## Welcome to MAT 250!

**Brightspace** for MAT 250.01 contains the course information:
Syllabus, Handouts, Announcements, etc.

**Gradescope** is the class homework platform.
Register for Gradescope using entry code **DJR6ZZ**

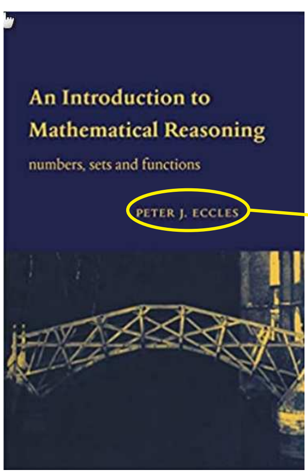
**Activities:** lectures
quizzes
homeworks (through Gradescope)
exams (two midterms and final)

| | | |
|---|---|---|
| **Grading:** | Midterm 1 (2/10) | 15% |
| | Midterm 2 | 20% |
| | Final (5/15) | 35% |
| | HW | 15% |
| | Quizzes | 10% |
| | Active in-class participation | 5% |

The **final grade** is the **maximum** of the score for final exam and the total grade calculated according to the scheme discribed above.



Our textbook: Its author:




## Liberal arts

Once upon a time, there were 7 liberal arts.

Freshmen studied 3 of them, called **Trivium**.

Sophomores studied the rest 4, called **Quadrivium**.

**High Faculties**:

1. Medicine
2. Law
3. Theology

**Liberal Arts**: (**Low Faculty**)

1. Grammar
2. Rhetoric
3. **Logic**

} **Trivium**

4. Geometry
5. Arithmetic
6. Astronomy
7. Music

} **Quadrivium**

All quadrivium subjects were considered Mathematics.

The Modern Mathematics has grown from the first two of them.

Mathematics relies on Logic, still.



## Allegory of the seven liberal arts

by Maerten de Vos (1590)

Find the representative for each liberal art.

# Definitions



### The role of a mathematical definition

Mathematics is an exact science. All the statements should be **precise**,
that is, to be understood in a **unique** way.

The precision (exactness, accuracy, clarity) is ensured by a careful usage of definitions.
A definition is an **agreement** about terms. A definition introduces a new word (or words), which will be understood exactly as it is stated in the definition.

A definition describes the **meaning** in which a certain word (or words) will be used.

It is important to know the definitions in their **exact** forms, not just to have an approximate idea.



### Structure of a definition

Like a fairy tale often begins with words "Once upon a time ...",
a typical definition in a well-written math book begins with a description of a context.

**Definition.** Let . . . <description of objects, universe, etc.>
<notation> is called <**name**>
if <statement>.

The word Definition is not necessary here.
The description is followed by one or several statements of names.
Names are emphasized typographically (by italic or bold).
The statements of names are followed by the conditions.

**Example:**
Let $X$, $Y$ and $Z$ be sets, and let $f : X \to Y$, $g : Y \to Z$ be maps.
A map $h : X \to Z$ is called the **composition** of $f$ and $g$
if $h(x) = g(f(x))$ for any $x \in X$.

This is a **descriptive** (or implicit) definition.
There are also **constructive** (or explicit) definitions.

## Variations

(1) Sometimes a description of context is omitted.

(2) The last two parts may be written in the opposite order:
If <condition>, then <description of names>.

(3) By a tradition, the **conditional** statement
must be understood as a **biconditional**.

(4) if the name is an adjective,
then instead of **is called** one may use **is said to be**.



## Constructive definitions

The scheme of a constructive definition looks as follows:

<description of objects>
<formula> is called <name>.

**Example.**
Let $X$, $Y$ and $Z$ be sets, and let $f : X \to Y$, $g : Y \to Z$ be maps.
Then the map $g \circ f : X \to Z$ defined by formula $g \circ f(x) = g(f(x))$ is called the **composition** of $f$ and $g$.

