

Maps: domain, codomain, range

MAT 250
Lecture 8, 10/02
Sets

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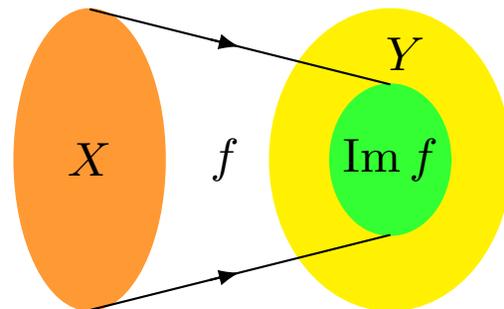
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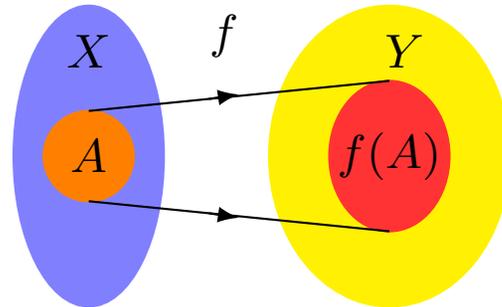
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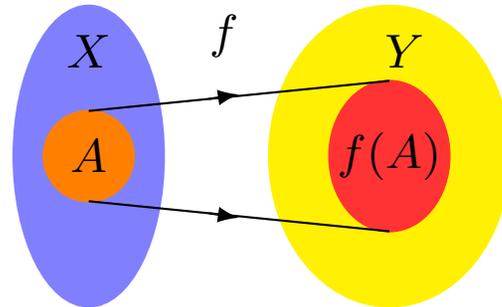
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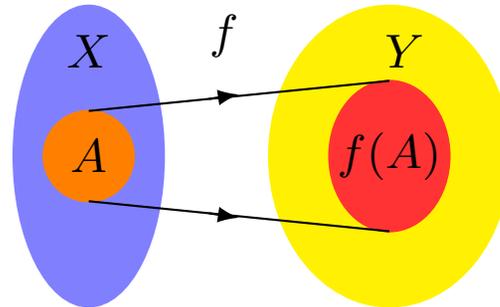


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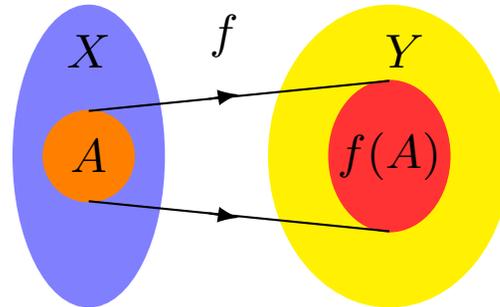


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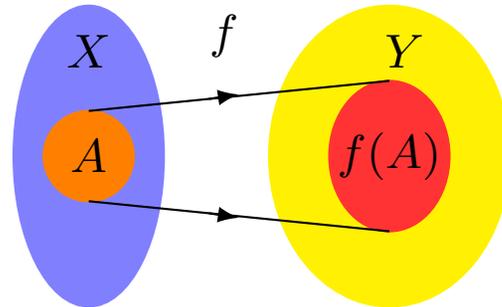


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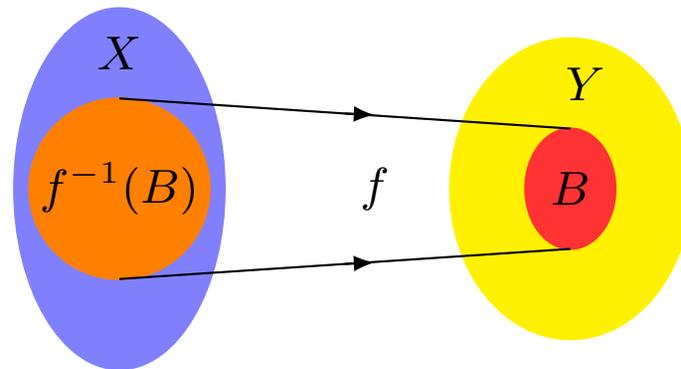
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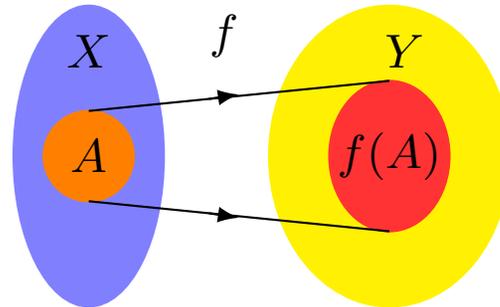
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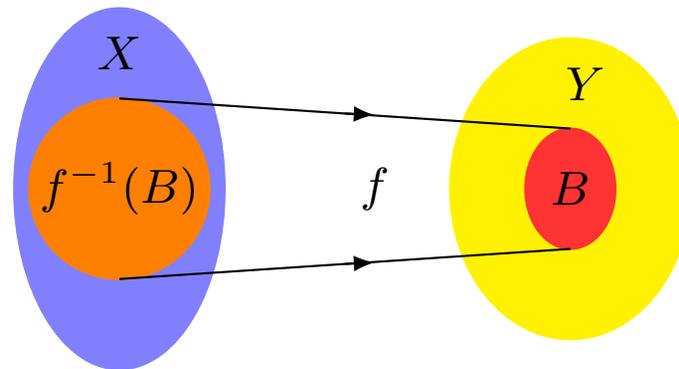
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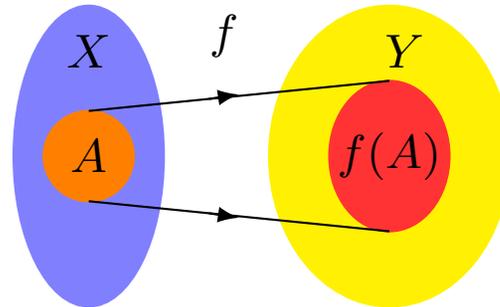


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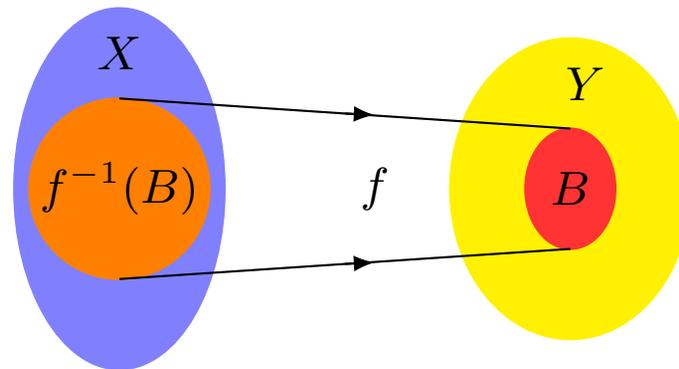
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Warning: f^{-1} is **not** the inverse map!

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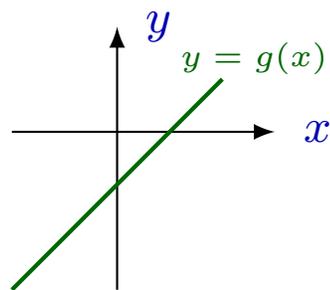
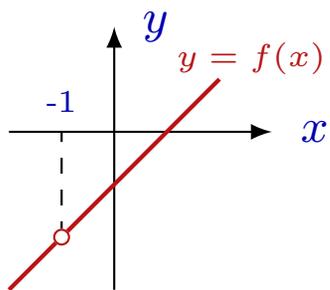
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Composition of maps

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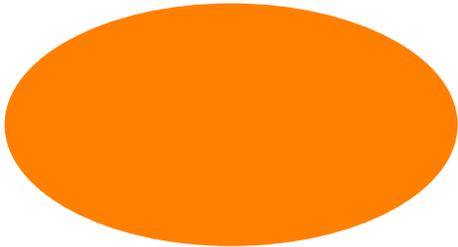
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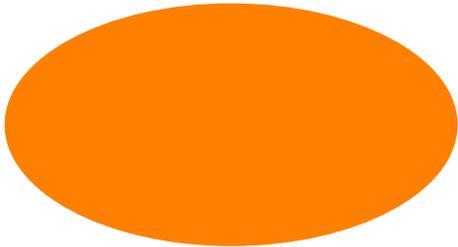


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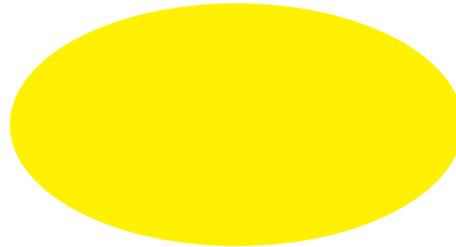
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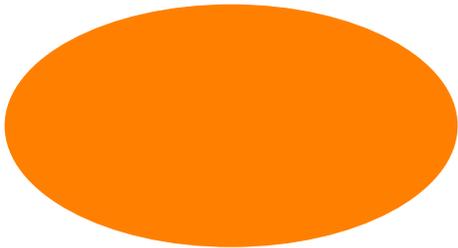


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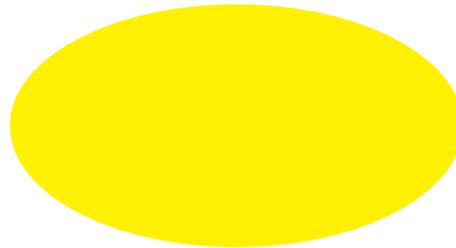
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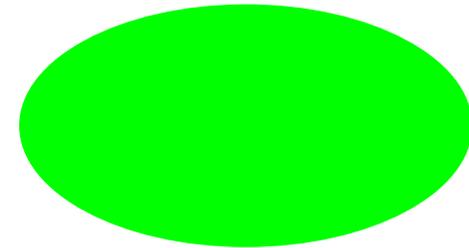
X



