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15

Partial Differential Equations and Boundary-Value Problems with Applications

Third Edition

Mark A. Pinsky

American Mathematical Society

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Preface

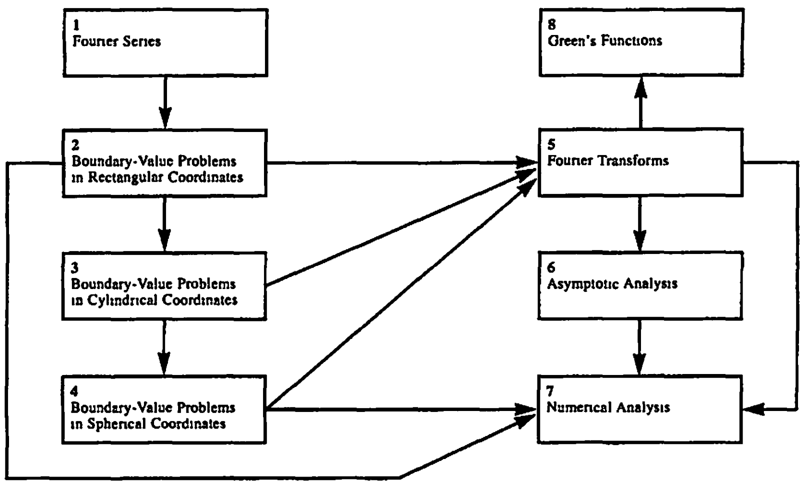
This third edition is an introduction to partial differential equations for students who have finished calculus through ordinary differential equations. The book provides physical motivation, mathematical method, and physical application. Although the first and last are the raison d'être for the mathematics, I have chosen to stress the systematic solution algorithms, based on the methods of separation of variables and Fourier series and integrals. My goal is to achieve a lucid and mathematically correct approach without becoming excessively involved in analysis per se. For example, I have stressed the interpretation of various solutions in terms of asymptotic behavior (for the heat equation) and geometry (for the wave equation).

This new edition builds upon the solid strengths of the previous editions and provides a more patient development of the core concepts. Chapters 0 and 1 have been reorganized and refined to provide more complete examples that will help students master the content. For example, the Sturm-Liouville theory has been rewritten and placed at the end of Chapter 1 just before it is used in Chapter 2. The coverage of infinite series and ordinary differential equations, formerly in Chapter 0, has been moved to appendixes. In addition, we have integrated the applications of Mathematica into the text because computer-assisted methods have become increasingly important in recent years. The previous edition of this text made Mathematica applications available for the first time in a book at this level, and this edition continues this coverage. Each section of the book contains numerous worked examples and a set of exercises. These exercises have been kept to a uniform level of difficulty, and solutions to nearly 450 of the 700 exercises in the text have been provided.

Chapter 0 is a brief introduction to the entire subject of partial differential equations and some technical material that is used frequently throughout the book. Chapters 1 to 4 contain the basic material on Fourier series and boundary-value problems in rectangular, cylindrical, and spherical coordinates. Bessel and Legendre functions are developed in Chapters 3 and 4 for those instructors who want a self-contained development of this material. Instructors who do not wish to use the material on boundary-value problems should cover only Secs. 3.1 and 4.1 in Chapters 3 and 4. These sections contain several interesting boundary-value problems that can be solved without the use of Bessel or Legendre functions.

Chapter 5 develops Fourier transforms and applies them to solve problems in unbounded regions. This material, which may be treated immediately following Chapter 2 if desired, uses real-variable methods. The student is referred to a subsequent course for complex-variable methods.

The student who has finished all the material through Chapter 5 will have a good working knowledge of the classical methods of solution. To complement these basic techniques, I have added chapters on asymptotic analysis (Chapter 6), numerical analysis (Chapter 7), and Green's functions (Chapter 8) for instructors who may have additional time or wish to omit some of the earlier material. The accompanying flowchart plots various paths through the book.



