2

Review of Tensors, Manifolds, and Vector Bundles

Most of the technical machinery of Riemannian geometry is built up using tensors; indeed, Riemannian metrics themselves are tensors. Thus we begin by reviewing the basic definitions and properties of tensors on a finite-dimensional vector space. When we put together spaces of tensors on a manifold, we obtain a particularly useful type of geometric structure called a "vector bundle," which plays an important role in many of our investigations. Because vector bundles are not always treated in beginning manifolds courses, we include a fairly complete discussion of them in this chapter. The chapter ends with an application of these ideas to tensor bundles on manifolds, which are vector bundles constructed from tensor spaces associated with the tangent space at each point.

Much of the material included in this chapter should be familiar from your study of manifolds. It is included here as a review and to establish our notations and conventions for later use. If you need more detail on any topics mentioned here, consult [Boo86] or [Spi79, volume 1].

Tensors on a Vector Space

Let V be a finite-dimensional vector space (all our vector spaces and manifolds are assumed real). As usual, V^* denotes the dual space of V—the space of *covectors,* or real-valued linear functionals, on V—and we denote the natural pairing $V^* \times V \to \mathbf{R}$ by either of the notations

$$(\omega, X) \mapsto \langle \omega, X \rangle \quad \text{or} \quad (\omega, X) \mapsto \omega(X)$$

for $\omega \in V^*$, $X \in V$.

A *covariant k-tensor* on V is a multilinear map

$$F\colon \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbf{R}.$$

Similarly, a *contravariant l-tensor* is a multilinear map

$$F\colon \underbrace{V^* \times \cdots \times V^*}_{l \text{ copies}} \to \mathbf{R}.$$

We often need to consider tensors of mixed types as well. A *tensor of type* $\binom{k}{l}$, also called a *k-covariant, l-contravariant tensor,* is a multilinear map

$$F\colon \underbrace{V^* \times \cdots \times V^*}_{l \text{ copies}} \times \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbf{R}.$$

Actually, in many cases it is necessary to consider multilinear maps whose arguments consist of k vectors and l covectors, but not necessarily in the order implied by the definition above; such an object is still called a tensor of type $\binom{k}{l}$. For any given tensor, we will make it clear which arguments are vectors and which are covectors.

The space of all covariant k-tensors on V is denoted by $T^k(V)$, the space of contravariant l-tensors by $T_l(V)$, and the space of mixed $\binom{k}{l}$-tensors by $T^k_l(V)$. The *rank* of a tensor is the number of arguments (vectors and/or covectors) it takes.

There are obvious identifications $T^k_0(V) = T^k(V)$, $T^0_l(V) = T_l(V)$, $T^1(V) = V^*$, $T_1(V) = V^{**} = V$, and $T^0(V) = \mathbf{R}$. A less obvious, but extremely important, identification is $T^1_1(V) = \operatorname{End}(V)$, the space of linear endomorphisms of V (linear maps from V to itself). A more general version of this identification is expressed in the following lemma.

Lemma 2.1. *Let V be a finite-dimensional vector space. There is a natural (basis-independent) isomorphism between $T^k_{l+1}(V)$ and the space of multilinear maps*

$$\underbrace{V^* \times \cdots \times V^*}_{l} \times \underbrace{V \times \cdots \times V}_{k} \to V.$$

Exercise 2.1. Prove Lemma 2.1. [Hint: In the special case $k = 1$, $l = 0$, consider the map $\Phi\colon \operatorname{End}(V) \to T^1_1(V)$ by letting ΦA be the $\binom{1}{1}$-tensor defined by $\Phi A(\omega, X) = \omega(AX)$. The general case is similar.]