Y



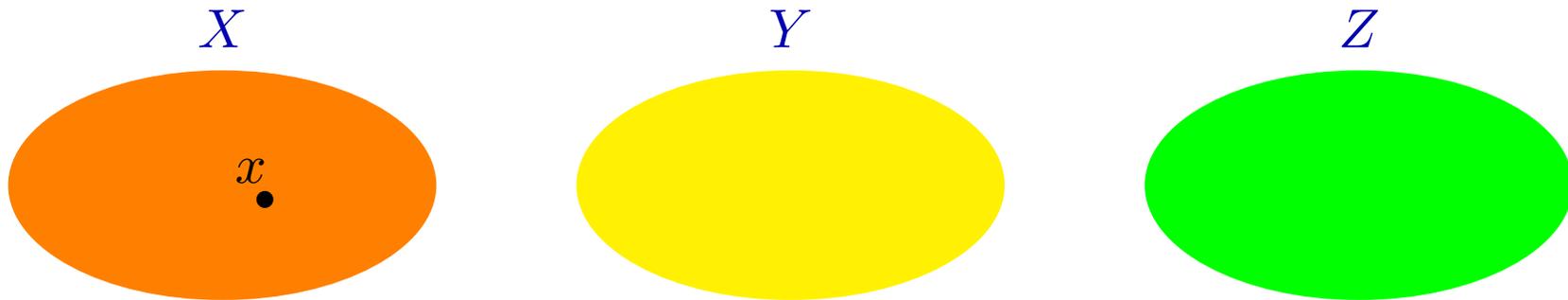
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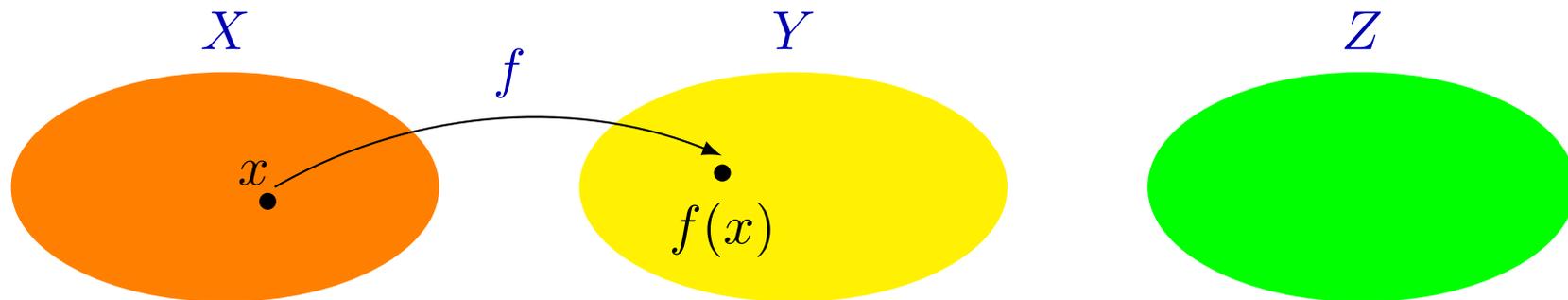
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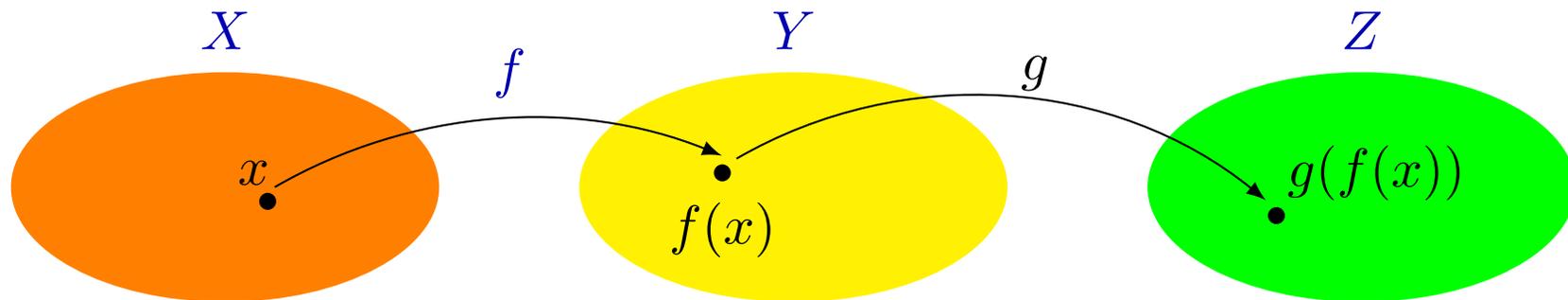
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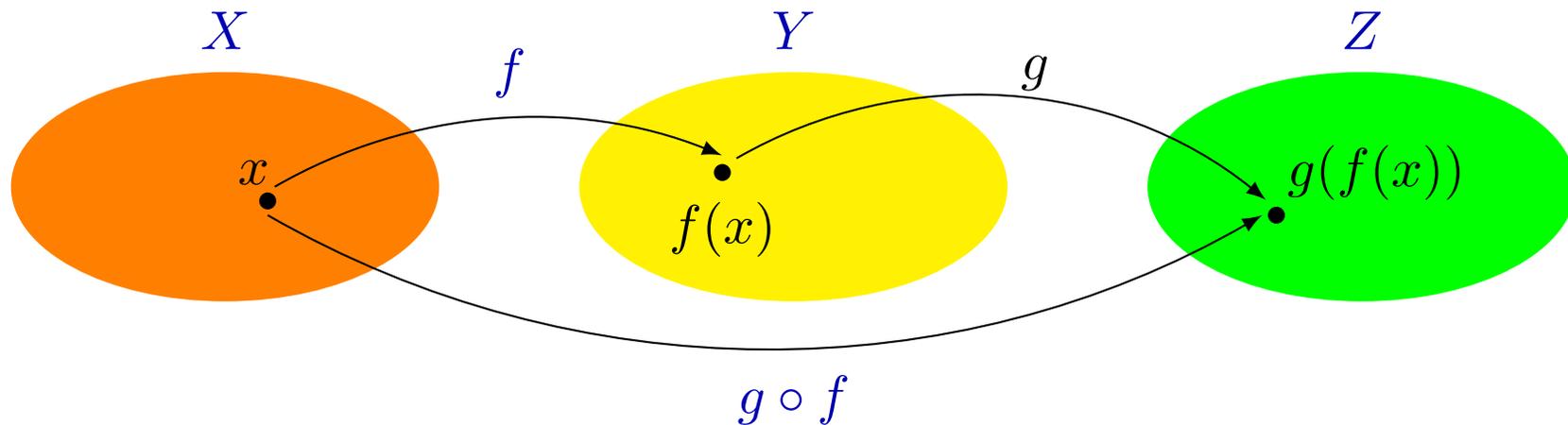
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Composition is associative

MAT 250
Lecture 8, 10/02
Sets

Composition is associative

Theorem.

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Examples of maps

MAT 250
Lecture 8, 10/02
Sets

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Give a **descriptive** definition of a constant map.

Examples of maps

- A **function** $y = f(x)$ is a map $f : D \rightarrow \mathbb{R}$,
where $D \subset \mathbb{R}$ is the domain of f .

Domain convention: when a function f is defined without specifying its domain, we assume that the domain is the **maximal** set of x -values for which $f(x)$ is defined.

- A **numerical sequence** $\mathbb{N} \rightarrow \mathbb{R}$, $n \mapsto a_n$ is a map.
- A **constant map**

Let X, Y be sets. Choose any $y_0 \in Y$ and define a map $f : X \rightarrow Y$ by $f(x) = y_0$ for all $x \in X$. This map is called a **constant** map.

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Or $\exists c \in Y \ \forall a \in X \ f(a) = c$.

Identity map

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□

Inclusion, restriction and submap

MAT 250
Lecture 8, 10/02
Sets

- Inclusion map

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This diagram is **commutative**, that is

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Characteristic function of a set

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Example: for $[2, 4] \subseteq \mathbb{R}$,

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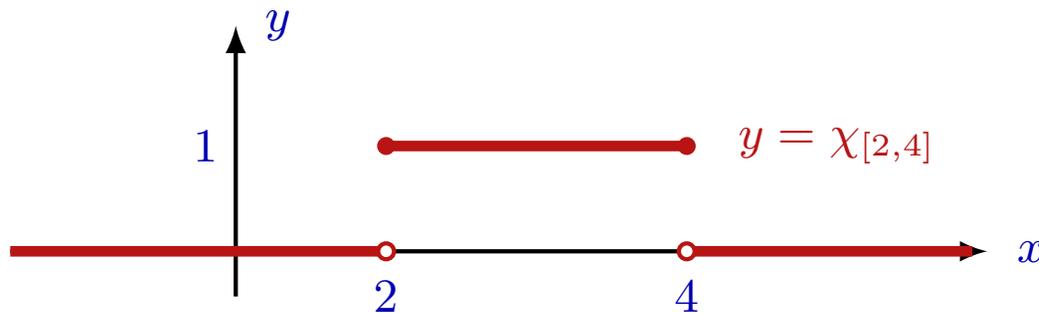
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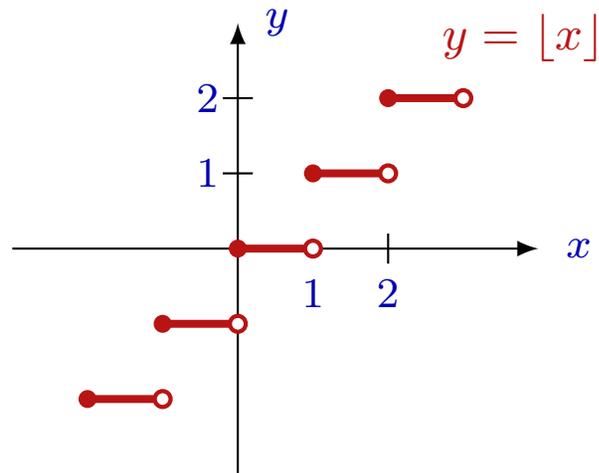
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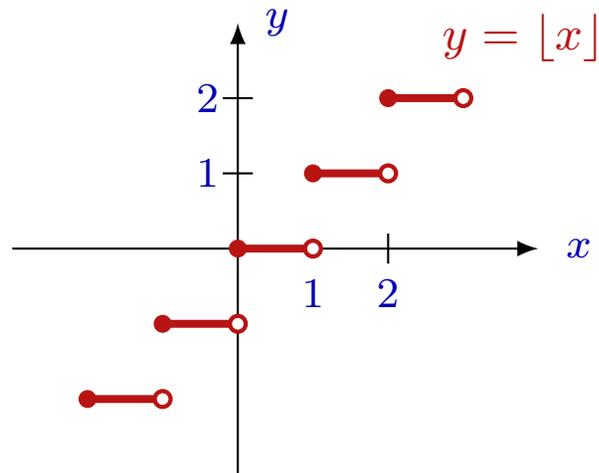
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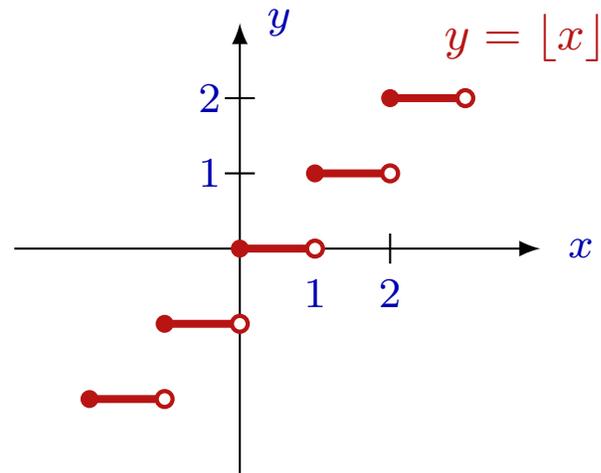


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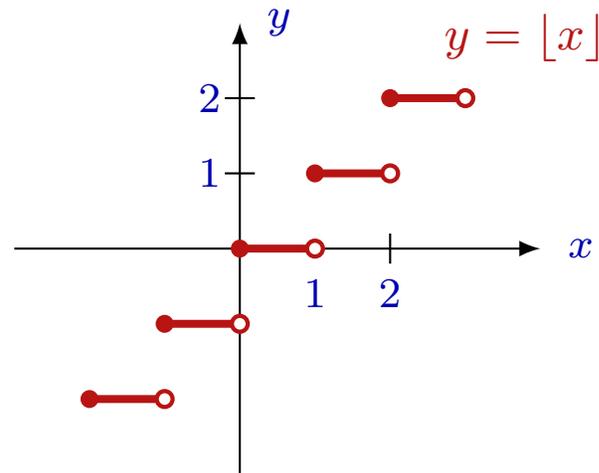
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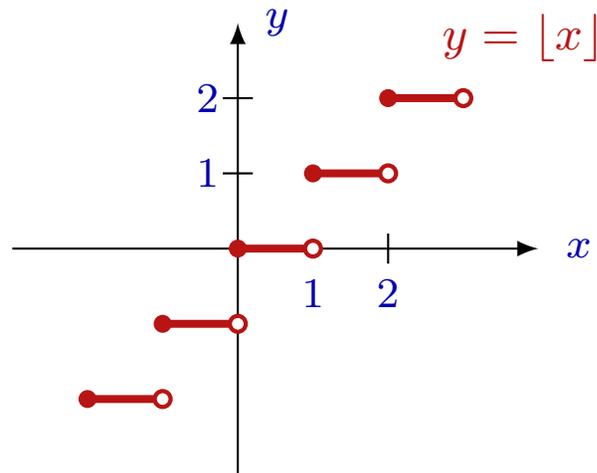
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Dirichlet function is **everywhere** discontinuous!

Injective maps

MAT 250
Lecture 8, 10/02
Sets

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Injective or not?

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Example.