### Example of a definition: divisibility

**Definition.** Let $d$ and $n$ be **integers** and $d \neq 0$. One says that
$d$ **divides** $n$ (or, equivalently, $n$ is **divisible** by $d$)
if $n = d \cdot k$ for some integer $k$.

**Notation:** $d \,|\, n$

**Remarks. 1.** The definition of divisibility is made in terms of multiplication, not division. Why? Is there a division?
How would it be with division?

**2.** Why $d \neq 0$? Why we can't divide by $0$?

Let us see **how** this **definition is used in** the **proof of a theorem**.

**Theorem.** Let $a, b$ and $c$ be integers, and $a \neq 0$.
If $a$ divides both $b$ and $c$, then $a$ divides $b + c$.

**Proof.** Since $a \,|\, b$, then, by definition of divisibility, $b = a \cdot k$ for some integer $k$. Since $a \,|\, c$, then $c = a \cdot l$ for some integer $l$. Therefore,

$$b + c = ak + al = a(k + l)\,.$$

Since $k + l$ is an integer, $a$ is a factor of $b + c$. Therefore, $a$ divides $b + c$. □

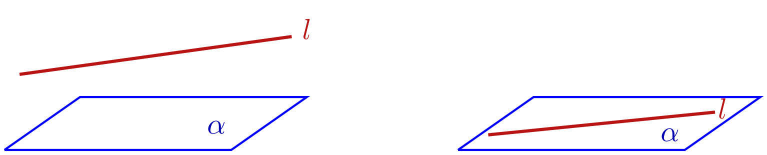


### A definition from geometry

**Definition.** Let $l$ be a line and $\alpha$ be a plane in the space.
The line $l$ is said to be **parallel** to the plane $\alpha$, if either $l$ doesn't intersect $\alpha$
or $l$ lies on $\alpha$.

**Notation:** $l \,||\, \alpha$

**Illustration:**




**Control question:** What does it mean that a line is **not** parallel to a plane?

By definition, $l \,||\, \alpha \iff \underbrace{l \cap \alpha = \varnothing}_{l \text{ doesn't intersect } \alpha} \;\vee\; \underbrace{l \subset \alpha}_{l \text{ lies on } \alpha}$

Therefore, $l \nparallel \alpha \iff \underbrace{l \cap \alpha \neq \varnothing}_{l \text{ intersects } \alpha} \;\wedge\; \underbrace{l \not\subset \alpha}_{l \text{ doesn't lies on } \alpha}$

## Non-parallel

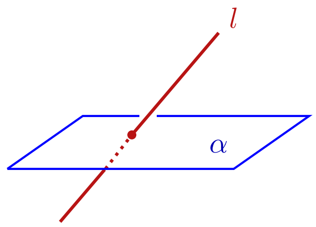
$$l \nparallel \alpha \iff \underbrace{l \cap \alpha \neq \varnothing}_{l \text{ intersects } \alpha} \;\wedge\; \underbrace{l \not\subset \alpha}_{l \text{ doesn't lie on } \alpha}$$

In words:
A line $l$ is **not** parallel to a plane $\alpha$ if $l$ intersects $\alpha$, but doesn't lie on $\alpha$.

A line which is not parallel to a plane is said to **transverse** the plane.

(The line and plane are said to be **transversal**. )

**Illustration:**






## Definition of limit

**Definition.** Let $f(x)$ be a function, $a$ and $L$ be real numbers.
$L$ is called a **limit** of $f$ as $x$ approaches $a$ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \qquad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon .$$

**Notations:** $L = \lim\limits_{x \to a} f(x)$ or $f(x) \underset{x \to a}{\longrightarrow} L$.

Why does this definition appear to be difficult?

