, and $\mathbb{Z} \setminus \mathbb{N} = \{0, -1, -2, -3, \dots\}$ is equipotent to $\mathbb{N}$.

However, $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

To prove this, let us construct a **bijection** $\mathbb{N} \to \mathbb{Z}$.

$$\begin{array}{cccccccccccc} \mathbb{N} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} & \dots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\ \mathbb{Z} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{-2} & \mathbf{3} & \mathbf{-3} & \mathbf{4} & \mathbf{-4} & \dots \end{array}$$

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In Calculus a similar phenomenon is known as "**indeterminacy** $\infty - \infty$".

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Corollary. Let X, Y be sets. Then $|X| \leq |Y|$, iff $\exists$ surjection $Y \to X$.

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Lemma 3. Inequality of cardinal numbers is total.

It suffices to prove that *for any sets* X, Y *, there exists map* $X \to Y$ *which is either injection or surjection.*

Proof. Let us try to build an injective map $X \to Y$, defining it on elements of X, one by one, and taking care of injectivity.

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What may prevent us from accomplishing this?

Either we would not find p. This would happen if $X \setminus A = \varnothing$, that is $A = X$, and then f is a desired injection $X \to Y$. In this case, $|X| \leq |Y|$.

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If nothing like this stops us, we continue and eventually come to constructing a map $X \to Y$ which is either injection or surjection. $\square$.

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Equivalence classes for grand orbit equivalence are called **grand orbits** of f.

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$$h(z) = \begin{cases} f(z) \text{ if } z \in X \\ g(z), \text{ if } z \in Y \end{cases} .$$

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A grand orbit O is recovered from any of its elements:
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The result is a map $X \to Y$ which maps bijectively each $O \cap X$ to $O \cap Y$.

Proof of C-B-S theorem. Part 2

Grand orbits in $Z = X \cup Y$ are of four types:

- periodic,
- non-periodic starting with an element of X, like $a, f(a), gf(a), fgf(a), \dots$
- non-periodic starting from an element of Y, like $b, g(b), fg(b), gfg(b), \dots$
- non-periodic infinite both forwards and backwards.

A grand orbit O is divided into $O \cap X$ and $O \cap Y$. The map f defines an injection $f_O : O \cap X \to O \cap Y$. This is a bijection, unless O is of the third type.

If O is of the third type,

then $(O \cap Y) \smallsetminus f(O \cap X)$ consists of the initial element of O.

But then g defines a map $g_O : O \cap Y \to O \cap X$ which is bijective.

Then $g_O^{-1} : O \cap X \to O \cap Y$ is a bijection.

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Due to Lemma, such presentation of a real number is unique.

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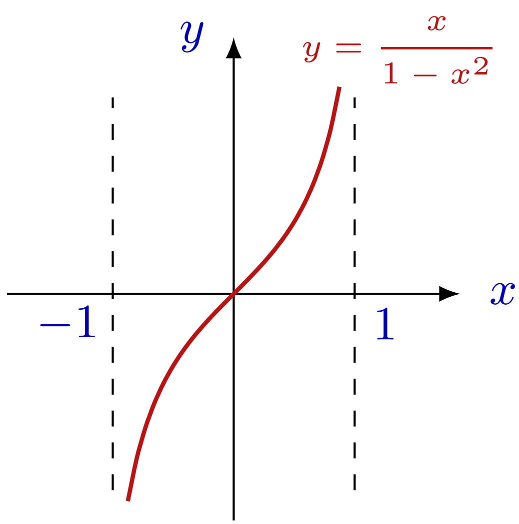

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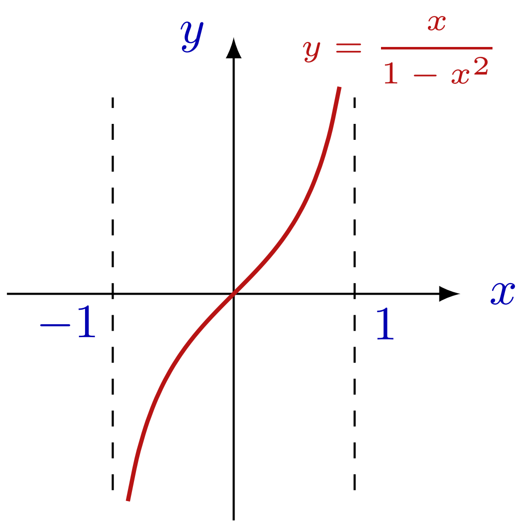