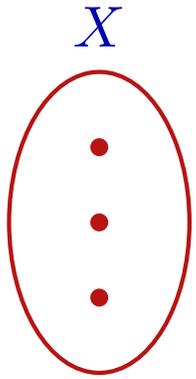
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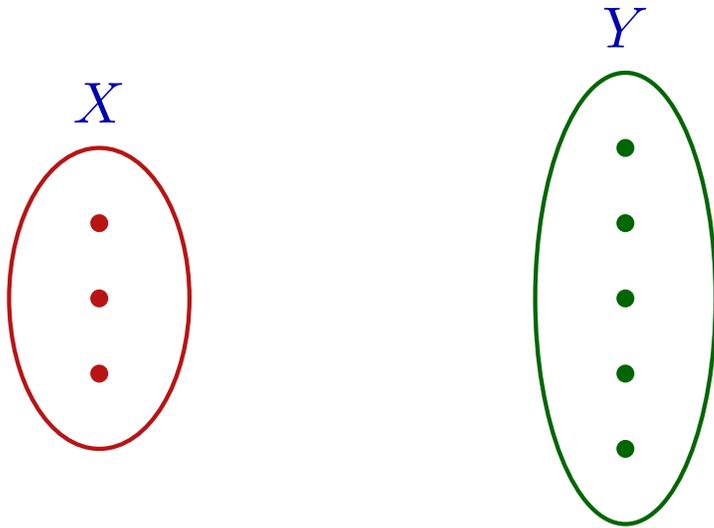
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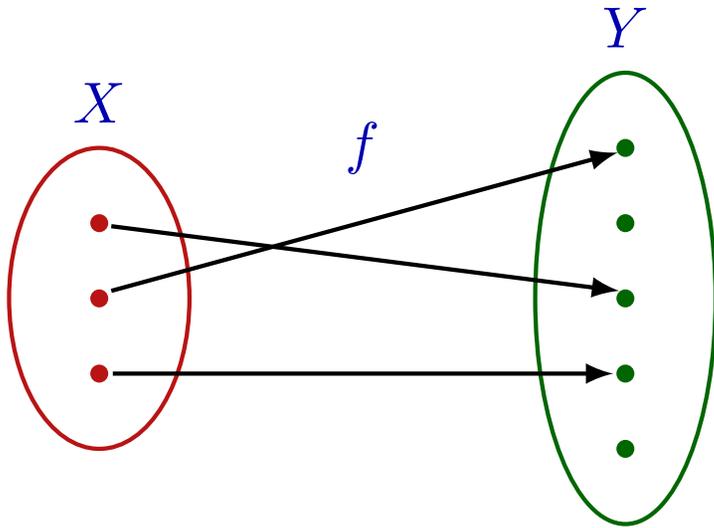
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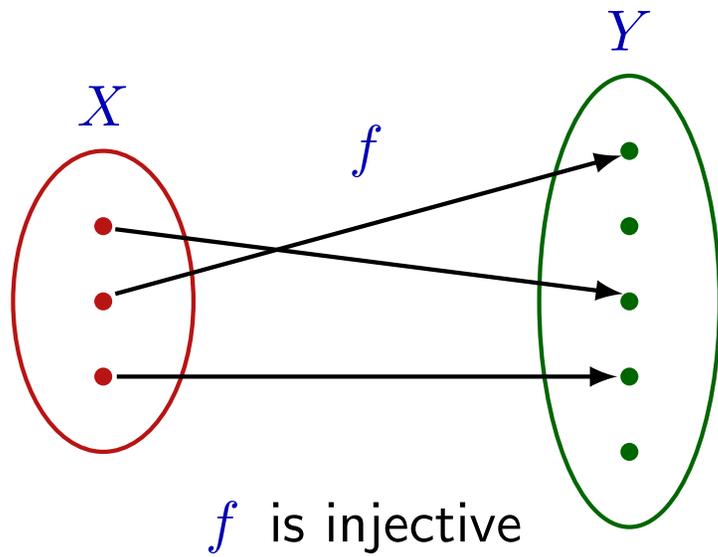
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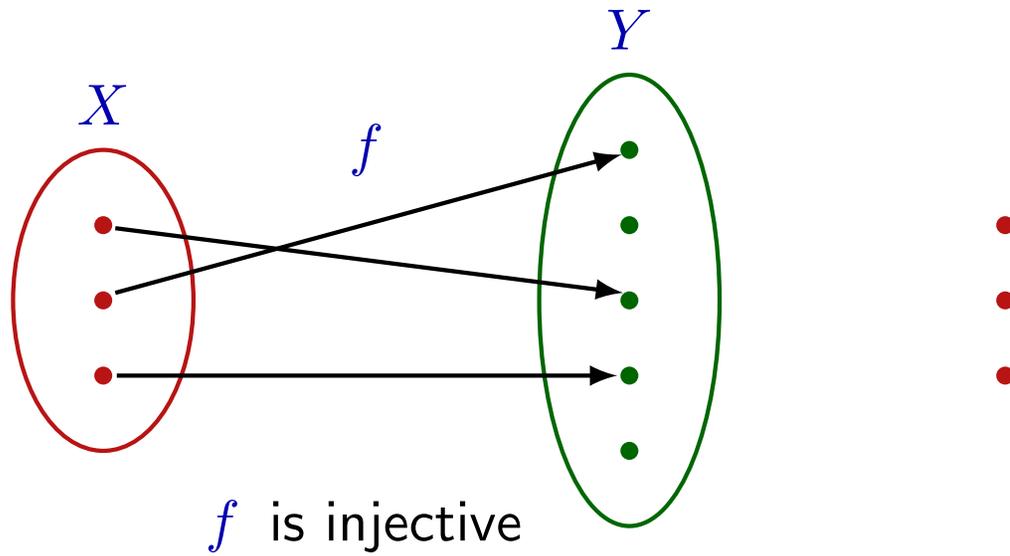
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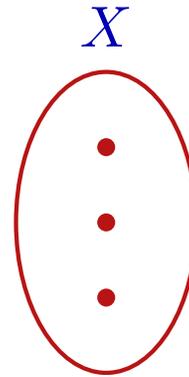
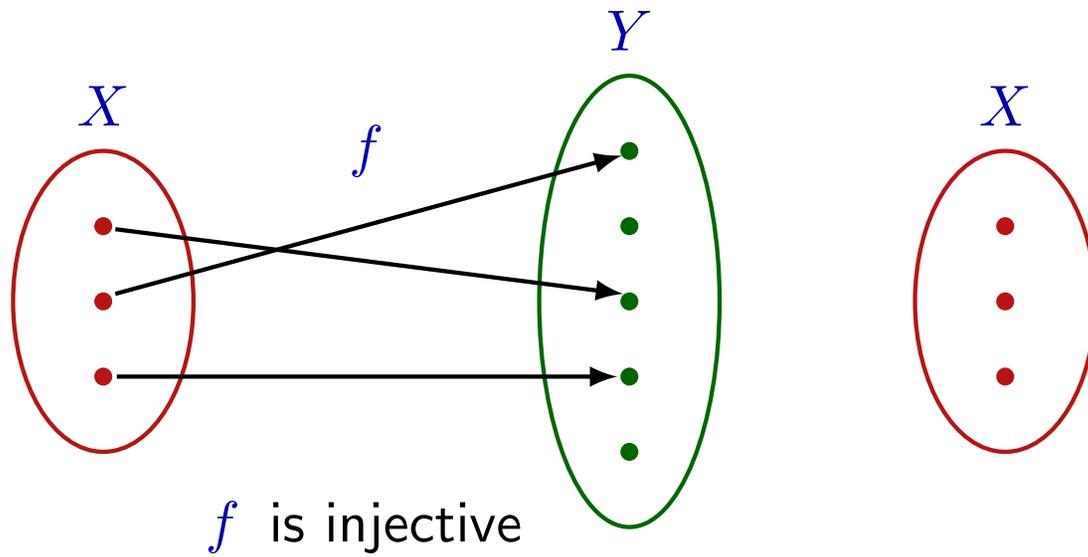
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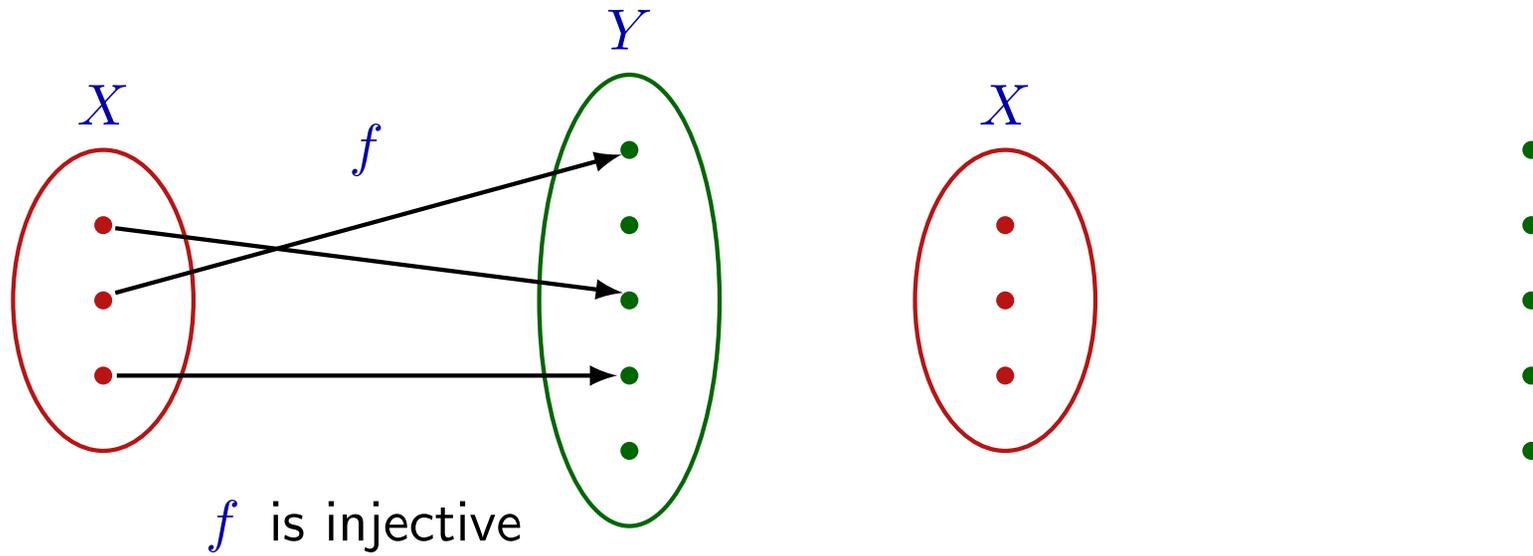
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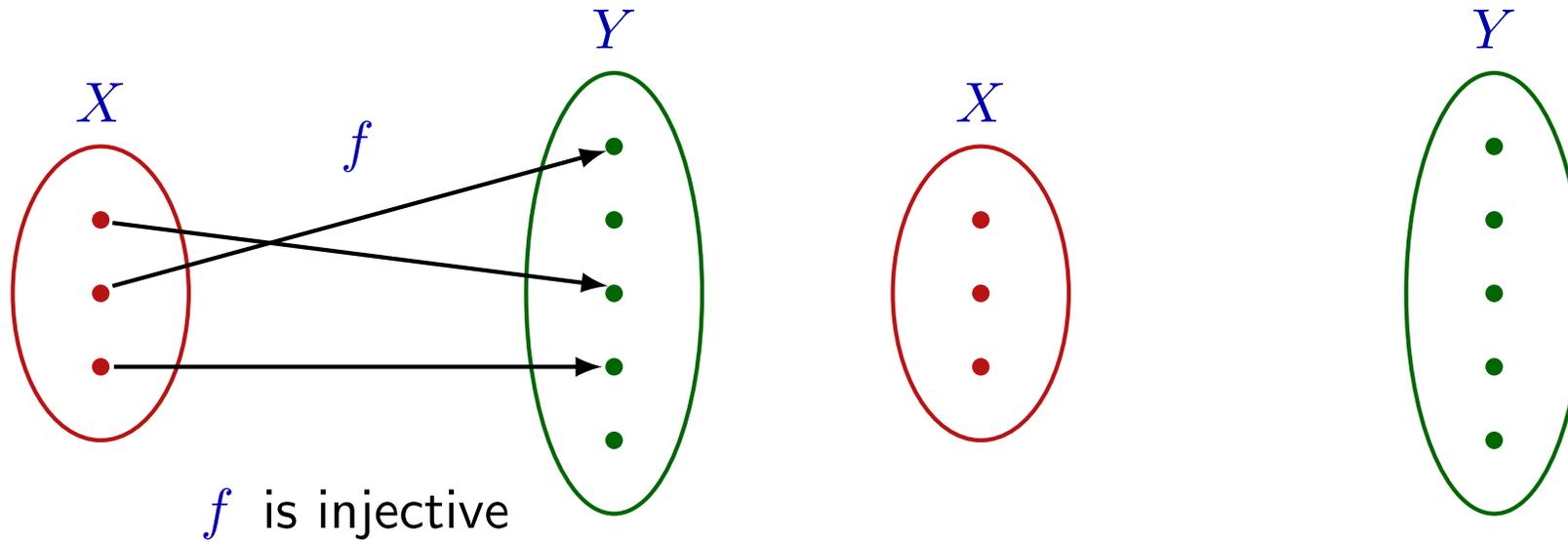
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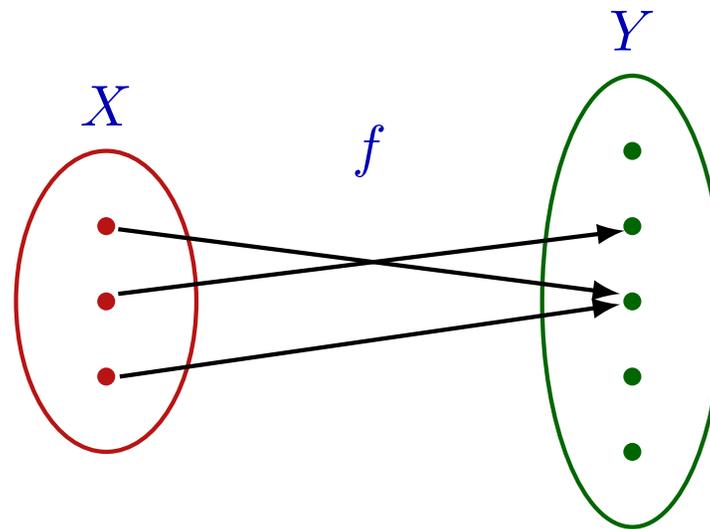
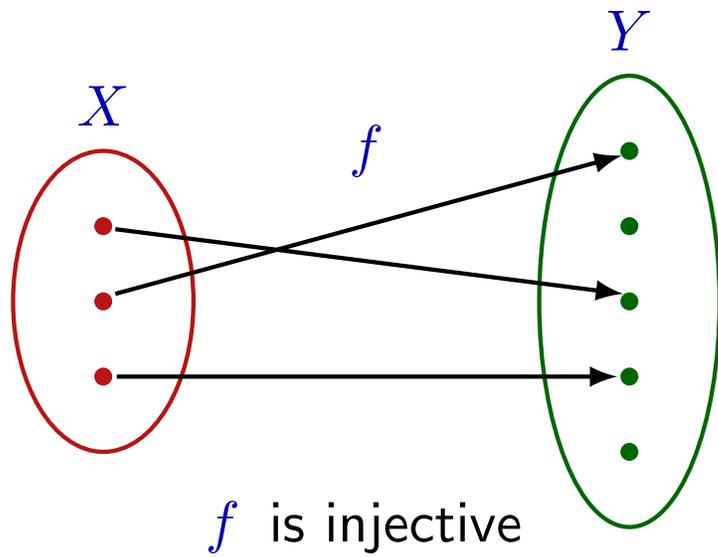
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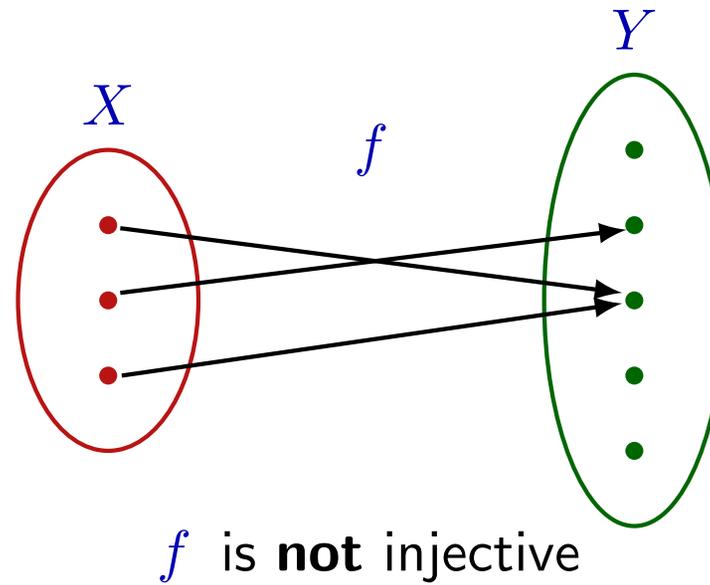
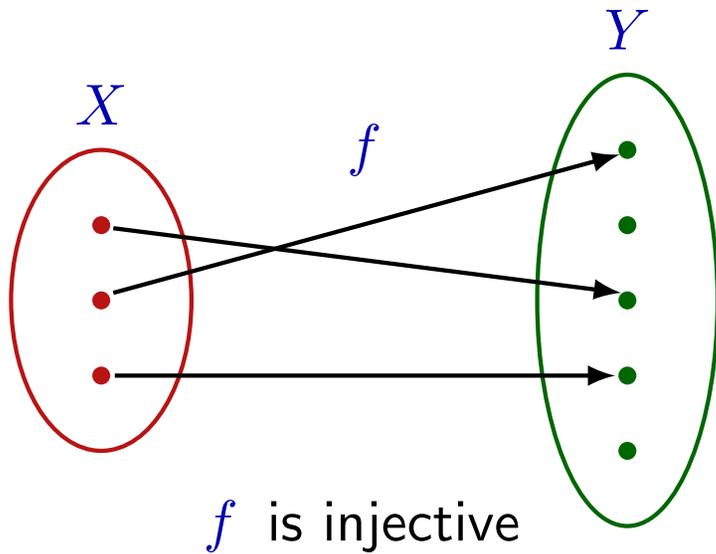
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Non-constant linear function is injective

