Lecture 12

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For objects A, B of a category C, the set of all morphisms $A \to B$ of the category C will be denoted by $\mathrm{mor}_C(A, B)$.

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Main Example. The category of sets.

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For rings P, Q, a map $f : P \to Q$ is said to be a **ring homomorphism**
if $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for any $x, y \in P$.

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The notion of isomorphism sets up a problem to classify objects in the category.

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Hilbert hotel ”Infinity”

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Hilbert Hotel revisited.

Constructing new categories

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Composition:

$$\left(\begin{tikzcd} A & X \arrow[l, "\gamma"] \arrow[d, "g"'] \\ B & Y \arrow[l, "\delta"'] \end{tikzcd} \arrow[from=1-1, to=2-1, "h"]\right) \circ \left(\begin{tikzcd} X \arrow[d, "g"] & V \arrow[l, "\alpha"] \arrow[d, "f"'] \\ Y & W \arrow[l, "\beta"'] \end{tikzcd}\right)$$

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Categories in Linear Algebra.

Categorial generalizations of injective and surjective

Working with maps, we often consider elements of the source and target sets. However often elements may be avoided and everything can be done on the categorical level, replacing elements by compositions of maps.

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A morphism $f : A \to B$ in a category C is called **monic**
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Let X, Y be posets, a map $X \to Y$ is called **monotonic**, or **monotone**, or **increasing** if $a \prec b \implies f(a) \prec f(b)$.

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Any reasonable general construction converting a set to other set can be upgraded to a functor.

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Exercise 2. Formulate and prove a similar identity for $(g \circ f)^*$.
f_* is a part of functor "Power set". It could be denoted as $\mathcal{P}(f)$.

But what is f^*? It has a "wrong" direction, not like in the definition of functor above. This is a **contravariant** functor. Above we defined **covariant** functors.

Two functors of power set

Any map $f : X \to Y$ induces maps

$$f_* : \mathcal{P}(X) \to \mathcal{P}(Y), \;\; A \mapsto f(A) \text{ for any } A \subset X \text{ and}$$

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(We remember that $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ is the preimage of B, and **not** the result of applying the inverse map f^{-1}.)

The maps f_* and f^* are well defined since

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A **contravariant functor** $C \to D$ is
a (covariant) functor from C to the category dual to D.