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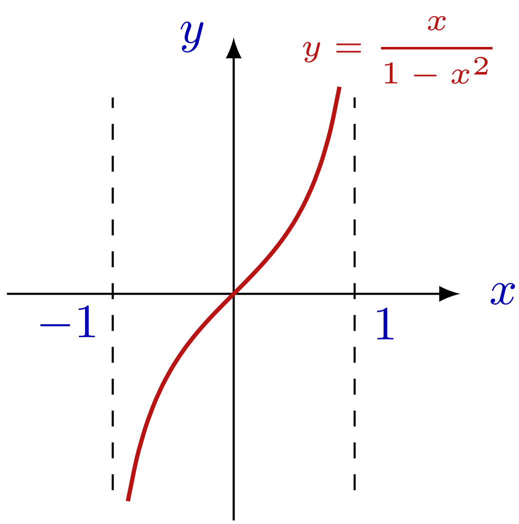

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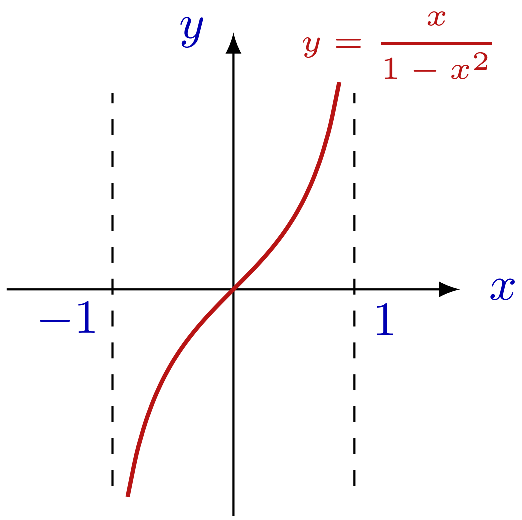

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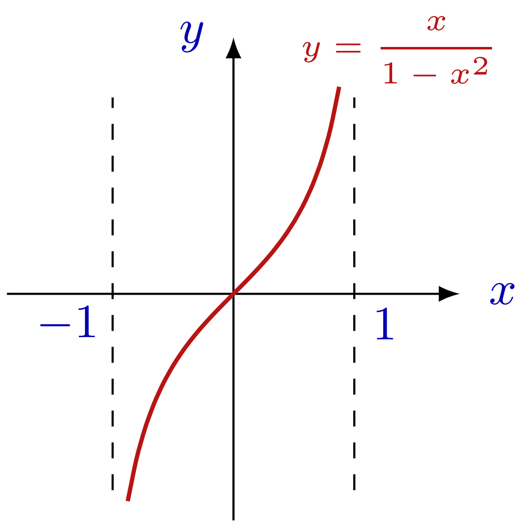

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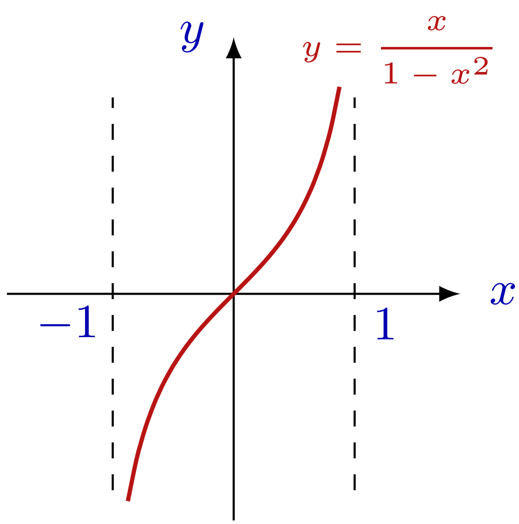

