

Lecture 12

Categories

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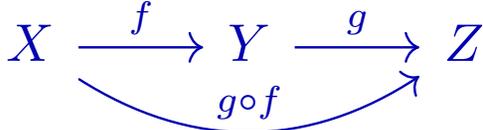
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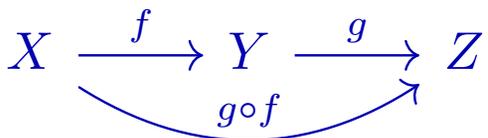
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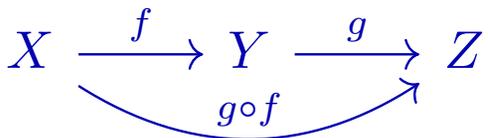
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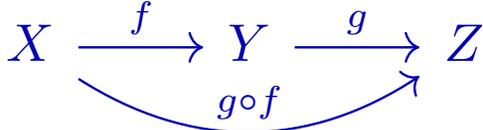
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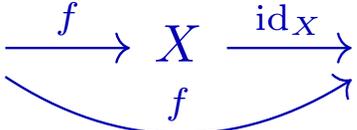
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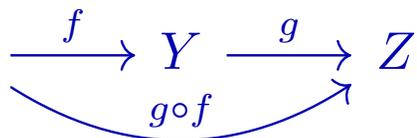
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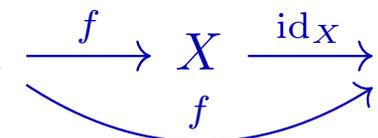
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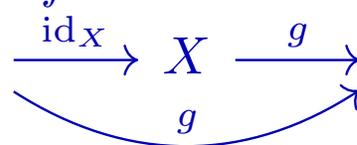
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For objects A, B of a category C ,

the set of all morphisms $A \rightarrow B$ of the category C will be denoted by $\text{mor}_C(A, B)$.

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For rings P, Q , a map $f : P \rightarrow Q$ is said to be a **ring homomorphism**

if $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for any $x, y \in P$.

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Toy categories

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How **small** a category may be?

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This is the **smallest** category.

Let us **keep requiring** single object, but **increase** the set of morphisms $A \rightarrow A$.

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The notion of isomorphism sets up a problem to classify objects in the category.

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Proof. Consider a composition of these maps:

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However,

cardinalities of infinite sets and the **numbers of elements** of finite sets have quite different properties.

Hilbert hotel "Infinity"

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Hilbert Hotel revisited.

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Let C be a category. **Dual** or **opposite** category is the category which has the same objects, but each morphism $f : A \rightarrow B$ is considered as a morphism $B \rightarrow A$ of the **opposite** direction. Composition $A \rightarrow B \rightarrow C$ turns to $C \rightarrow B \rightarrow A$. Verify that this is a category, indeed.

Category of morphisms. Here is another way to cook up a new category from C : Objects are morphisms of C .

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Categories in Linear Algebra.

Working with maps, we often consider elements of the source and target sets. However often elements may be avoided and everything can be done on the categorical level, replacing elements by compositions of maps.

A morphism $f : A \rightarrow B$ in a category \mathcal{C} is called **monic** if for any morphisms $\alpha_1, \alpha_2 : X \rightarrow A$ the equality $f \circ \alpha_1 = f \circ \alpha_2$ implies $\alpha_1 = \alpha_2$.

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Let X, Y be posets, a map $X \rightarrow Y$ is called
monotonic, or **monotone**, or **increasing** if $a < b \implies f(a) < f(b)$.

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If $a < b$ is false, then there is no morphism $a \rightarrow b$.

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Any reasonable general construction converting a set to other set can be upgraded to a functor.

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A **contravariant functor** $C \rightarrow D$ is

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