There is a natural product, called the *tensor product,* linking the various tensor spaces over V; if $F \in T^k_l(V)$ and $G \in T^p_q(V)$, the tensor $F \otimes G \in T^{k+p}_{l+q}(V)$ is defined by

$$\begin{aligned} F \otimes G(\omega^1, \dots, \omega^{l+q}, X_1, \dots, X_{k+p}) \\ = F(\omega^1, \dots, \omega^l, X_1, \dots, X_k) G(\omega^{l+1}, \dots, \omega^{l+q}, X_{k+1}, \dots, X_{k+p}). \end{aligned}$$

If $(E_1, \ldots, E_n)$ is a basis for V, we let $(\varphi^1, \ldots, \varphi^n)$ denote the corresponding dual basis for V^*, defined by $\varphi^i(E_j) = \delta^i_j$. A basis for $T^k_l(V)$ is given by the set of all tensors of the form

$$E_{j_1} \otimes \cdots \otimes E_{j_l} \otimes \varphi^{i_1} \otimes \cdots \otimes \varphi^{i_k}, \tag{2.1}$$

as the indices i_p, j_q range from 1 to n. These tensors act on basis elements by

$$E_{j_1} \otimes \cdots \otimes E_{j_l} \otimes \varphi^{i_1} \otimes \cdots \otimes \varphi^{i_k}(\varphi^{s_1}, \ldots, \varphi^{s_l}, E_{r_1}, \ldots, E_{r_k}) \\ = \delta^{s_1}_{j_1} \cdots \delta^{s_l}_{j_l} \delta^{i_1}_{r_1} \cdots \delta^{i_k}_{r_k}.$$

Any tensor $F \in T^k_l(V)$ can be written in terms of this basis as

$$F = F^{j_1 \ldots j_l}_{i_1 \ldots i_k} E_{j_1} \otimes \cdots \otimes E_{j_l} \otimes \varphi^{i_1} \otimes \cdots \otimes \varphi^{i_k}, \tag{2.2}$$

where

$$F^{j_1 \ldots j_l}_{i_1 \ldots i_k} = F(\varphi^{j_1}, \ldots, \varphi^{j_l}, E_{i_1}, \ldots, E_{i_k}).$$

In (2.2), and throughout this book, we use the *Einstein summation convention* for expressions with indices: if in any term the same index name appears twice, as both an upper and a lower index, that term is assumed to be summed over all possible values of that index (usually from 1 to the dimension of the space). We always choose our index positions so that vectors have lower indices and covectors have upper indices, while the *components* of vectors have upper indices and those of covectors have lower indices. This ensures that summations that make mathematical sense always obey the rule that each repeated index appears once up and once down in each term to be summed.

If the arguments of a mixed tensor F occur in a nonstandard order, then the horizontal as well as vertical positions of the indices are significant and reflect which arguments are vectors and which are covectors. For example, if B is a $\binom{2}{1}$-tensor whose first argument is a vector, second is a covector, and third is a vector, its components are written

$$B_i{}^j{}_k = B(E_i, \varphi^j, E_k). \tag{2.3}$$

We can use the result of Lemma 2.1 to define a natural operation called *trace* or *contraction,* which lowers the rank of a tensor by 2. In one special case, it is easy to describe: the operator $\operatorname{tr}\colon T^1_1(V) \to \mathbf{R}$ is just the trace of F when it is considered as an endomorphism of V. Since the trace of an endomorphism is basis-independent, this is well defined. More generally, we define $\operatorname{tr}\colon T^{k+1}_{l+1}(V) \to T^k_l(V)$ by letting $\operatorname{tr} F(\omega^1, \ldots, \omega^l, V_1, \ldots, V_k)$ be the trace of the endomorphism

$$F(\omega^1, \ldots, \omega^l, \cdot, V_1, \ldots, V_k, \cdot) \in T^1_1(V).$$

In terms of a basis, the components of $\operatorname{tr} F$ are

$$(\operatorname{tr} F)_{i_1\ldots i_k}^{j_1\ldots j_l} = F_{i_1\ldots i_k m}^{j_1\ldots j_l m}.$$

Even more generally, we can contract a given tensor on any pair of indices as long as one is contravariant and one is covariant. There is no general notation for this operation, so we just describe it in words each time it arises. For example, we can contract the tensor B with components given by (2.3) on its first and second indices to obtain a covariant 1-tensor A whose components are $A_k = B_i{}^i{}_k$.

Exercise 2.2. Show that the trace on any pair of indices is a well-defined linear map from $T_{l+1}^{k+1}(V)$ to $T_l^k(V)$.