Logical Dependence of Chapters

Chapters 1 and 2 form the heart of the book. They begin with the theory of Fourier series, including a complete discussion of convergence, Parseval's theorem, and the Gibbs phenomenon. We work with the class of piecewise smooth functions, which are infinitely differentiable except at a finite number of points, where all derivatives have left and right limits. Despite the generous dose of theory, it is expected that the student will learn to compute Fourier coefficients and to use Parseval's theorem to estimate the mean square error in approximating a function by the partial sum of its Fourier series. Chapter 1 concludes with Sturm-Liouville theory, which will be used in Chapter 2 and repeatedly throughout the book.

Chapter 2 takes up the systematic study of the wave equation and the heat equation. It begins with steady-state and time-periodic solutions of the heat equation in Sec. 2.1, including applications to heat transfer and to geophysics, and follows with the study of initial-value problems in Secs. 2.2 and 2.3, which are treated by a five-stage method. This systematic breakdown allows the student to separate the steady-state solution from the transient solution (found by the separation-of-variables algorithm) and to verify the uniqueness and asymptotic behavior of the solution as well as to compute the relaxation time. I have found that students can easily appreciate and understand this method, which combines mathematical precision and clear physical interpretation. The five-stage method is used throughout the book, in Secs. 2.5, 3.4, and 4.1. Chapter 2 also includes the wave equation for the vibrating string (Sec. 2.4), solved both by the Fourier series and by the d'Alembert formula. Both methods have advantages and disadvantages, which are discussed in detail. My derivations of both the wave equation and the heat equation are from a three-dimensional viewpoint, which I feel is less artificial and more elegant than many treatments that begin with a one-dimensional formulation.

Following Chapter 2, there is a wide choice in the direction of the course. Those instructors who wish to give a complete treatment of boundary-value problems in cylindrical and spherical coordinates, including Bessel and Legendre functions, will want to cover all of Chapters 3 and 4. Other instructors may ignore this material completely and proceed directly to Chapter 5, on Fourier transforms. An intermediate path might be to cover Secs. 3.1 and/or 4.1, which treat (respectively) Laplace's equation in polar coordinates and spherically symmetric solutions of the heat equation in three dimensions. Neither topic requires any special functions beyond those encountered in trigonometric Fourier series.

Chapter 5 treats Fourier transforms using the complex exponential notation. This is a natural extension of the complex form of the Fourier series, which is covered in Sec. 1.5. Using the Fourier transform, I reduce the heat, Laplace, wave, and telegraph equations to ordinary differential equations with constant coefficients, which can be solved by elementary methods. In many cases, these Fourier representations of the solutions can be rewritten as explicit representations (by what is usually known as the Green function method). The method of images for solving problems on a semi-infinite axis is naturally developed here. The Green functions methods are developed more systematically in Chapter 8. After preparing the one-dimensional case, I give a self-contained treatment of the explicit representation of the solution of Poisson's equation in two and three dimensions. In addition to the traditional physical applications, the Black-Scholes model of option pricing from financial mathematics is included.

Throughout the book I emphasize the asymptotic analysis of *series* solutions of boundary-value problems. Chapter 6 gives an elementary account of asymptotic analysis of *integrals,* in particular the Fourier integral representations of the solutions obtained in Chapter 5. The methods include integration by parts, Laplace's method, and the method of stationary phase. These culminate in an asymptotic analysis of the telegraph equation, which illustrates the group velocity of a wave packet.

No introduction to partial differential equations would be complete without some discussion of approximate solutions and numerical methods. Chapter 7 gives the student some working knowledge of the finite difference solution of the heat equation and Laplace's equation in one and two space dimensions. The material on variational methods first relates differential equations to variational problems and then outlines some direct methods that may be used to arrive at approximate solutions, including the finite element method.

This book was developed from course notes for Mathematics C91-1 in the Integrated Science Program at Northwestern University. The course has been taught to college juniors since 1977; Chapters 1 to 5 are covered in a 10-week quarter. I am indebted to my colleagues Leonard Evens, Robert Speed, Paul Auvil, Gene Birchfield, and Mark Ratner for providing valuable suggestions on the mathematics and its applications. The first draft was written in collaboration with Michael Hopkins. The typing was done by Vicki Davis and Julie Mendelson. The solutions were compiled with the assistance of Mark Scherer. Valuable technical advice was further provided by Edward Reiss and Stuart Antman.