MAT 250
Lecture 8, 10/02
Sets

Non-constant linear function is injective

Theorem.

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Theorem. A linear map $f : \mathbb{R} \rightarrow \mathbb{R}$

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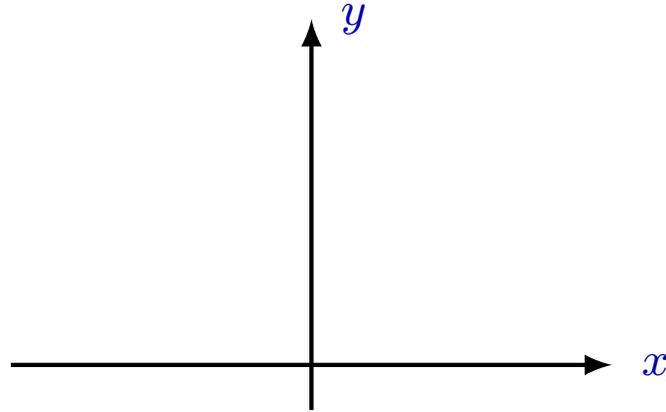
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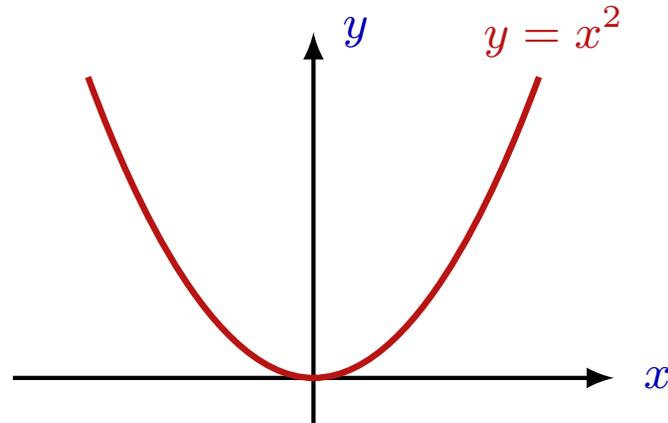
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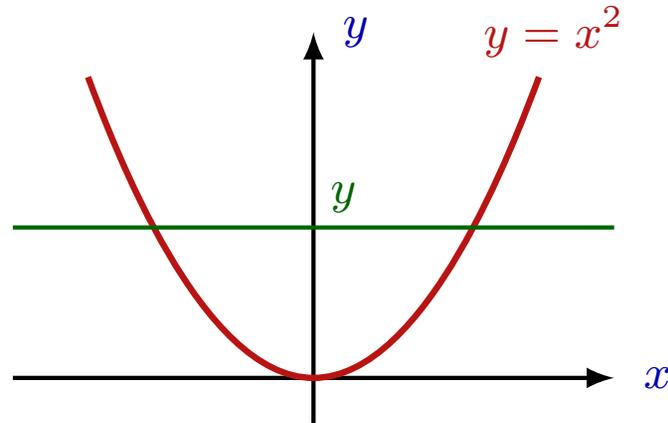
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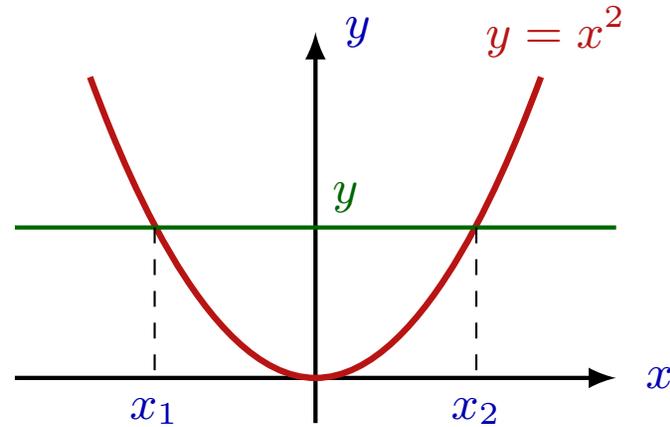
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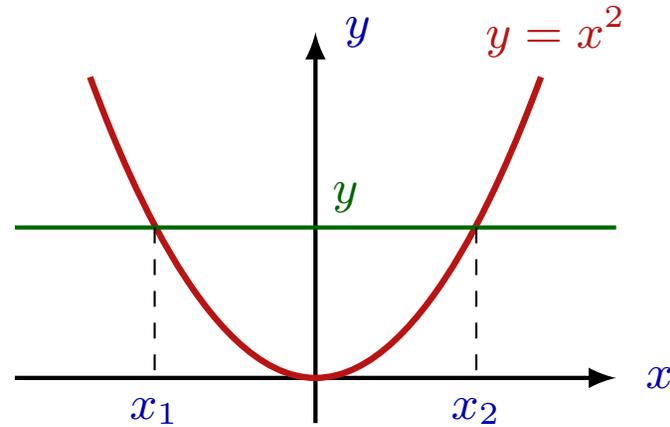
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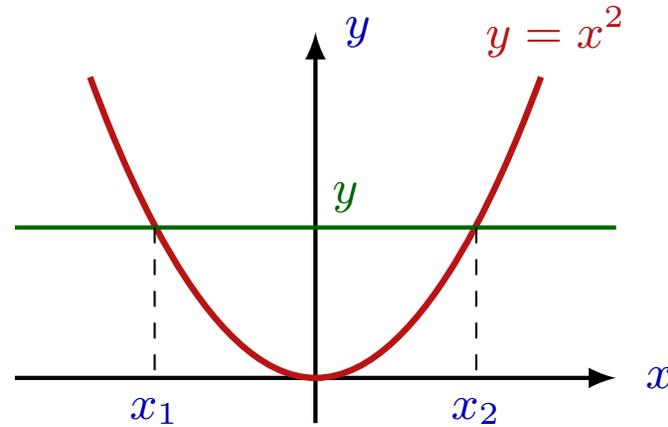
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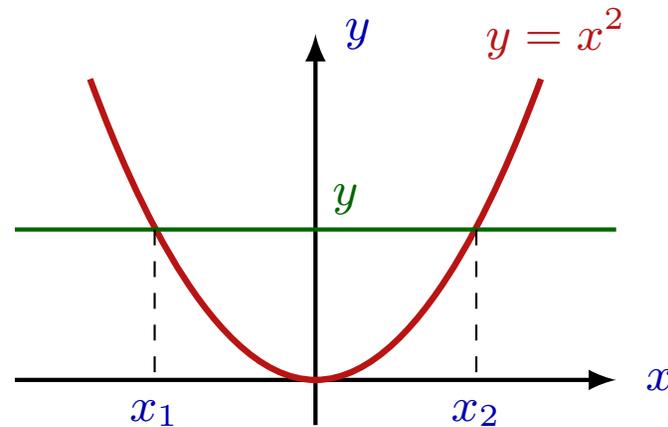
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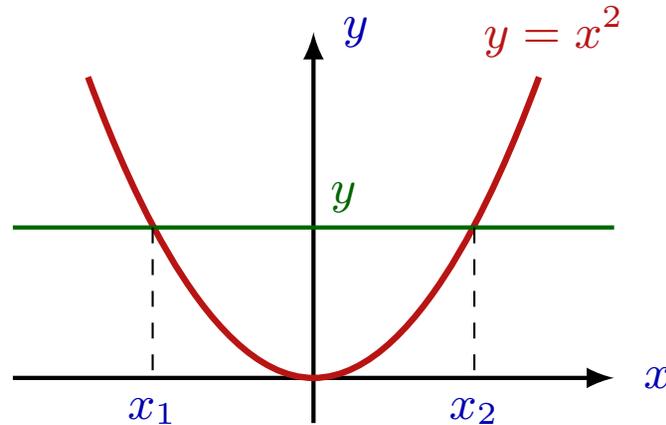


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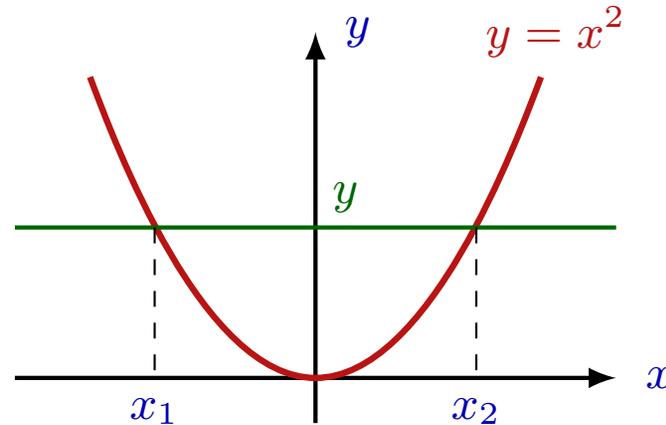
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Remark: The restriction $f|_{\mathbb{R}_+}$ is injective.

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Surjective maps

MAT 250
Lecture 8, 10/02
Sets

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Surjective or not?

MAT 250
Lecture 8, 10/02
Sets

Surjective or not?