A class of tensors that plays a special role in differential geometry is that of *alternating tensors:* those that change sign whenever two arguments are interchanged. We let $\Lambda^k(V)$ denote the space of covariant alternating k-tensors on V, also called *k-covectors* or *(exterior) k-forms*. There is a natural bilinear, associative product on forms called the *wedge product,* defined on 1-forms $\omega^1, \ldots, \omega^k$ by setting

$$\omega^1 \wedge \cdots \wedge \omega^k(X_1, \ldots, X_k) = \det(\langle \omega^i, X_j \rangle),$$

and extending by linearity. (There is an alternative definition of the wedge product in common use, which amounts to multiplying our wedge product by a factor of $1/k!$. The choice of which definition to use is a matter of convention, though there are various reasons to justify each choice depending on the context. The definition we have chosen is most common in introductory differential geometry texts, and is used, for example, in [Boo86, Cha93, dC92, Spi79]. The other convention is used in [KN63] and is more common in complex differential geometry.)

Manifolds

Now we turn our attention to manifolds. Throughout this book, all our manifolds are assumed to be smooth, Hausdorff, and second countable; and *smooth* always means C^∞, or infinitely differentiable. As in most parts of differential geometry, the theory still works under weaker differentiability assumptions, but such considerations are usually relevant only when treating questions of hard analysis that are beyond our scope.

We write local coordinates on any open subset $U \subset M$ as $(x^1, \ldots, x^n)$, (x^i), or x, depending on context. Although, formally speaking, coordinates constitute a map from U to $\mathbf{R}^n$, it is more common to use a coordinate chart to *identify* U with its image in $\mathbf{R}^n$, and to identify a point in U with its coordinate representation (x^i) in $\mathbf{R}^n$.

For any $p \in M$, the *tangent space* T_pM can be characterized either as the set of derivations of the algebra of germs at p of C^∞ functions on M (i.e., tangent vectors are "directional derivatives"), or as the set of equivalence classes of curves through p under a suitable equivalence relation (i.e., tangent vectors are "velocities"). Regardless of which characterization is taken as the definition, local coordinates (x^i) give a basis for T_pM consisting of the partial derivative operators $\partial/\partial x^i$. When there can be no confusion about which coordinates are meant, we usually abbreviate $\partial/\partial x^i$ by the notation ∂_i.

On a finite-dimensional vector space V with its standard smooth manifold structure, there is a natural (basis-independent) identification of each tangent space T_pV with V itself, obtained by identifying a vector $X \in V$ with the directional derivative

$$Xf = \left.\frac{d}{dt}\right|_{t=0} f(p + tX).$$

In terms of the coordinates (x^i) induced on V by any basis, this is just the usual identification $(x^1, \ldots, x^n) \leftrightarrow x^i\partial_i$.

In this book, we always write coordinates with upper indices, as in (x^i). This has the consequence that the differentials dx^i of the coordinate functions are consistent with the convention that covectors have upper indices. Likewise, the coordinate vectors $\partial_i = \partial/\partial x^i$ have lower indices if we consider an upper index "in the denominator" to be the same as a lower index.

If $\widetilde{M}$ is a smooth manifold, a *submanifold* (or *immersed submanifold*) of $\widetilde{M}$ is a smooth manifold M together with an injective immersion $\iota\colon M \to \widetilde{M}$. Identifying M with its image $\iota(M) \subset \widetilde{M}$, we can consider M as a subset of $\widetilde{M}$, although in general the topology and smooth structure of M may have little to do with those of $\widetilde{M}$ and have to be considered as extra data. The most important type of submanifold is that in which the inclusion map ι is an *embedding,* which means that it is a homeomorphism onto its image with the subspace topology. In that case, M is called an *embedded submanifold* or a *regular submanifold.*

Suppose M is an embedded n-dimensional submanifold of an m-dimensional manifold $\widetilde{M}$. For every point $p \in M$, there exist *slice coordinates* $(x^1, \ldots, x^m)$ on a neighborhood $\widetilde{\mathcal{U}}$ of p in $\widetilde{M}$ such that $\widetilde{\mathcal{U}} \cap M$ is given by $\{x : x^{n+1} = \cdots = x^m = 0\}$, and $(x^1, \ldots, x^n)$ form local coordinates for M (Figure 2.1). At each $q \in \widetilde{\mathcal{U}} \cap M$, T_qM can be naturally identified as the subspace of $T_q\widetilde{M}$ spanned by the vectors $(\partial_1, \ldots, \partial_n)$.