In preparation of this new edition, I received valuable comments and suggestions from Andrew Bernoff, Joseph B. Keller, Thaddeus Ladd, Jeff Miller, Carl Prather, Robert Seeley, and Marshall Slemrod. I also acknowledge the reviewing services of the following individuals: David Bao, University of Houston; William O. Bray, University of Maine; Peter Colwell, Iowa State University; Kenneth A. Heimes, Iowa State University; Yinxi Huang, University of Memphis; Mohammad Kozemi, University of North Carolina–Charlotte; and William Mays, Gloucester Community College (NJ).

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In the preparation of the first edition I was encouraged by John Corrigan of the McGraw-Hill College Division. Preparing the second edition of this text, I benefited from the editorial services of Karen M. Hughes and Richard Wallis. Most recently, for this new edition with Waveland Press, it has been a pleasure to work with Jan Fisher and the staff of Publication Services. The current printing was completed with the editorial assistance of Miron Bekker, Harry R. Hughes, Monica Sharpnack, Nancy Stanton, and Alphonse Sterling.

Mark A. Pinsky

Contents

CHAPTER 0

PRELIMINARIES

INTRODUCTION

This chapter serves as an overview, with some motivation of the origins of partial differential equations and some of the mathematical methods that will be used repeatedly throughout the book. In particular, the technique of *separation of variables* is introduced in Sec. 0.2, and the concept of *orthogonal functions* is introduced in Sec. 0.3 and illustrated through relevant examples. Previous work in vector calculus, infinite series, and ordinary differential equations is reviewed in the appendixes.

0.1. Partial Differential Equations

In this section we introduce the notion of a partial differential equation and illustrate it with various examples.

0.1.1. What is a partial differential equation? From the purely mathematical point of view, a partial differential equation (PDE) is an equation that relates a function u of several variables $x_1, \ldots, x_n$ and its partial derivatives. This is distinguished from an *ordinary differential equation*, which pertains to functions of *one variable*. For example, if a function of two variables is denoted $u(x, y)$, then one may consider the following as examples of partial differential equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace's equation)}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(the wave equation)}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0 \qquad \text{(the heat equation)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x, y) \qquad \text{(Poisson's equation)}$$

In order to simplify the notation, we will often use subscripts to denote the various partial derivatives, so that $u_x = \partial u/\partial x$, $u_{xx} = \partial^2 u/\partial x^2$, and so forth. In this notation, the above four examples are written, respectively,

$$u_{xx} + u_{yy} = 0, \qquad u_{xx} - u_{yy} = 0, \qquad u_{xx} - u_y = 0, \qquad u_{xx} + u_{yy} = g$$

The *order* of a PDE is indicated by the highest-order derivative that appears. All of the above four examples are PDEs of second order.

In the case of a function of several variables $u(x_1, \ldots, x_n)$, the most general second-order partial differential equation can be written

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_1}, u_{x_1 x_2}, \ldots, u_{x_n x_n}) = 0$$

where the dots imply the other partial derivatives that may occur. In case $n = 1$ we obtain the second-order ordinary differential equation $F(x, u, u', u'') = 0$. The necessary information on ordinary differential equations is reviewed in Appendix A.1.

Another important concept pertaining to a PDE is that of *linearity*. This is most easily described in the context of a differential operator $\mathcal{L}$ applied to a function u. Examples of differential operators are $\mathcal{L}u = \partial u/\partial x$, $\mathcal{L}u = 3u + \sin y\, \partial u/\partial x$, and $\mathcal{L}u = u\, \partial^2 u/\partial x^2$. The operator is said to be *linear* if for any two functions u, v and any constant c,

$$\mathcal{L}(u + v) = \mathcal{L}u + \mathcal{L}v, \qquad \mathcal{L}(cu) = c\mathcal{L}u$$

A PDE is said to be *linear* if it can be written in the form

$$\mathcal{L}u = g \tag{0.1.1}$$

where $\mathcal{L}$ is a linear differential operator and g is a given function. In case $g = 0$, (0.1.1) is said to be *homogeneous*. For example, three of the above examples (Laplace's equation, the wave equation, and the heat equation) are linear homogeneous PDEs. The most general linear second-order PDE in two variables is written

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = g(x, y)$$

where the functions a, b, c, d, e, f, g are given.