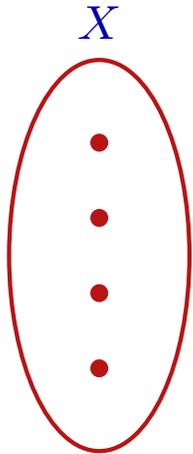
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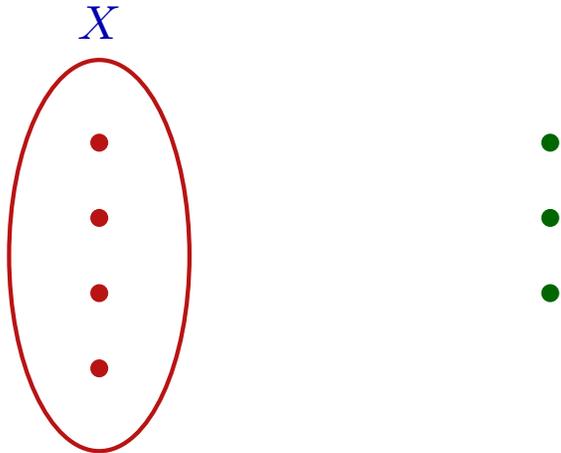
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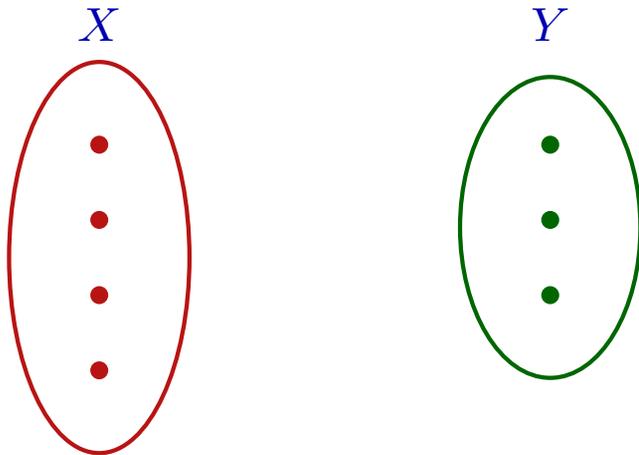
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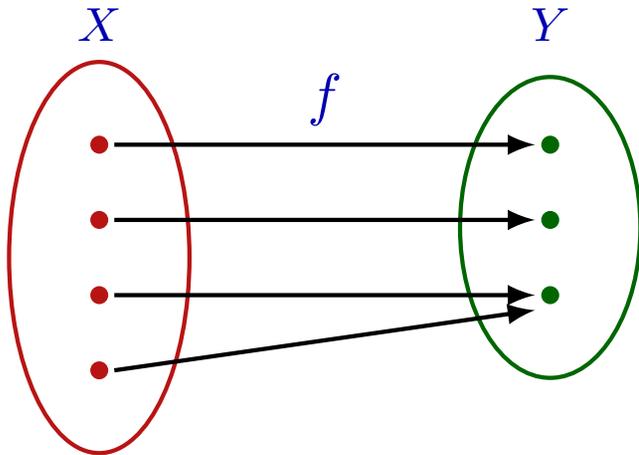
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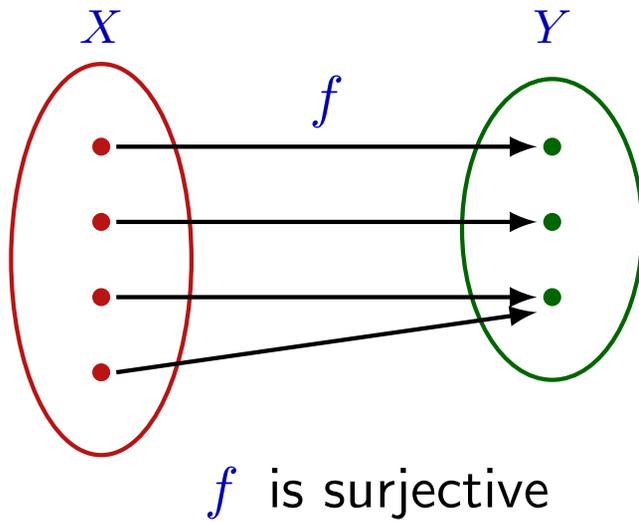
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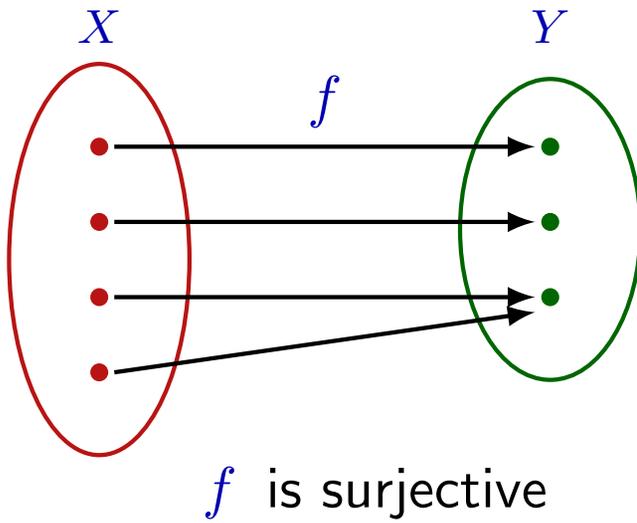
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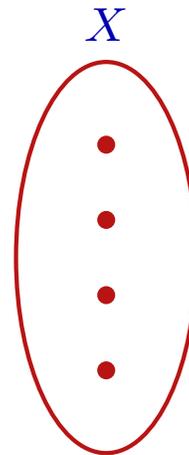
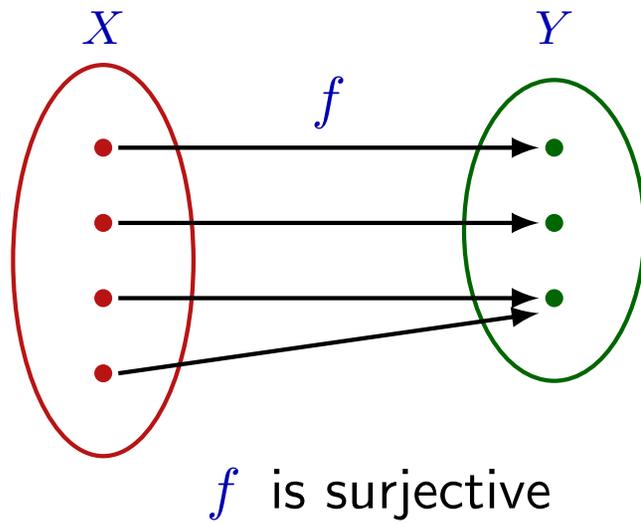
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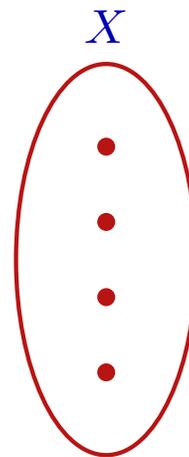
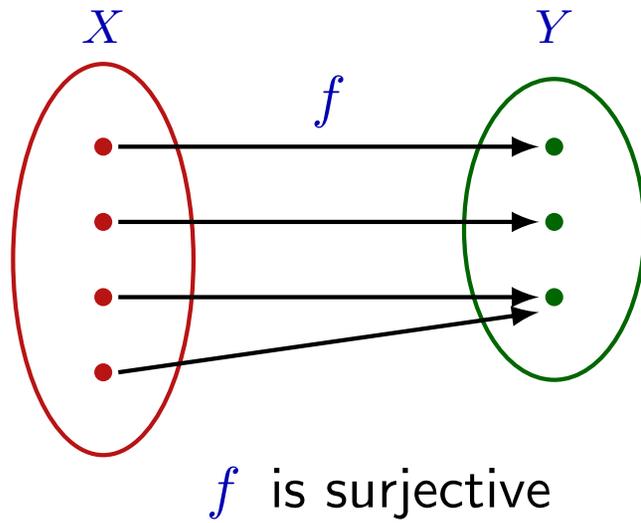
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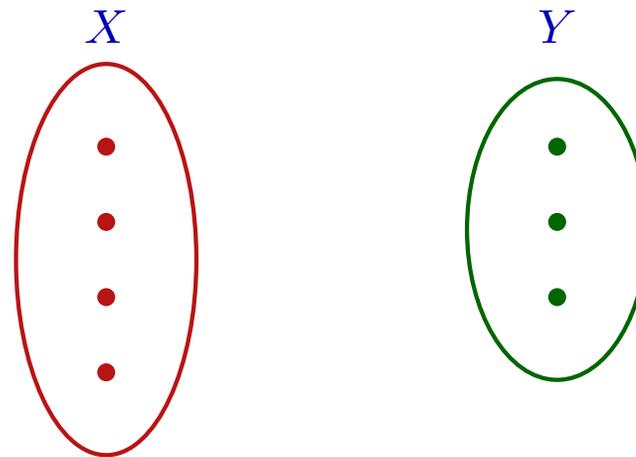
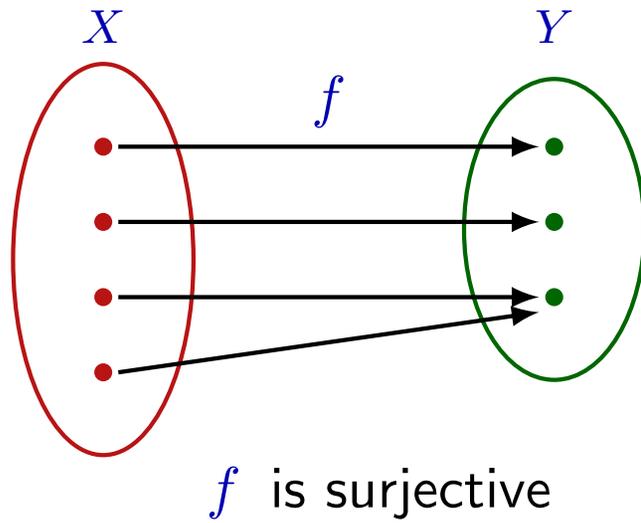
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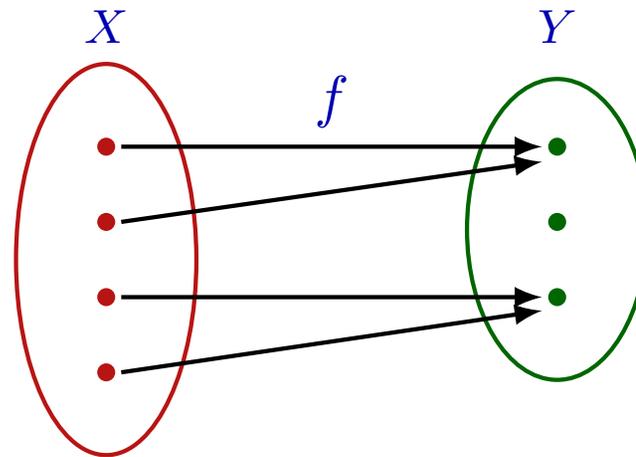
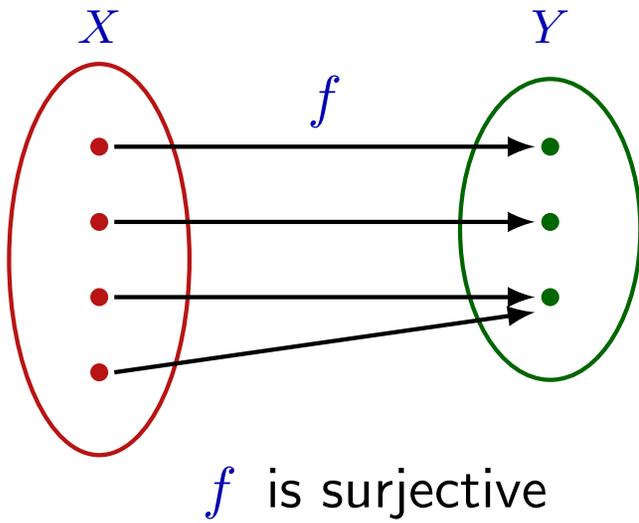
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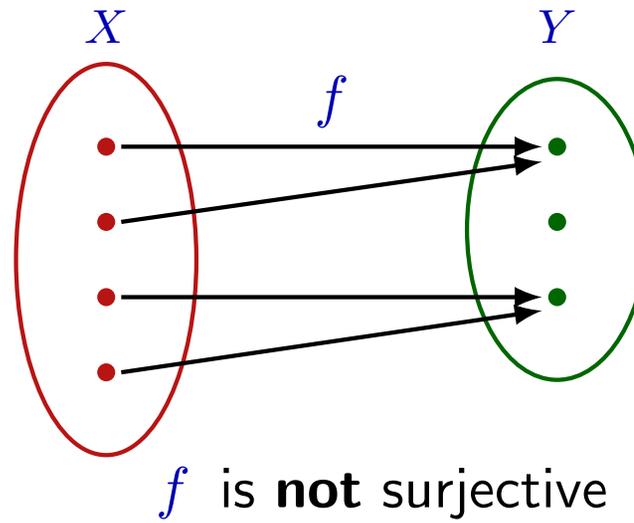
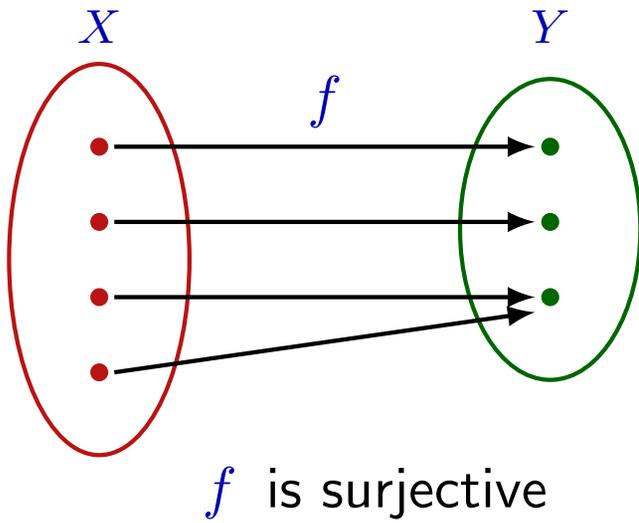
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Linear function is surjective, quadratic function is not

