Sets

A set and its elements

In any intellectual activity,
one of the most profound actions is **gathering objects in groups**, performed in mind, with no action in the physical world.
A group can be a **subject of thoughts and arguments**, and can be **included** into other groups.

In Mathematics, creation those groups and manipulating with them is organized and regulated by the **naive set theory**.

This is rather a **language**, than a theory.
The **first words** in this language are **set** and **element**.

A **set** is a collection of objects which are called **elements**.
A set **consists** of (and is **defined** by) its elements.

Notations and synonims

Notation: $x \in S$ "x is an **element** of a set S"
"x **belongs** to S"
"A **consists** of its elements"
" A is **formed** by its elements"

Other notations: $S \ni x$, $S \not\ni x$, $x \notin S$.

Do not confuse "$\in$" and "ε".

Standard number sets

$\mathbb{N} = \mathbb{Z}^+ = \{1, 2, 3, , \dots\}$ **natural** numbers (or **positive integers**)

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, , \dots\}$ **integers**

$\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, \; q \neq 0\right\}$ **rational** numbers

$\mathbb{R}$ **real** numbers

$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}, \; i^2 = -1\}$ **complex** numbers

Equal sets

Definition. Two sets are called **equal** if they have the same elements.

Notation: $X = Y$.

By definition, $X = Y \iff \forall x \; (x \in X \iff x \in Y)$.

Example 1. $X = \{1, 2\}$

$Y = \{n \in \mathbb{N} \mid n < 3\}$

$Z = \{x \in \mathbb{R} \mid x^2 - 3x + 2 = 0\}$

$X = Y = Z$ since they consists of the same elements: 1 and 2 .

Example 2. $\{1, 2, 2\} = \{1, 2\}$, since a set is defined by its elements.

Example 3. $\{1, 2, 3\} = \{3, 2, 1\}$

Example 4. $\{1, \{1\}\} \neq \{1\}$

Example 5. $\{1, 2, 3\} \neq \{\{1\}, 2, 3\}$

Empty set

Definition. An **empty set** is a set with no elements.

Notation: $\varnothing$

Is $\varnothing = \{\varnothing\}$? No!

empty box $\neq$ a box containing an empty box.

Subsets

Definition. A set A is a **subset** of a set B if any element of A is an element of B.

Notation: $A \subset B$, or $A \subseteq B$, or $B \supset A$, or $B \supseteq A$.

The signs "$\subset$","$\subseteq$", "$\supset$" and "$\supseteq$" are called **inclusion** symbols. Commonly $\subset$ and $\subseteq$ are used in the same sense.

By definition, $A \subset B \iff \forall x \, (x \in A \implies x \in B)$.

Warning: distinguish the signs "$\in$" and "$\subset$"

Example. $A = \{1, 2, 3\}$.

correct:	$1 \in A$	wrong:	$\{1\} \in A$
wrong:	$1 \subset A$	correct:	$\{1\} \subset A$

Example. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Subsets

Proposition. For any set A, $\varnothing \subset A$ and $A \subset A$.

Definition. Let $A \subset B$. A is called a **proper subset** of B,

if $A \neq \varnothing$ and $A \neq B$.

Theorem. Let A and B be sets. Then

$$A = B \iff A \subset B \land B \subset A$$

Proof. Write a proof.

Theorem (transitivity of inclusion). Let A, B and C be sets. Then

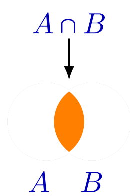
$$A \subset B \land B \subset C \implies A \subset C.$$

Proof. Write a proof.

Intersection and Union

Intersection

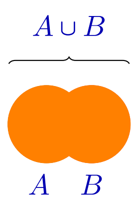
$A \cap B = \{x \mid x \in A \land x \in B\}$



Venn diagram

Union

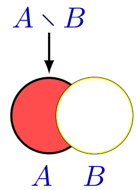
$A \cup B = \{x \mid x \in A \lor x \in B\}$



Difference

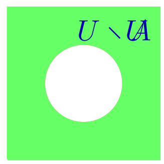
Difference and Complement

$A \setminus B = \{x \mid x \in A \wedge x \notin B\}$



Complement

$A^C = \underbrace{U}_{\text{universe}} \setminus A = \{x \in U \mid x \notin A\}$



Simplest set-theoretical identities

Let A be an arbitrary set. Then

$A \cap A = A$, $A \cup A = A$, $A \setminus A = \varnothing$,

$A \cap \varnothing = \varnothing$, $A \cup \varnothing = A$, $A \setminus \varnothing = A$.