– Unknown letters: $\varepsilon$, $\delta$ from **Greek alphabet**:

$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \theta,\ \iota,\ \kappa,\ \lambda,\ \mu,\ \nu,\ \xi,\ o,\ \pi,\ \rho,\ \sigma,\ \tau,\ \upsilon,\ \varphi,\ \chi,\ \psi,\ \omega$
$\mathrm{A},\ \mathrm{B},\ \Gamma,\ \Delta,\ \mathrm{E},\ \mathrm{Z},\ \mathrm{H},\ \Theta,\ \mathrm{I},\ \mathrm{K},\ \Lambda,\ \mathrm{M},\ \mathrm{N},\ \Xi,\ \mathrm{O},\ \Pi,\ \mathrm{P},\ \Sigma,\ \mathrm{T},\ \Upsilon,\ \Phi,\ \mathrm{X},\ \Psi,\ \Omega$

– Three quantifiers

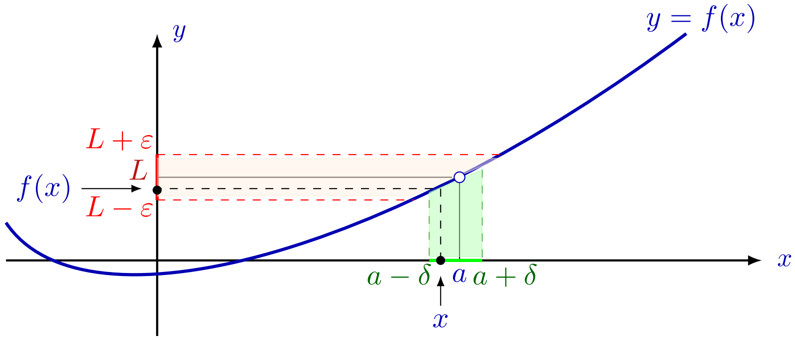
– Two inequalities

– One implication

## Understanding the definition of limit

How to understand what **exactly** the definition says?

$$L = \lim_{x \to a} f(x) \iff \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \qquad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \,.$$




For any $x$ such that $x \in (a - \delta, a + \delta)$, we have $f(x) \in (L - \varepsilon, L + \varepsilon)$.



## Working with the definition of limit

What does it mean that $L \neq \lim_{x \to a} f(x)$?

$$L = \lim_{x \to a} f(x) \iff \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \qquad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \,.$$

$$L \neq \lim_{x \to a} f(x) \iff \exists \varepsilon > 0 \quad \forall \delta > 0 \quad \exists x \qquad 0 < |x - a| < \delta \,\wedge\, |f(x) - L| \geq \varepsilon \,.$$

In words:

A number $L$ is **not** a limit of a function $f(x)$ at a point $a$, if there exists a positive number $\varepsilon$, such that for any positive number $\delta$ one can find $x$, such that $0 < |x - a| < \delta$, but $|f(x) - L| \geq \varepsilon$.

**Exercise 1.** Use the definition of limit to prove that $\lim_{x \to 3} (2x + 1) = 7$.

**Exercise 2.** Use the definition of limit to prove that $\lim_{x \to 0} \left( \sin \frac{1}{x} \right) \neq 0$.

## Can one simplify the definition of limit?

Yes, at some cost. At the cost of an extra definition.

Let $a \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. Then the interval $(a-\varepsilon, a+\varepsilon)$ is called the $\varepsilon$-**neighborhood** of $a$.

$L$ is called a **limit** of $f$ as $x$ approaches $a$ if
for any $\varepsilon$-neighborhood $V$ of $L$ there exists a $\delta$-neighborhood $U$ of $a$ such that $f(U \setminus \{a\}) \subset V$.
Not easy enough? Then take one more definition:

Let $a \in \mathbb{R}$. A set $U$ is a **neighborhood** of $a$ iff
there exists $\varepsilon > 0$ such that $U$ contains the $\varepsilon$-neighborhood of $a$. Now

$L$ is a **limit** of $f$ as $x$ approaches $a$ iff for each neighborhood $V$ of $L$
$f^{-1}(V) \cup \{a\}$ is a neighborhood of $a$.

The notion of **limit** can be replaced by the notion of **continuity**:

A function $f$ is said to be **continuous** at $a$ if
the preimage $f^{-1}(U)$ of any neighborhood $U$ of $f(a)$ is a neighborhood of $a$.