MAT 250
Lecture 8, 10/02
Sets

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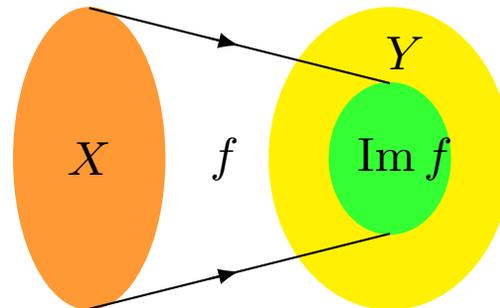
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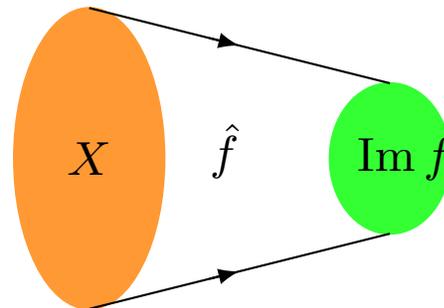
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Bijjective maps

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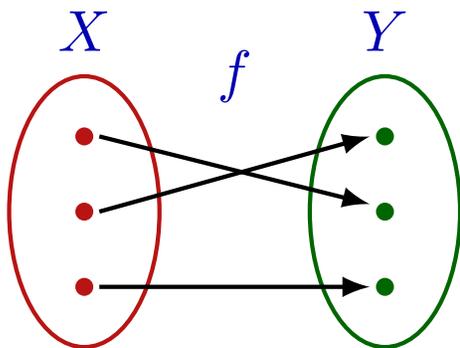
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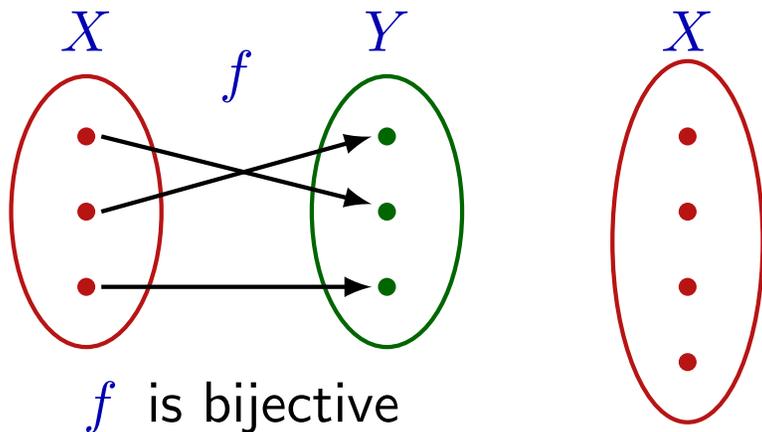
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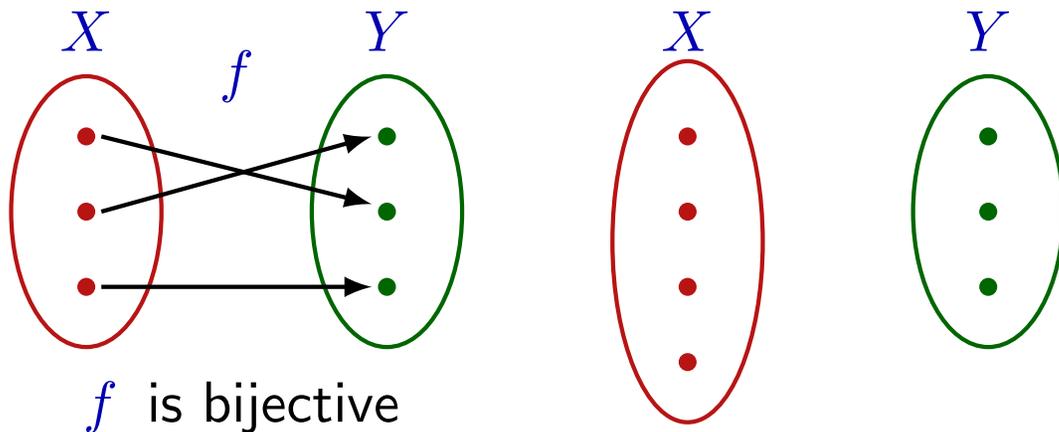
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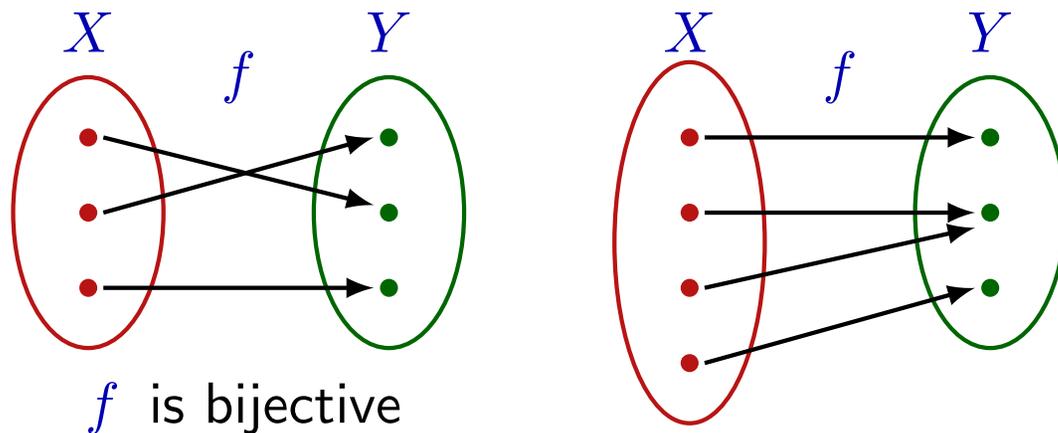
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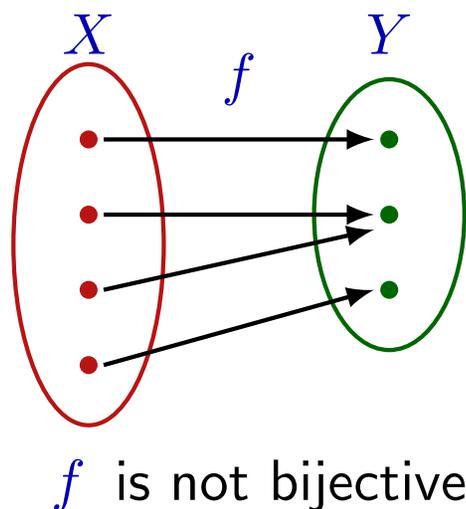
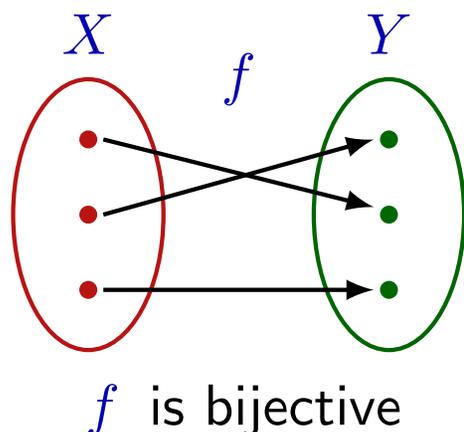


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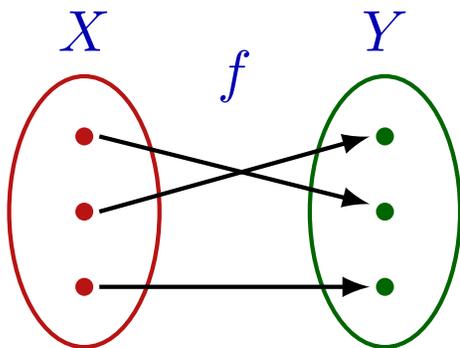
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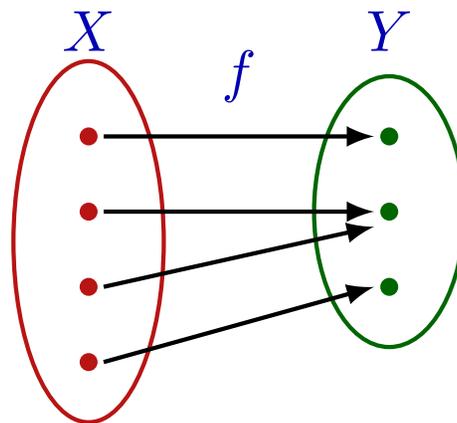
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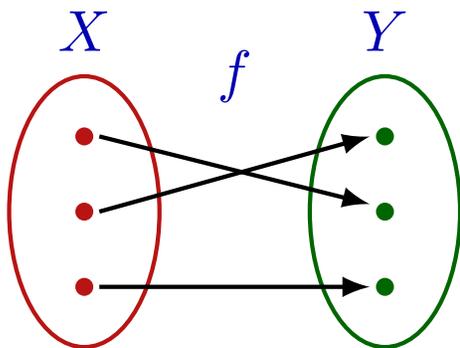
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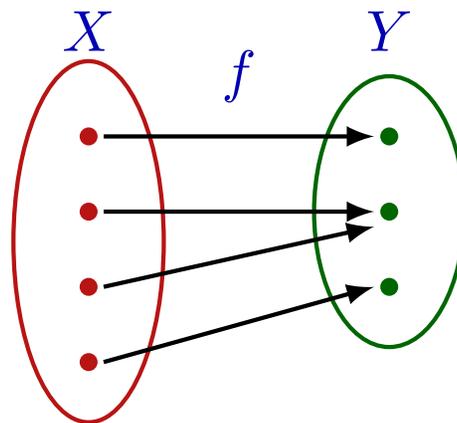
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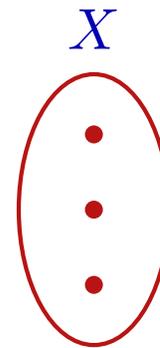
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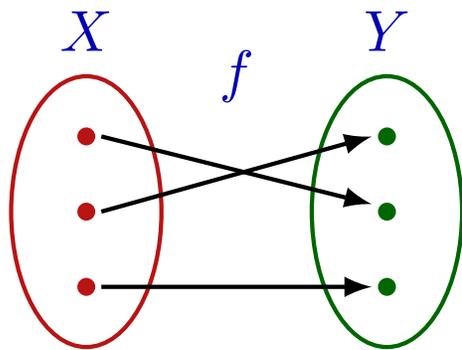