Definition. Sets A and B are called **disjoint** if $A \cap B = \varnothing$.

Set-builder notation

The subset of a set A consisting of the elements x that satisfy a condition $P(x)$ is denoted by $\{x \in A \mid P(x)\}$.

For example, $\{x \in \mathbb{N} \mid x < 5\} = \{1, 2, 3, 4\}$.

This **set-builder notation** unveils a close relation between **predicates** and **sets**:

Every predicate $P(x)$ defines a subset $\{x \in A \mid P(x)\}$ of A.

Vice versa, every subset $B \subset A$ gives rise to a predicate $x \in B$.

Logic vs. set theory

Logic	pred. P	$\neg P$	$\wedge$	$\vee$	$\Longrightarrow$	$\Longleftrightarrow$	contradiction	tautology
Sets	set A	A^c	$\cap$	$\cup$	$\subset$	$=$	$\varnothing$	universe

Warning: Use correct signs!

Let P, Q be propositions, and A, B be sets.

correct: $P \wedge Q$, $A \cap B$ ☺

incorrect: $P \cap Q$, $A \wedge B$ ☹

Propositions and sets

Let $P(x)$ be a predicate (proposition depending on variable x),

where $x \in \underbrace{U}_{\text{universe}}$

Then $A = \{x \mid P(x)\}$ be a set.

Logic	Sets
$P(x)$	$A = \{x \mid P(x)\}$
$\exists x\ P(x)$	$A \neq \varnothing$
$\forall x\ P(x)$	$A = U$
$\neg\neg P \iff P$	$(A^c)^c = A$
$P \wedge \neg P$ is a contradiction	$A \cap A^c = \varnothing$
$P \vee \neg P$ is a tautology	$A \cup A^c = U$
$\neg(P \wedge Q) \iff \neg P \vee \neg Q$	$(A \cap B)^c = A^c \cup B^c$ De Morgan's law

Basic set-theoretic identities

- **Commutativity** of $\cap$ and $\cup$: for any sets A and B,
  $A \cap B = B \cap A$ and $A \cup B = B \cup A$.
- **Associativity** of $\cap$ and $\cup$: for any sets A, B and C,
  $(A \cap B) \cap C = A \cap (B \cap C)$ and $(A \cup B) \cup C = A \cup (B \cup C)$.
- **Distributivities**: for any sets A, B and C,
  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ and $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
- **De Morgans' laws:** for any sets A and B,
  $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$.

Proving set-theoretic identities: De Morgan's law

Example 1. Prove **De Morgan's** law: $(A \cap B)^c = A^c \cup B^c$

Proof. Let us prove first that $(A \cap B)^c \subset A^c \cup B^c$.

Let $x \in (A \cap B)^c$ Then $x \notin A \cap B \implies \neg(x \in A \wedge x \in B)$

$\implies x \notin A \vee x \notin B \implies x \in A^c \vee x \in B^c \implies x \in A^c \cup B^c$.

So $\forall x \in (A \cap B)^c$, we have $x \in A^c \cup B^c$.

Therefore, $(A \cap B)^c \subset A^c \cup B^c \quad (*)$.

Prove now that $A^c \cup B^c \subset (A \cap B)^c$.

$x \in A^c \cup B^c \implies$
$x \in A^c \vee x \in B^c \implies x \notin A \vee x \notin B \implies \neg(x \in A \wedge x \in B)$

$\implies x \notin A \cap B \implies x \in (A \cap B)^c$

Therefore, $A^c \cup B^c \subset (A \cap B)^c \quad (**)$
Combining $(*)$ and $(**)$, we get $(A \cap B)^c = A^c \cup B^c$. □

Could it be done faster?

Yes, all our arguments are biderectional.