EXERCISES 0.1.1

1. Write down the most general linear first-order PDE in two variables. How many given functions are necessary to specify the PDE?
2. Write down the most general linear first-order PDE in three variables. How many given functions are necessary to specify the PDE?
3. Write down the most general linear first-order homogeneous PDE in two variables. How many given functions are necessary to specify the PDE?
4. Write down the most general linear first-order homogeneous PDE in three variables. How many given functions are necessary to specify the PDE?

5. Define the operator $\mathcal{L}$ by the formula $\mathcal{L}u(x,y) = d(x,y)u_x + e(x,y)u_y + f(x,y)u$. Show that $\mathcal{L}$ is a linear differential operator.
6. Define the operator $\mathcal{L}$ by the formula $\mathcal{L}u(x,y) = a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy}$. Show that $\mathcal{L}$ is a linear differential operator.
7. Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear differential operators. Show that $\mathcal{L}_1+\mathcal{L}_2$ is also a linear differential operator.

0.1.2. Superposition principle and subtraction principle. In the study of ordinary differential equations, it is often possible to write the general solution in a closed form, in terms of arbitrary constants and a set of particular solutions. This is not possible in the case of partial differential equations. To see this in more detail, we cite the example of the second-order equation $u_{xx} = 0$ for the unknown function $u(x,y)$. Integrating once reveals that $u_x(x,y) = C(y)$, while a second integration reveals that $u(x,y) = xC(y) + D(y)$, where C and D are *arbitrary functions.* Clearly, there are infinitely many different choices for each of C and D, so that this solution cannot be specified in terms of a finite number of arbitrary constants. Stated otherwise, the space of solutions is infinite-dimensional.

In order to work effectively with a linear PDE, we must develop rules for combining known solutions. The following principle is basic to all of our future work.

PROPOSITION 0.1.1. **(Superposition principle for homogeneous equations).** *If $u_1, \ldots, u_N$ are solutions of the same linear homogeneous PDE $\mathcal{L}u = 0$, and $c_1, \ldots, c_N$ are constants (real or complex), then $c_1u_1 + \cdots + c_Nu_N$ is also a solution of the PDE.*

Proof. The proof of this depends on the property of linearity. Indeed, we have $\mathcal{L}(u_i) = 0$ for $i = 1, \ldots, n$. Hence

$$\mathcal{L}(c_1u_1 + \cdots + c_Nu_N) = c_1\mathcal{L}(u_1) + \cdots + c_N\mathcal{L}(u_N) = 0 \quad \bullet$$

For example, one may verify that for any constant k, the function $u(x,y) = e^{kx}\cos ky$ is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$. Therefore, by the superposition principle, the function $u(x,y) = e^{-x}\cos y + 2e^{-3x}\cos 3y - 5e^{-\pi x}\cos \pi y$ is also a solution of Laplace's equation.

The superposition principle does not apply to nonhomogeneous equations. For example, if u_1 and u_2 are solutions of the Poisson equation $u_{xx} + u_{yy} = 1$, then the function $u_1 + u_2$ is the solution of a different equation, namely, $u_{xx} + u_{yy} = 2$. Nevertheless, we have the following important general principle that allows one to relate nonhomogeneous equations to homogeneous equations.

PROPOSITION 0.1.2. **(Subtraction principle for nonhomogeneous equations).** *If u_1 and u_2 are solutions of the same linear nonhomogeneous equation $\mathcal{L}u = g$, then the function $u_1 - u_2$ is a solution of the associated homogeneous equation $\mathcal{L}u = 0$.*

Proof. We have

$$\mathcal{L}(u_1 - u_2) = \mathcal{L}u_1 - \mathcal{L}u_2 = 0 \quad \bullet$$

For example, if u_1 and u_2 are both solutions of the Poisson equation $u_{xx}+u_{yy} = 1$, then $u_1 - u_2$ is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.