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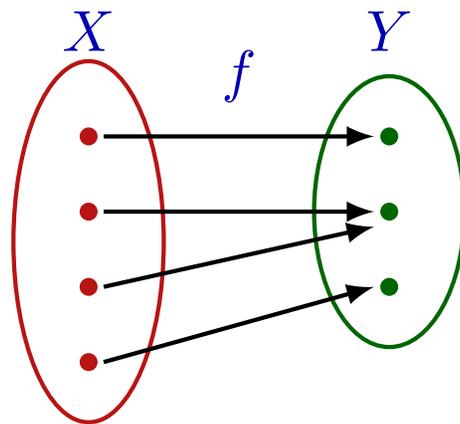
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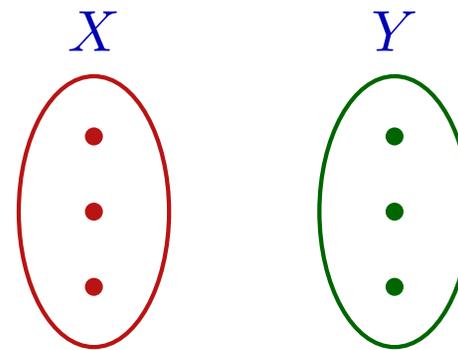
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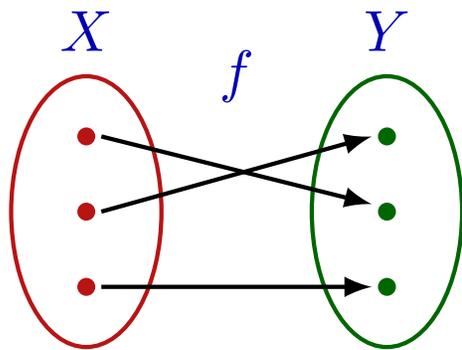
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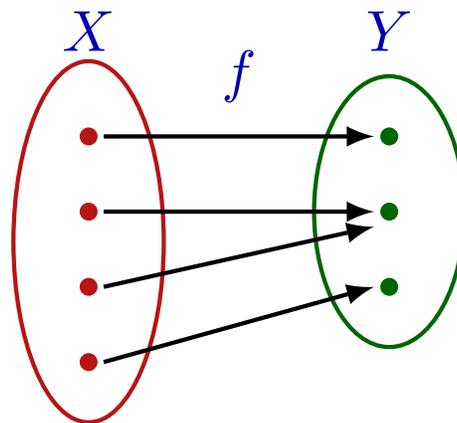
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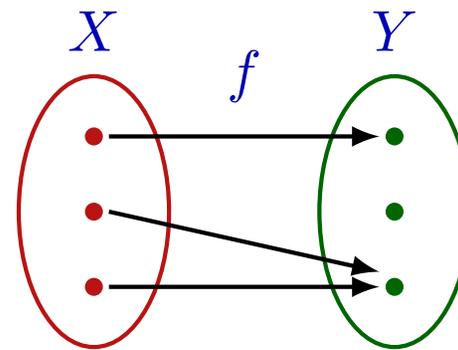
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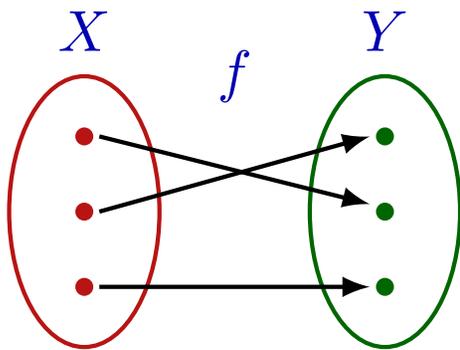
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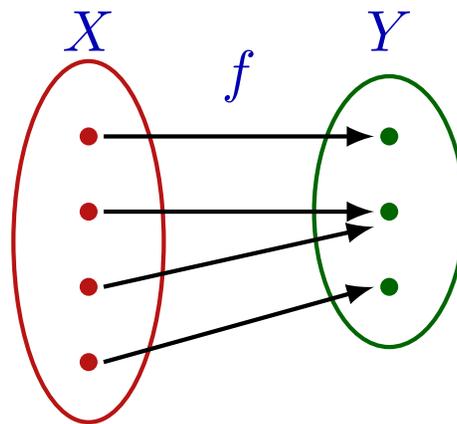
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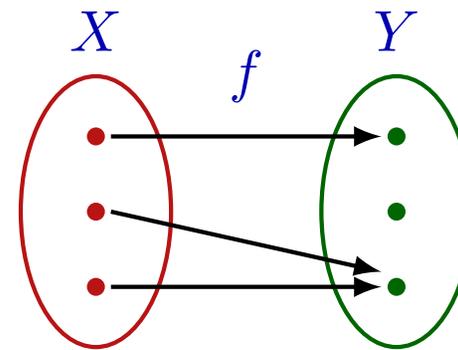
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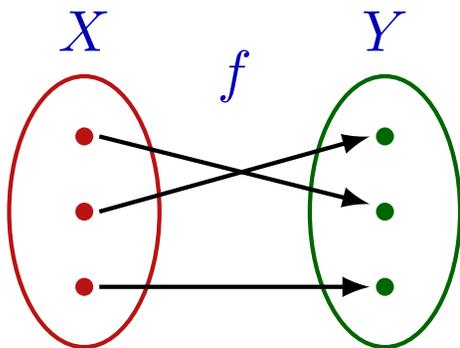
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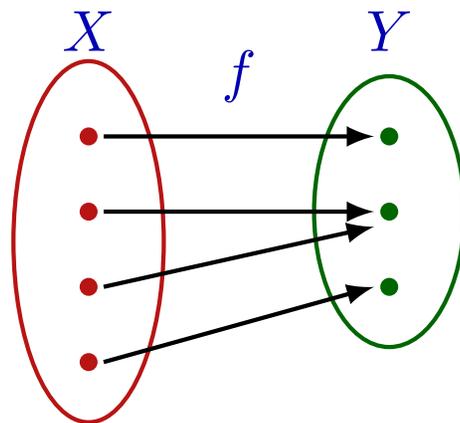
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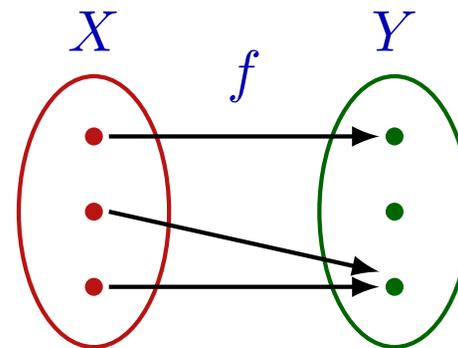
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A linear map $f : \mathbb{R} \rightarrow \mathbb{R}$

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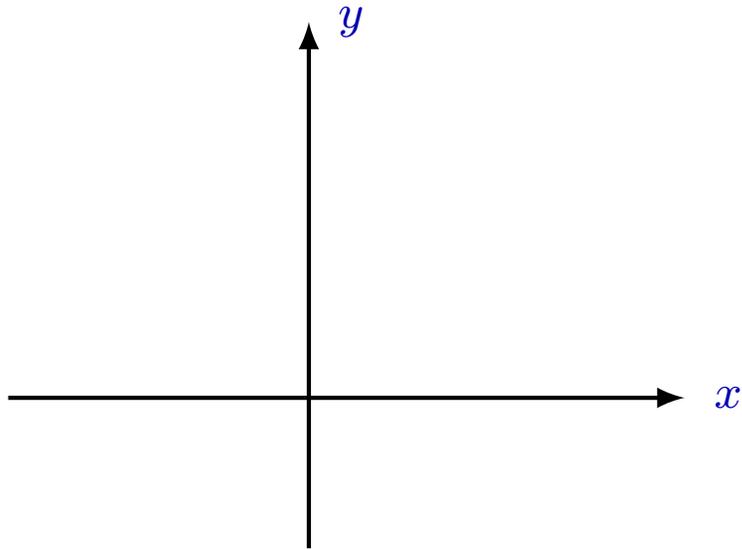
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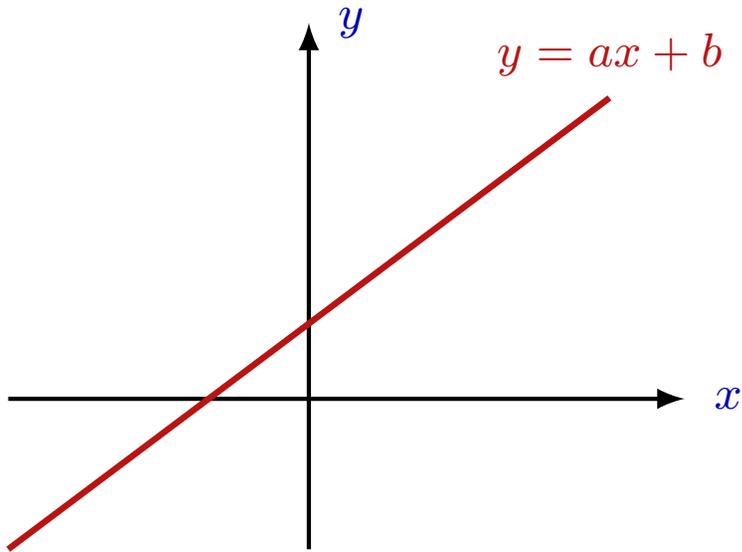
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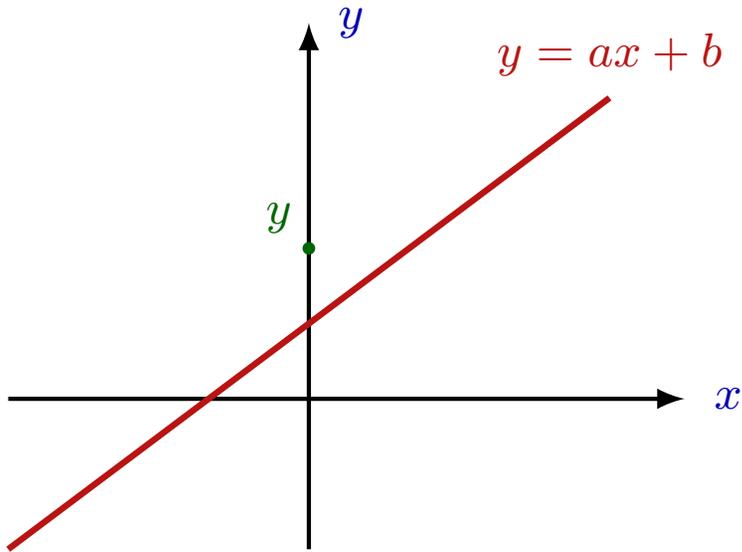
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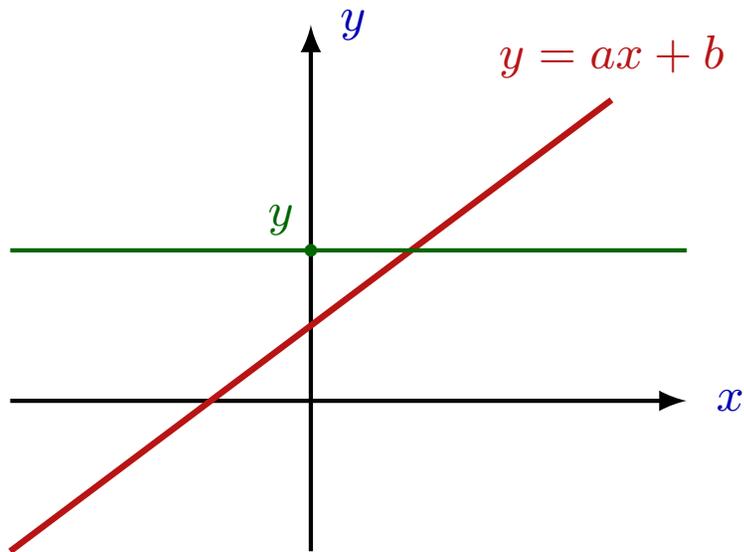
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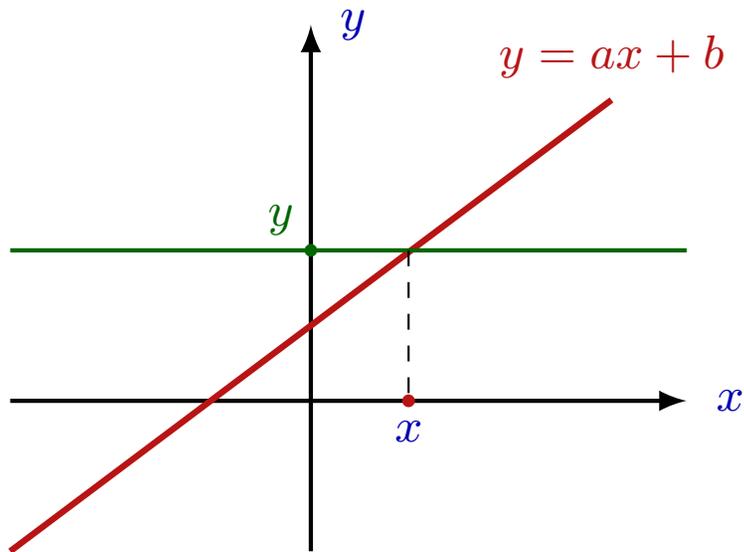
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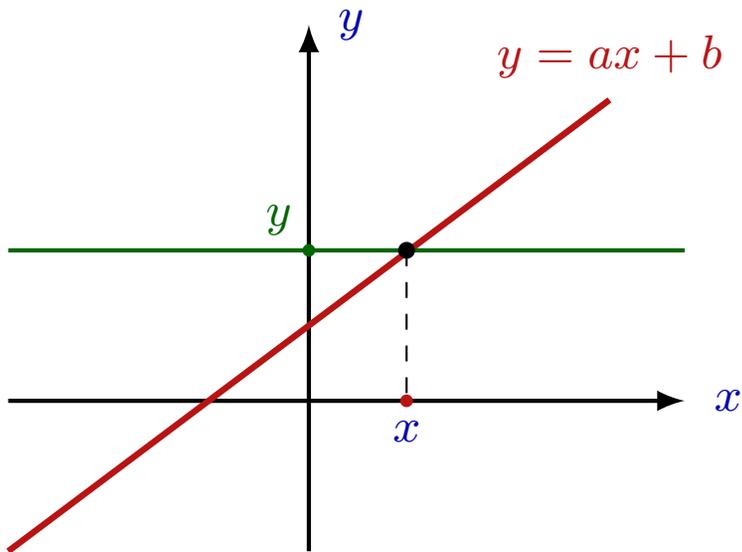
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Henri Cartan