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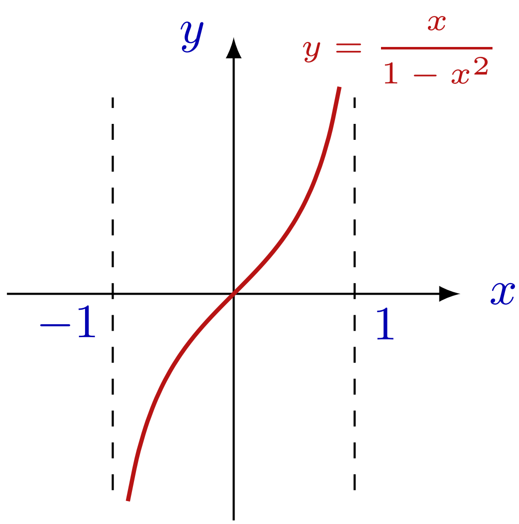

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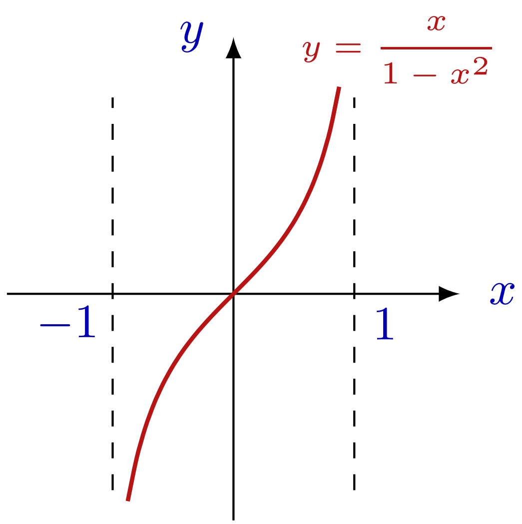

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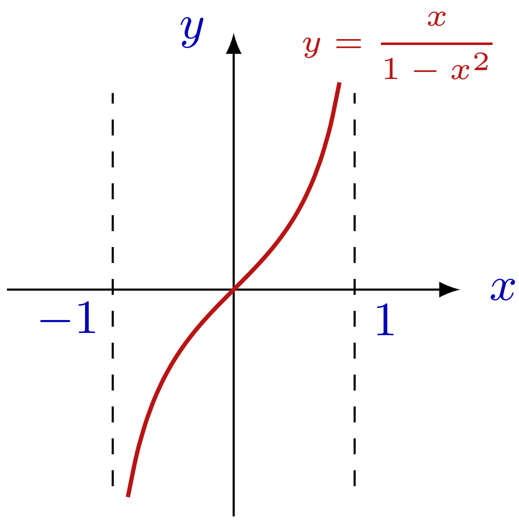

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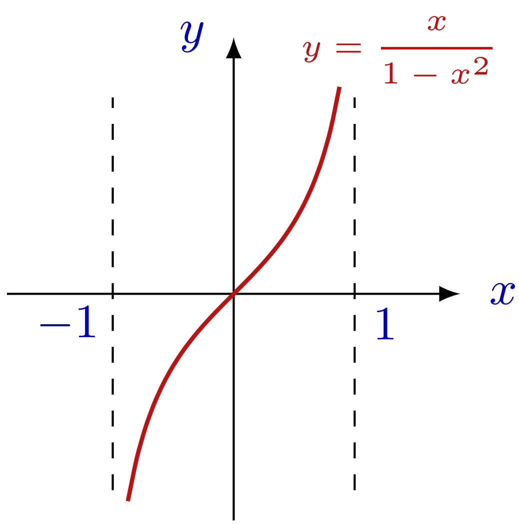

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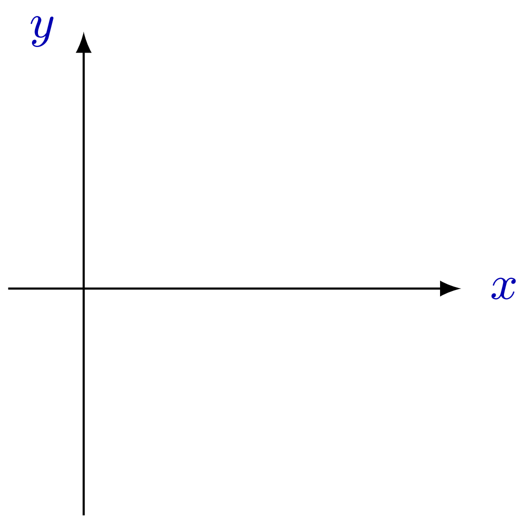

