
Section 11

Categories of maps

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Families Γ and Δ of sets are equivalent

if there exists a bijection $\Gamma \rightarrow \Delta$ formed of bijections.

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Maps $X \xrightarrow{f} X$ and $Y \xrightarrow{g} Y$ are called **conjugate** if they are isomorphic in the category of self-maps, i.e., there exists a (conjugating) bijection $h : X \rightarrow Y$ such that $g = h^{-1} \circ f \circ h$.

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Therefore, *Being conjugate is an equivalence relation.*



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Similarly, if $X \xrightarrow{f} X$ and $Y \xrightarrow{g} Y$ are conjugate via $h : X \rightarrow Y$, then grand orbits of f are mapped by $h : X \rightarrow Y$ to grand orbits of g .

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Corollary 3. *For a map $X \rightarrow X$ of a finite set X ,
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Control question: Is this true for infinite sets?

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Proof. Exercise.

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Classification of self-maps in a finite set

Let X be a finite set.

Find invariants which determine a self-map $f : X \rightarrow X$ up to isomorphism.

How many classes of self-maps are out there?

List the isomorphism classes of maps $X \xrightarrow{f} X$.

For each $f : X \rightarrow X$, there is a partition of X to grand-orbits.

A grand orbit is non-empty.

Each self-map is a disjoint union of self-maps with a single grand orbit.

Theorem. *In a finite set with a self-map,
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Then $|X| \geq |U| = |\mathbb{N}|$, which contradicts to the assumption that X is finite. $\square$

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Arithmetic operations match: $|\mathbb{N}_p \times \mathbb{N}_q| = |\mathbb{N}_{pq}|$, etc.

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Pigeonhole Principle

MAT 250
Section 11
Categories of maps

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Pigeonhole principle. Examples 1-5

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Pigeonhole principle. Examples 6-9

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Example 6. Five points are placed randomly
inside an equilateral triangle with a side of length 1 inch.
Prove that among these five points
there are two points which are at most $1/2$ inches apart.

Example 7. Prove that there exists a power of 3 that ends up with digits 01.

Example 8. Prove that if 51 points are placed randomly in a unit square,
then one can always find three points
that can be covered by a circle of radius $1/7$.

Example 9. Given 6 points inside a unit circle. Prove that some two of them
are within 1 unit from each other.