André Weil



René de Possel



Charles Ehresmann



Laurent Schwartz



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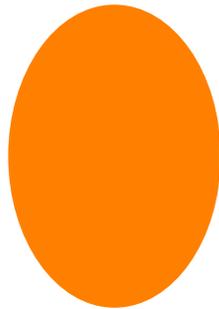
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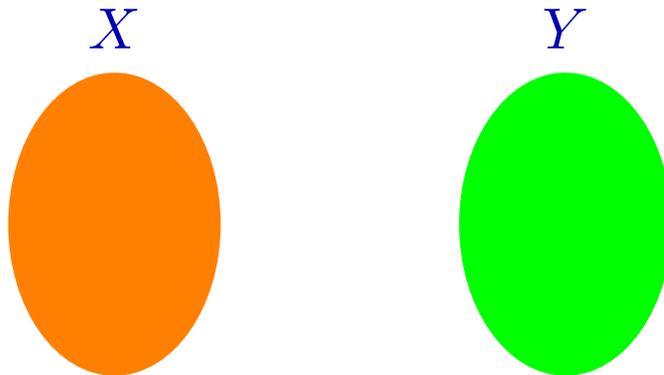


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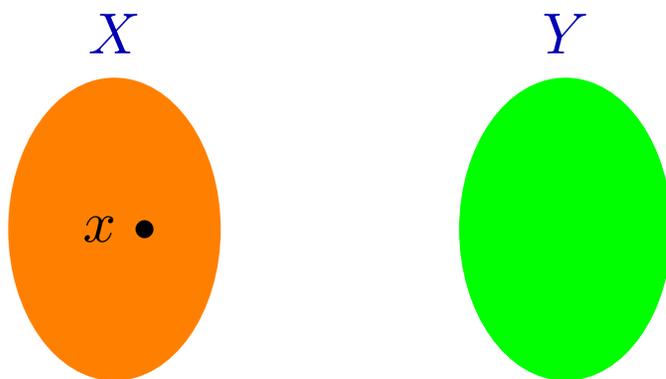


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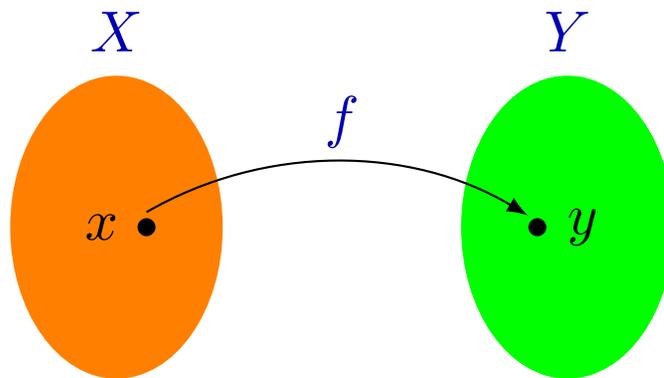


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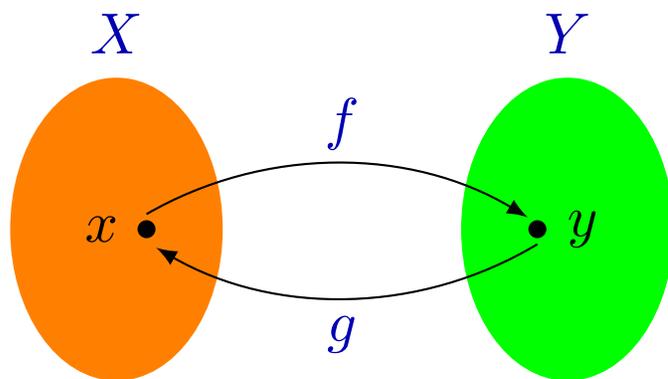


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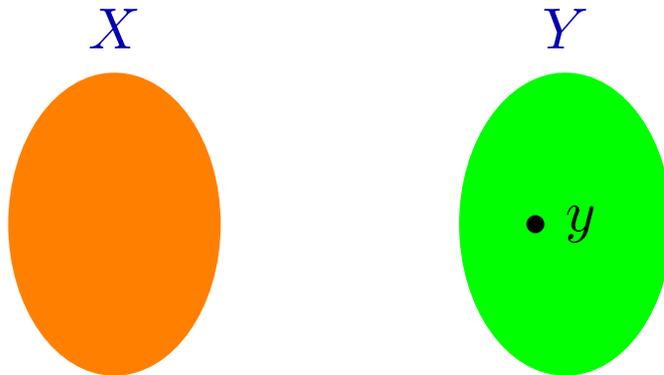


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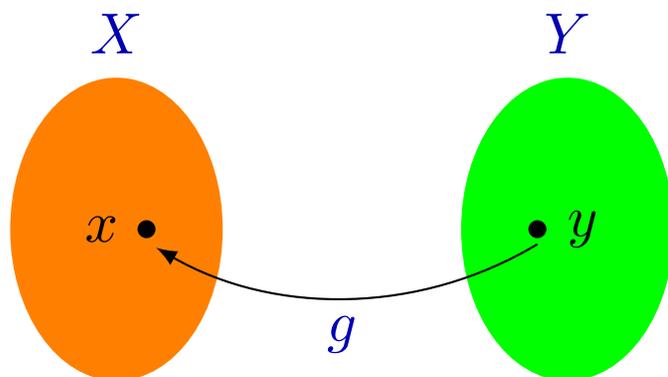


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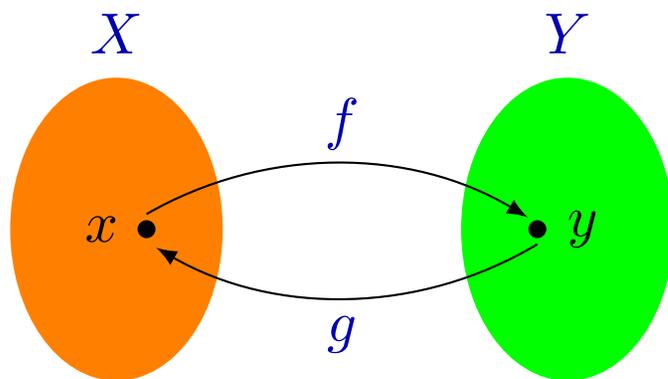


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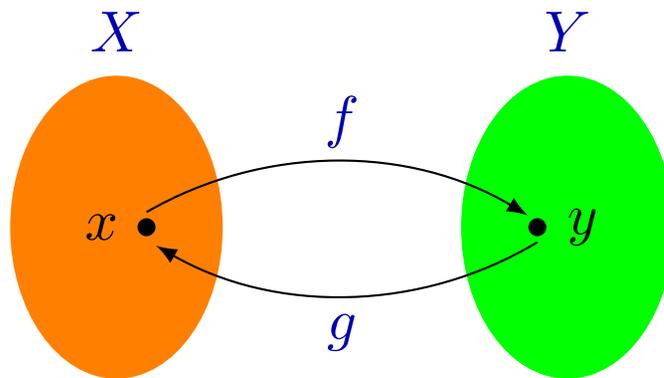
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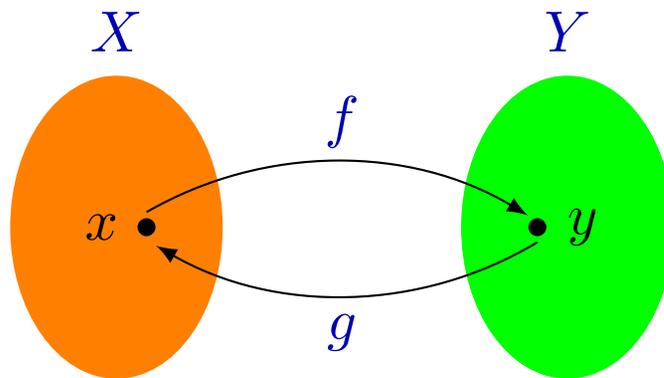
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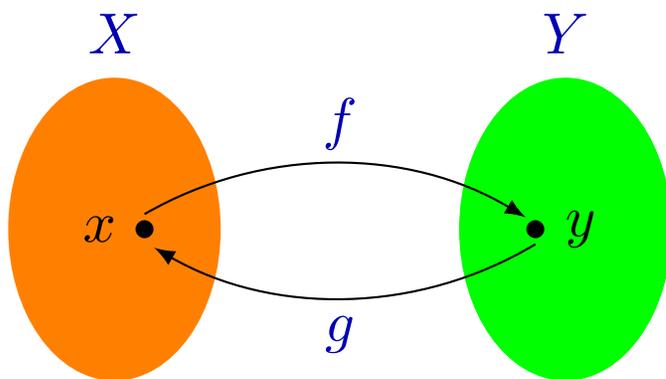
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$g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$, that is

$(g \circ f)(x) = x$ for any $x \in X$, and

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Warning.

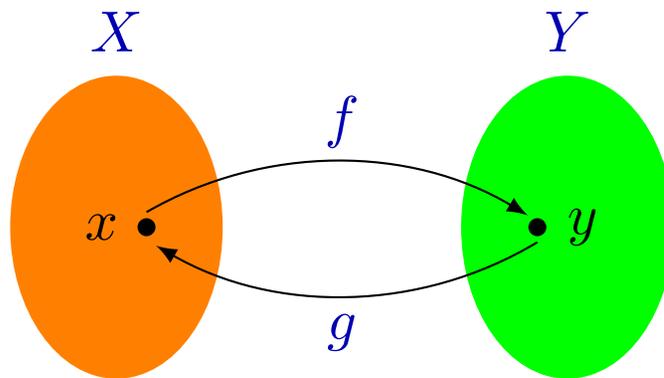
Inverse map

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Definition. A map is called **invertible** if it has an **inverse**.

Warning. Not all maps are invertible!

Inverse is unique

Theorem.

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Therefore, $g = h$ and the inverse map is unique. □

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Since the inverse map is **unique**, it deserves a **functional notation**.

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The inverse for f is denoted by f^{-1} .

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$$\boxed{f^{-1} \circ f = \text{id}_X \quad \text{and} \quad f \circ f^{-1} = \text{id}_Y}$$

Bijection = invertible map

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Apply f^{-1}

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Apply f^{-1} (it exists since f is invertible)

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By this, f is injective.

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To show **surjectivity**, take any $y \in Y$ and apply f^{-1} .

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To show **surjectivity**, take any $y \in Y$ and apply f^{-1} . Let $x = f^{-1}(y)$.

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such that $f(x)$

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So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

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So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

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So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

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By this, f is surjective.

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By this, f is injective.

To show **surjectivity**, take any $y \in Y$ and apply f^{-1} . Let $x = f^{-1}(y)$.

So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

$$\text{such that } f(x) = f(f^{-1}(y)) = y.$$

By this, f is surjective.

We have proved that f is injective and surjective,

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So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

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By this, f is surjective.

We have proved that f is injective and surjective, therefore, f is **bijective**.

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By this, f is surjective.

We have proved that f is injective and surjective, therefore, f is **bijective**.

The half of the proof is done!

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Assume now that f is a **bijection**,

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Define a map $g : Y \rightarrow X$

Bijection = invertible map

Assume now that f is a **bijection**, and prove that f is **invertible**.

By definition of bijectivity, $\forall y \in Y \ \exists! x \in X \ y = f(x)$.

Define a map $g : Y \rightarrow X$ by the formula $g(y) = x$,

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2. $f^{-1}(B)$ denotes the preimage of a set B under under any f
(not necessarily invertible).

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Therefore, f^{-1} is invertible (and by this, is a bijection) and $(f^{-1})^{-1} = f$. □

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Corollary 3. A composition of bijections is a bijection, that is,
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□

Inversion of the exponential function

MAT 250
Lecture 8, 10/02
Sets

Example 1.

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These identities are used as

the **definition** of logarithmic function as the inverse for exponential function,
or the other way around:

as the definition of the exponential function as the inverse for logarithmic function.

Inversion of the tangent

MAT 250
Lecture 8, 10/02
Sets

Example 2.

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It may be understood as $\tan^{-1} x = \frac{1}{\tan x} = \cot x$.

To avoid this ambiguity, always use $\arctan x$

as a notation for the inverse function for $\tan x$.

Inversion of the sin function

MAT 250
Lecture 8, 10/02
Sets

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Example 4. What is \arccos ?