## Definition of ring (from Algebra)

**Motivation.** We know that the set of integers is **closed** with respect to the operations of addition and multiplication. It means that

$$\forall\, a, b \in \mathbb{Z} \quad a+b \in \mathbb{Z} \quad \text{and} \quad ab \in \mathbb{Z}.$$

Addition and multiplication in $\mathbb{Z}$ possess several important properties, like **associativity** and **distributivity**.
Besides the integers, there are many other sets of mathematical objects
for which there are operations of addition and multiplication possessing the same properties. For example, polynomials or matrices.

It is natural to gather all such sets equipped with operations under the same roof.
It is done in the definition of **ring**.

## Definition of ring

**Definition.** A **ring** $R$ is a set with two operations, addition and multiplication, denoted by $+$ and $\cdot$, satisfying the following properties:

**1.** $\forall\, a, b \in R \quad a+b \in R \quad$ ( $R$ is **closed** with respect to $+$ )
**2.** $\forall\, a, b \in R \quad a \cdot b \in R \quad$ ( $R$ is **closed** with respect to $\cdot$ )
**3.** $\forall\, a, b, c \in R \quad (a+b)+c = a+(b+c) \quad$ ( $+$ is **associative**)
**4.** $\forall\, a, b \in R \quad a+b = b+a \quad$ ( $+$ is **commutative**)
**5.** $\exists\, 0 \in R \quad \forall\, a \in R \quad a+0 = a \quad$ (there exists an **additive identity** in $R$ )
**6.** $\forall\, a \in R \;\; \exists -a \in R \;\; a+(-a) = 0$ (each element in $R$ has an **additive inverse**)
**7.** $\forall\, a, b, c \in R \quad (a \cdot b) \cdot c = a \cdot (b \cdot c) \quad$ ( $\cdot$ is **associative**)
**8.** $\forall\, a, b, c \in R \quad a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a$
(multiplication **distributes** over addition)

- If, additionally, $\forall\, a, b \in R \quad a \cdot b = b \cdot a \quad$ ( $\cdot$ is **commutative**), then $R$ is called a **commutative** ring.
- If, additionally, $\exists\, 1 \in R \quad \forall\, a \in R \quad 1 \cdot a = a \cdot 1 = a$

(there exists a **multiplicative identity**), then $R$ is called a ring with **unity**.

The properties are called the **axioms** of a ring.



## Examples of rings

**1.** $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are commutative rings with unity.

**2.** $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ is a ring of even integers (Commutative? With unity?)

**3.** $\mathbb{Z}[x]$, polynomials in variable $x$ with integer coefficients, form a ring. (Commutative? With unity?)

**4.** $\mathbb{Q}[x]$, $\mathbb{R}[x]$, $\mathbb{Z}[x, y]$, etc. are rings of polynomials.

**5.** $M_n(\mathbb{R})$, square $n \times n$ matrices with real coefficients form a ring. (Commutative? With unity?)

**6.** $\mathbb{Z}_m$, residues modulo $m$ (to be discussed later in the course) form a ring.

**7.** $\mathcal{F} = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$, real valued functions with the operations of
addition $(f+g)(x) = f(x) + g(x)$ and multiplication $(f \cdot g)(x) = f(x) \cdot g(x)$ form a ring.

**Important:** To prove that each of the listed above objects is a ring, we have to verify all ring axioms.

## How to use the definition of ring

Let us see how the definition of ring is used in the proof of a theorem.

**Theorem.** In any ring $R$, $a \cdot 0 = 0$ for all $a \in R$.

**Proof.**

$a \cdot 0 = a \cdot 0 + 0$ by axiom **5**

$= a \cdot 0 + (a \cdot 0 + (-a \cdot 0))$ by axiom **6**

$= (a \cdot 0 + a \cdot 0) + (-a \cdot 0)$ by axiom **3**

$= a \cdot (0 + 0) + (-a \cdot 0)$ by axiom **8**

$= a \cdot 0 + (-a \cdot 0)$ by axiom **5**

$= 0$ by axiom **6**

20 / 20