Exercise 2.3. Suppose $M \subset \widetilde{M}$ is an embedded submanifold.

(a) If f is any smooth function on M, show that f can be extended to a smooth function on $\widetilde{M}$ whose restriction to M is f. [Hint: Extend f locally in slice coordinates by letting it be independent of $(x^{n+1}, \ldots, x^m)$, and patch together using a partition of unity.]

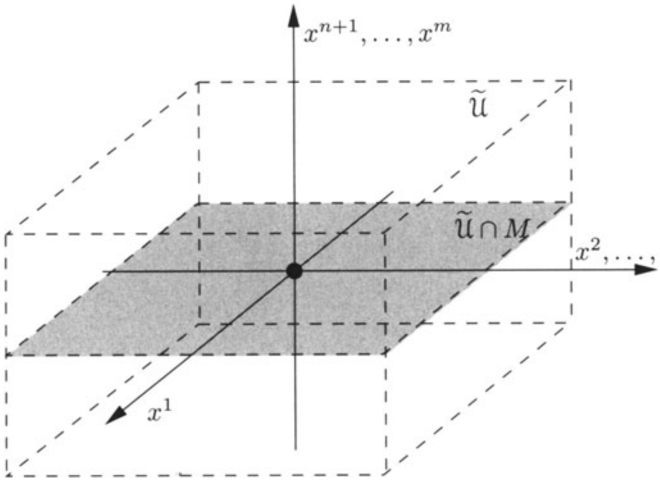

FIGURE 2.1. Slice coordinates.

(b) Show that any vector field on M can be extended to a vector field on $\widetilde{M}$.

(c) If $\widetilde{X}$ is a vector field on $\widetilde{M}$, show that $\widetilde{X}$ is tangent to M at points of M if and only if $\widetilde{X} f = 0$ whenever $f \in C^\infty(\widetilde{M})$ is a function that vanishes on M.

Vector Bundles

When we glue together the tangent spaces at all points on a manifold M, we get a set that can be thought of both as a union of vector spaces and as a manifold in its own right. This kind of structure is so common in differential geometry that it has a name.

A (*smooth*) *k-dimensional vector bundle* is a pair of smooth manifolds E (the *total space*) and M (the *base*), together with a surjective map $\pi \colon E \to M$ (the *projection*), satisfying the following conditions:

(a) Each set $E_p := \pi^{-1}(p)$ (called the *fiber* of E over p) is endowed with the structure of a vector space.

(b) For each $p \in M$, there exists a neighborhood U of p and a diffeomorphism $\varphi \colon \pi^{-1}(U) \to U \times \mathbf{R}^k$ (Figure 2.2), called a *local trivialization*

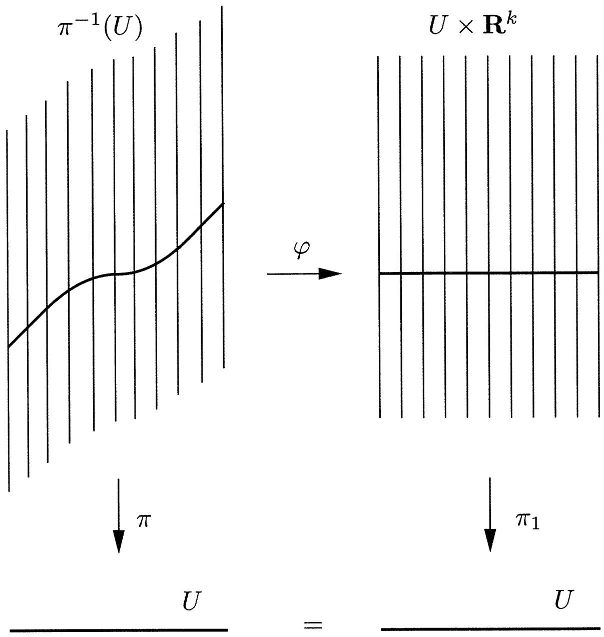

FIGURE 2.2. A local trivialization.

of E, such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbf{R}^k \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ U & = & U \end{array}$$

(where π_1 is the projection onto the first factor).