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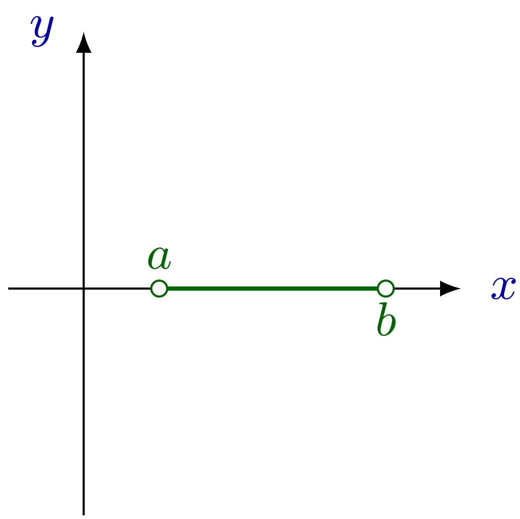

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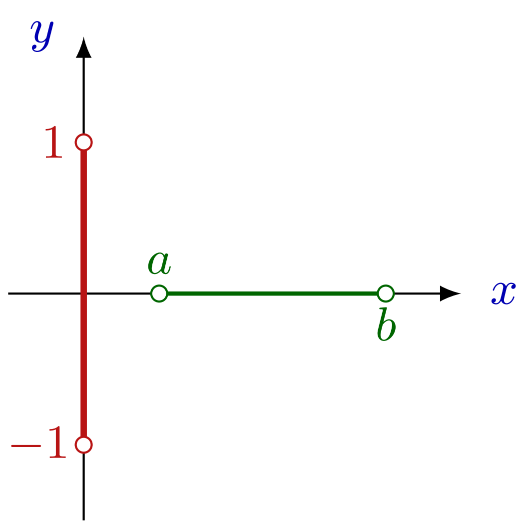

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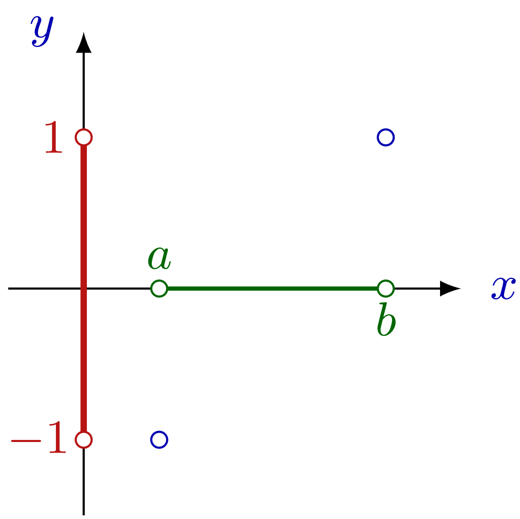

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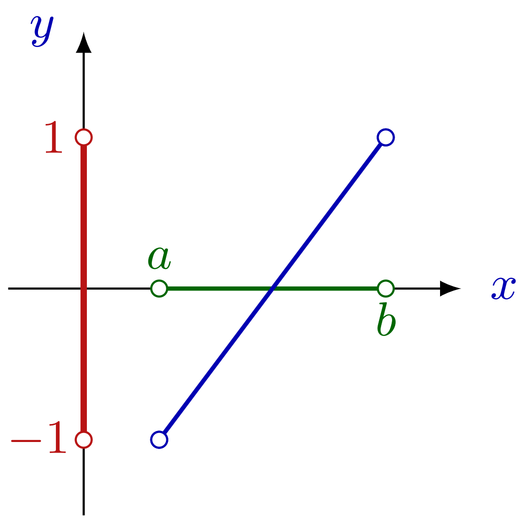

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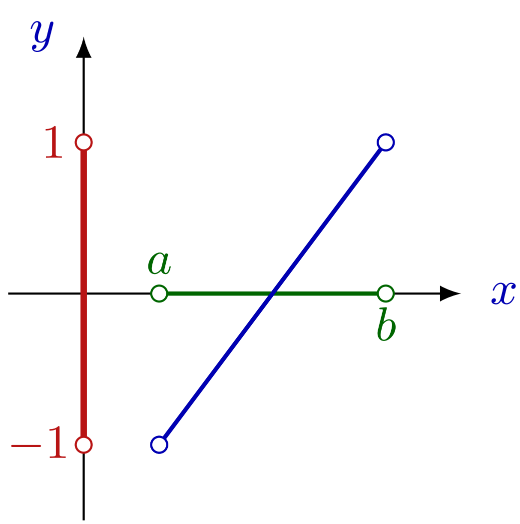