Indeed, $x \in (A \cap B)^c \iff x \notin A \cap B \iff \neg(x \in A \wedge x \in B)$

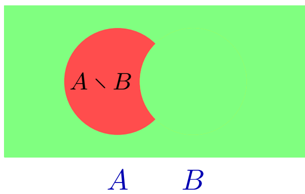
$\iff x \notin A \vee x \notin B \iff x \in A^c \vee x \in B^c \iff x \in A^c \cup B^c$.
So $\forall x \;\; x \in (A \cap B)^c \iff x \in A^c \cup B^c$.

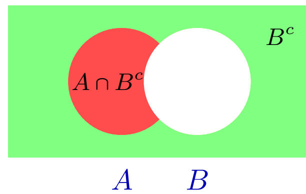
Therefore, $(A \cap B)^c = A^c \cup B^c$

How to prove set-theoretic identities

Example 2. Prove that $A \setminus B = A \cap B^c$ for any sets A, B.

Illustration (not a proof!):





Proof. Alternative 1 (element-wise)

$A \setminus B = \{x \mid x \in A \wedge x \notin B\} = \{x \mid x \in A \wedge x \in B^c\} = A \cap B^c$ □

How to prove set-theoretic identities

Alternative 2 (two sets are equal iff each of them is a subset of the other)

To prove $A \setminus B = A \cap B^c$, we prove that

$A \setminus B \subset A \cap B^c$ and

$A \setminus B \supset A \cap B^c$

Indeed,

$$\left.\begin{array}{l} A \setminus B \subset A \\ A \setminus B \subset U \setminus B = B^c \end{array}\right\} \implies A \setminus B \subset A \cap B^c$$

$$\left.\begin{array}{l} A \cap B^c \subset A \\ A \cap B^c \subset B^c = U \setminus B \end{array}\right\} \implies A \cap B^c \subset A \cap (U \setminus B) = (A \cap U) \setminus B = A \setminus B$$

We have got that $A \setminus B \subset A \cap B^c$ and $A \setminus B \supseteq A \cap B^c$.

Therefore, $A \setminus B = A \cap B^c$. □

How to prove set-theoretic identities

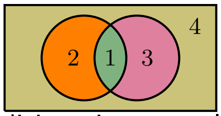
Alternative 3 (by truth table)

	$x \in A$	$x \in B$	$x \notin B$	$\underbrace{x \in A \wedge x \notin B}_{x \in A \smallsetminus B}$	$\underbrace{x \in A \wedge x \in B^c}_{x \in A \cap B^c}$
1	T	T	F	F	F
2	T	F	T	T	T
3	F	T	F	F	F
4	F	F	T	F	F

Since the last two columns of the truth table are identical, $A \smallsetminus B = A \cap B^c$.

Remark. The universe can be presented as a **disjoint union**

$U = (A \cap B) \cup (A \smallsetminus B) \cup (B \smallsetminus A) \cup (A \cup B)^c$



What does this formula remind you? Is it related to disjunctive normal form?

How to prove set-theoretic identities

Example 3. Prove that $A \subset B \iff A \smallsetminus B = \varnothing$ for any sets A, B.

Proof. Let $A \subset B$. Then
$\forall x \in A \quad x \in A \implies x \in B \implies x \notin B^c$.
So any x in A doesn't belong to B^c. Therefore, $A \cap B^c = \varnothing$.

But $A \cap B^c = A \smallsetminus B$,

hence $A \smallsetminus B = \varnothing$
We have proven that $A \subset B \implies A \smallsetminus B = \varnothing \quad (*)$

Prove now the opposite implication.

Let $A \smallsetminus B = \varnothing$. Then $A \cap B^c = \varnothing$. Therefore,

$\forall x \in A \quad x \in A \implies x \notin B^c \implies x \in B$. By this, $A \subset B$.

Therefore, $A \smallsetminus B = \varnothing \implies A \subset B \quad (**)$

Combining $(*)$ and $(**)$, we get $A \subset B \iff A \smallsetminus B = \varnothing$. □

Inclusion vs. belonging

Despite obvious similarity between the symbols $\in$ and $\subset$,

the concepts are quite different.

$$x \in A \iff \{x\} \subset A$$

$A \subset A$ for any A, but $A \notin A$ for any reasonable A.

Belonging is not transitive: $(a \in B) \wedge (B \in C) \nRightarrow a \in C$

while inclusion is: $(A \subset B) \wedge (B \subset C) \implies A \subset C$.