(c) The restriction of φ to each fiber, $\varphi\colon E_p \to \{p\} \times \mathbf{R}^k$, is a linear isomorphism.

Whether or not you have encountered the formal definition of vector bundles, you have certainly seen at least two examples: the *tangent bundle* TM of a smooth manifold M, which is just the disjoint union of the tangent spaces T_pM for all $p \in M$, and the *cotangent bundle* T^*M, which is the disjoint union of the cotangent spaces $T_p^*M = (T_pM)^*$. Another example that is relatively easy to visualize (and which we formally define in Chapter 8) is the *normal bundle* to a submanifold $M \subset \mathbf{R}^n$, whose fiber at each point is the normal space N_pM, the orthogonal complement of T_pM in $\mathbf{R}^n$.

It frequently happens that we are given a collection of vector spaces, one for each point in a manifold, that we would like to "glue together" to form a vector bundle. For example, this is how the tangent and cotangent bundles are defined. There is a shortcut for showing that such a collection forms a vector bundle without first constructing a smooth manifold structure on the total space. As the next lemma shows, all we need to do is to exhibit the maps that we wish to consider as local trivializations and check that they overlap correctly.

Lemma 2.2. *Let M be a smooth manifold, E a set, and $\pi\colon E \to M$ a surjective map. Suppose we are given an open covering $\{U_\alpha\}$ of M together with bijective maps $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{R}^k$ satisfying $\pi_1 \circ \varphi_\alpha = \pi$, such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the composite map*

$$\varphi_\alpha \circ \varphi_\beta^{-1}\colon U_\alpha \cap U_\beta \times \mathbf{R}^k \to U_\alpha \cap U_\beta \times \mathbf{R}^k$$

is of the form

$$\varphi_\alpha \circ \varphi_\beta^{-1}(p, V) = (p, \tau(p)V) \tag{2.4}$$

for some smooth map $\tau\colon U_\alpha \cap U_\beta \to GL(k, \mathbf{R})$. Then E has a unique structure as a smooth k-dimensional vector bundle over M for which the maps φ_α are local trivializations.

Proof. For each $p \in M$, let $E_p = \pi^{-1}(p)$. If $p \in U_\alpha$, observe that the map $(\varphi_\alpha)_p\colon E_p \to \{p\} \times \mathbf{R}^k$ obtained by restricting φ_α is a bijection. We can define a vector space structure on E_p by declaring this map to be a linear isomorphism. This structure is well defined, since for any other set U_β containing p, (2.4) guarantees that $(\varphi_\alpha)_p \circ (\varphi_\beta)_p^{-1} = \tau(p)$ is an isomorphism.

Shrinking the sets U_α and taking more of them if necessary, we may assume each of them is diffeomorphic to some open set $\widetilde{U}_\alpha \subset \mathbf{R}^n$. Following φ_α with such a diffeomorphism, we get a bijection $\pi^{-1}(U_\alpha) \to \widetilde{U}_\alpha \times \mathbf{R}^k$, which we can use as a coordinate chart for E. Because (2.4) shows that the φ_αs overlap smoothly, these charts determine a locally Euclidean topology and a smooth manifold structure on E. It is immediate that each map φ_α is a diffeomorphism with respect to this smooth structure, and the rest of the conditions for a vector bundle follow automatically. □

The smooth $GL(k, \mathbf{R})$-valued maps τ of the preceding lemma are called *transition functions* for E.

As an illustration, we show how to apply this construction to the tangent bundle. Given a coordinate chart $(U, (x^i))$ for M, any tangent vector $V \in T_xM$ at a point $x \in U$ can be expressed in terms of the coordinate basis as $V = v^i \partial/\partial x^i$ for some n-tuple $v = (v^1, \ldots, v^n)$. Define a bijection $\varphi\colon \pi^{-1}(U) \to U \times \mathbf{R}^n$ by sending $V \in T_xM$ to (x, v). Where two coordinate charts (x^i) and $(\tilde{x}^i)$ overlap, the respective coordinate basis vectors are related by

$$\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j},$$

and therefore the same vector V is represented by

$$V = \tilde{v}^j \frac{\partial}{\partial \tilde{x}^j} = v^i \frac{\partial}{\partial x^i} = v^i \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j}.$$