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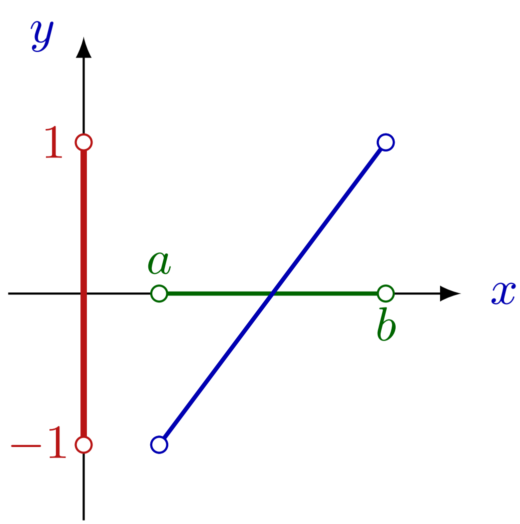

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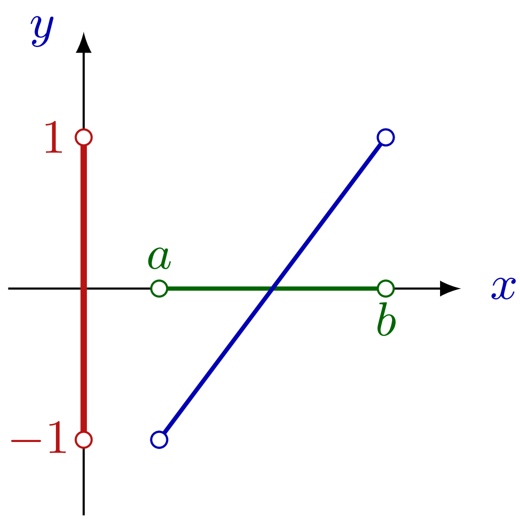

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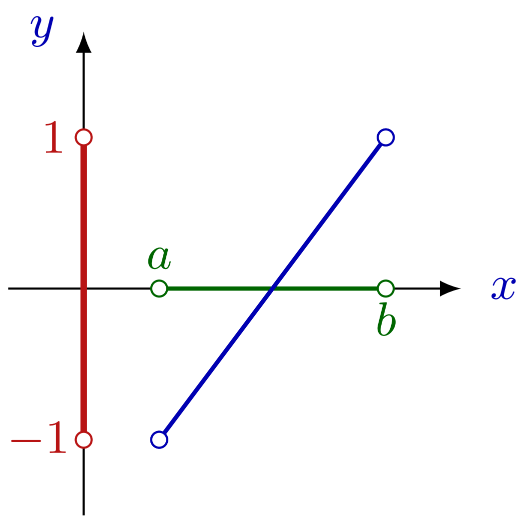

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Proof. $\mathbb{R} \setminus (0,1) \subseteq \mathbb{R} \implies |\mathbb{R} \setminus (0,1)| \leq |\mathbb{R}|$.

$$(2,3) \subseteq \mathbb{R} \setminus (0,1) \implies \underset{\displaystyle |\mathbb{R}|}{\overset{\|}{|(2,3)|}} \leq |\mathbb{R} \setminus (0,1)|.$$

So $|\mathbb{R} \setminus (0,1)| \leq |\mathbb{R}|$ and $|\mathbb{R}| \leq |\mathbb{R} \setminus (0,1)|$,

Applications of Cantor-Schröder-Bernstein theorem

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Indeed, $(0,1) \subseteq [0,1] \implies |(0,1)| \le |[0,1]|$.

$[0,1] \subseteq (-1,2) \implies |[0,1]| \le |(-1,2)| = |(0,1)|$.

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Theorem (Cantor). *For any set* X , $|X| < |\mathcal{P}(X)|$.

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If $a \notin A$, then it is not true that $a \notin F(a)$, that is, $a \in F(a) = A$.
It's a contradiction.
Therefore, the assumption about existing a bijection $X \to \mathcal{P}(X)$ was wrong, and there is no such a bijection. $\square$

Huge sets

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For **infinite** X, define $2^{|X|}$

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Corollary. $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$.

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