This means that $\tilde{v}^j = v^i \partial \tilde{x}^j / \partial x^i$, so the corresponding local trivializations φ and $\widetilde{\varphi}$ are related by

$$\widetilde{\varphi} \circ \varphi^{-1}(x, v) = \widetilde{\varphi}(V) = (x, \tilde{v}) = (x, \tau(x) v),$$

where $\tau(x)$ is the $GL(n, \mathbf{R})$-valued function $\partial \tilde{x}^j / \partial x^i$. It is now immediate from Lemma 2.2 that these are the local trivializations for a vector bundle structure on TM.

It is useful to note that this construction actually gives explicit coordinates (x^i, v^i) on $\pi^{-1}(U)$, which we will refer to as *standard coordinates* for the tangent bundle.

If $\pi \colon E \to M$ is a vector bundle over M, a *section* of E is a map $F \colon M \to E$ such that $\pi \circ F = Id_M$, or, equivalently, $F(p) \in E_p$ for all p. It is said to be a *smooth section* if it is smooth as a map between manifolds. The next lemma gives another criterion for smoothness that is more easily verified in practice.

Lemma 2.3. *Let $F \colon M \to E$ be a section of a vector bundle. F is smooth if and only if the components $F_{i_1 \ldots i_k}^{j_1 \ldots j_l}$ of F in terms of any smooth local frame $\{E_i\}$ on an open set $U \in M$ depend smoothly on $p \in U$.*

Exercise 2.4. Prove Lemma 2.3.

The set of smooth sections of a vector bundle is an infinite-dimensional vector space under pointwise addition and multiplication by constants, whose zero element is the *zero section* ζ defined by $\zeta_p = 0 \in E_p$ for all $p \in M$. In this book, we use the script letter corresponding to the name of a vector bundle to denote its space of sections. Thus, for example, the space of smooth sections of TM is denoted $\mathcal{T}(M)$; it is the space of smooth vector fields on M. (Many books use the notation $\mathcal{X}(M)$ for this space, but our notation is more systematic, and seems to be becoming more common.)

Tensor Bundles and Tensor Fields

On a manifold M, we can perform the same linear-algebraic constructions on each tangent space T_pM that we perform on any vector space, yielding tensors at p. For example, a $\binom{k}{l}$-tensor at $p \in M$ is just an element of $T^k_l(T_pM)$. We define the *bundle of $\binom{k}{l}$-tensors* on M as

$$T^k_l M := \coprod_{p \in M} T^k_l(T_pM),$$

where $\coprod$ denotes the disjoint union. Similarly, the *bundle of k-forms* is

$$\Lambda^k M := \coprod_{p \in M} \Lambda^k(T_pM).$$

There are the usual identifications $T_1M = TM$ and $T^1M = \Lambda^1 M = T^*M$.

To see that each of these tensor bundles is a vector bundle, define the projection $\pi\colon T^k_l M \to M$ to be the map that simply sends $F \in T^k_l(T_pM)$ to p. If (x^i) are any local coordinates on $U \subset M$, and $p \in U$, the coordinate vectors $\{\partial_i\}$ form a basis for T_pM whose dual basis is $\{dx^i\}$. Any tensor $F \in T^k_l(T_pM)$ can be expressed in terms of this basis as

$$F = F^{j_1 \dots j_l}_{i_1 \dots i_k}\, \partial_{j_1} \otimes \cdots \otimes \partial_{j_l} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_k}.$$

Exercise 2.5. For any coordinate chart $(U, (x^i))$ on M, define a map φ from $\pi^{-1}(U) \subset T^k_l M$ to $U \times \mathbf{R}^{n^{k+l}}$ by sending a tensor $F \in T^k_l(T_xM)$ to $(x, (F^{j_1 \dots j_l}_{i_1 \dots i_k})) \in U \times \mathbf{R}^{n^{k+l}}$. Show that $T^k_l M$ can be made into a smooth vector bundle in a unique way so that all such maps φ are local trivializations.

A *tensor field* on M is a smooth section of some tensor bundle $T^k_l M$, and a *differential k-form* is a smooth section of $\Lambda^k M$. To avoid confusion between the point $p \in M$ at which a tensor field is evaluated and the vectors and covectors to which it is applied, we usually write the value of a tensor field F at $p \in M$ as $F_p \in T^k_l(T_pM)$, or, if it is clearer (for example if F itself has one or more subscripts), as $F|_p$. The space of $\binom{k}{l}$-tensor fields is denoted by $\mathcal{T}^k_l(M)$, and the space of covariant k-tensor fields (smooth sections of T^kM) by $\mathcal{T}^k(M)$. In particular, $\mathcal{T}^1(M)$ is the space of 1-forms. We follow the common practice of denoting the space of smooth real-valued functions on M (i.e., smooth sections of T^0M) by $C^\infty(M)$.

Let $(E_1, \dots, E_n)$ be any *local frame* for TM, that is, n smooth vector fields defined on some open set U such that $(E_1|_p, \dots, E_n|_p)$ form a basis for T_pM at each point $p \in U$. Associated with such a frame is the *dual coframe,* which we denote $(\varphi^1, \dots, \varphi^n)$; these are smooth 1-forms satisfying $\varphi^i(E_j) = \delta^i_j$. In terms of any local frame, a $\binom{k}{l}$-tensor field F can be written in the form (2.2), where now the components $F^{j_1 \dots j_l}_{i_1 \dots i_k}$ are to be interpreted as functions on U. In particular, in terms of a coordinate frame $\{\partial_i\}$ and its dual coframe $\{dx^i\}$, F has the coordinate expression

$$F_p = F^{j_1 \dots j_l}_{i_1 \dots i_k}(p)\, \partial_{j_1} \otimes \cdots \otimes \partial_{j_l} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_k}. \tag{2.5}$$

Exercise 2.6. Let $F\colon M \to T^k_l M$ be a section. Show that F is a smooth tensor field if and only if whenever $\{X_i\}$ are smooth vector fields and $\{\omega^j\}$ are smooth 1-forms defined on an open set $U \subset M$, the function $F(\omega^1, \dots, \omega^l, X_1, \dots, X_k)$ on U, defined by

$$F(\omega^1, \dots, \omega^l, X_1, \dots, X_k)(p) = F_p(\omega^1_p, \dots, \omega^l_p, X_1|_p, \dots, X_k|_p),$$

is smooth.

An important property of tensor fields is that they are multilinear over the space of smooth functions. Given a tensor field $F \in \mathcal{T}^k_l(M)$, vector fields $X_i \in \mathcal{T}(M)$, and 1-forms $\omega^j \in \mathcal{T}^1(M)$, Exercise 2.6 shows that the function $F(X_1, \ldots, X_k, \omega^1, \ldots, \omega^l)$ is smooth, and thus F induces a map

$$F\colon \mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M) \times \mathcal{T}(M) \times \cdots \times \mathcal{T}(M) \to C^\infty(M).$$

It is easy to check that this map is *multilinear over* $C^\infty(M)$, that is, for any functions $f, g \in C^\infty(M)$ and any smooth vector or covector fields α, β,

$$F(\ldots, f\alpha + g\beta, \ldots) = fF(\ldots, \alpha, \ldots) + gF(\ldots, \beta, \ldots).$$

Even more important is the converse: as the next lemma shows, any such map that is multilinear over $C^\infty(M)$ defines a tensor field.

Lemma 2.4. (Tensor Characterization Lemma) *A map*

$$\tau\colon \mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M) \times \mathcal{T}(M) \times \cdots \times \mathcal{T}(M) \to C^\infty(M)$$

is induced by a $\binom{k}{l}$*-tensor field as above if and only if it is multilinear over* $C^\infty(M)$*. Similarly, a map*

$$\tau\colon \mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M) \times \mathcal{T}(M) \times \cdots \times \mathcal{T}(M) \to \mathcal{T}(M)$$

is induced by a $\binom{k}{l+1}$*-tensor field as in Lemma 2.1 if and only if it is multilinear over* $C^\infty(M)$*.*

Exercise 2.7. Prove Lemma 2.4.