The subtraction principle allows us to find the general solution of a nonhomogeneous equation $\mathcal{L}u = g$ once we know a particular solution of the equation and the general solution of the related homogeneous equation $\mathcal{L}u = 0$. The result is expressed as follows.

Corollary. *The general solution of the linear partial differential equation $\mathcal{L}u = g$ can be written in the form*

$$u = U + v$$

where U is a particular solution of the equation $\mathcal{L}U = g$ and v is the general solution of the related homogeneous equation $\mathcal{L}v = 0$.

We illustrate with an example.

EXAMPLE 0.1.1. *Find the general solution $u(x,y)$ of the equation $u_{xx} = 2$.*

Solution. It is immediately verified that the function $u = x^2$ is a solution of the given equation. The general solution of the associated homogeneous equation $u_{xx} = 0$ is $u(x,y) = xg(y) + h(y)$. Therefore the general solution of the nonhomogeneous equation is $u(x,y) = x^2 + xg(y) + h(y)$. $\bullet$

EXERCISES 0.1.2

1. Show that for any constant k, the function $u(x,y) = e^{kx}\cos ky$ is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.
2. Show that for any constant k, the function $u(x,y) = e^{kx}e^{k^2 y}$ is a solution of the heat equation $u_{xx} - u_y = 0$.
3. Show that for any constant k, the function $u(x,y) = e^{kx}e^{-ky}$ is a solution of the wave equation $u_{xx} - u_{yy} = 0$.
4. Show that for any constant k, the function $u(x,y) = (k/2)x^2 + (1-k)y^2/2$ is a solution of Poisson's equation $u_{xx} + u_{yy} = 1$.

0.1.3. Sources of PDEs in classical physics. Many laws of physics are expressed mathematically as differential equations. The student of elementary mechanics is familiar with Newton's second law of motion, which expresses the acceleration of a system in terms of the forces on the system. In the case of one or more point particles, this translates into a system of ordinary differential equations when the force law is known.

For example, a single spring with Hooke's law of elastic restoration and no frictional forces gives rise to the linear equation of the harmonic oscillator, which is well studied in elementary courses. A system of particles that interact through several springs gives rise to a second-order system of differential equations, which may be resolved into its normal modes— each of which undergoes simple harmonic motion. Newton's law of gravitational attraction gives rise to a more complicated system of nonlinear ordinary differential equations. Generally speaking, whenever we have a finite number of point particles, the mathematical model is a system of ordinary differential equations, where *time* plays the role of independent variable and the positions/velocities of the particles are the dependent variables. In Chapter 2, we will give the complete derivation of the wave equation, which governs the motion of a tightly streched vibrating string.

For time-dependent systems in one spatial dimension, we will use the notation $u(x;t)$ to denote the unknown function that is a solution of the PDE. In the case of two or three spatial dimensions we will use the repective notations $u(x,y;t)$ and $u(x,y,z;t)$ to denote the solution of the PDE.

In the following subsection we will give a simplified derivation of the one-dimensional heat equation. The complete derivation of the heat equation as it applies to three-dimensional systems is found in Chapter 2.

0.1.4. The one-dimensional heat equation. Consider a one-dimensional rod that is capable of conducting heat, and for which we can measure the temperature $u(x;t)$ at the position x at time instant t. We assume that this function has continuous partial derivatives of orders 1 and 2. In order to motivate the discussion, we first consider a finite system of equally spaced points $x_1 < x_2 < \cdots < x_N$. We expect that the temperature will remain constant as a function of time if there is a *local equilibrium*, meaning that the temperature $u(x_i;t)$ is equal to the average of its neighbors; in symbols,

$$\frac{\partial u}{\partial t}(x_i;t) = 0 \qquad \text{if} \qquad u(x_i;t) = \frac{1}{2}u(x_{i-1};t) + \frac{1}{2}u(x_{i+1};t)$$

For example, if the point x_i is at 50 degrees and the neighbor to the left is at 40 degrees while the neighbor to the right is at 60 degrees, then we expect no change in temperature.

On the other hand, if this condition of local equilibrium is not satisfied, then we may expect that the temperature will change, in relation to the amount of disequilibrium. Certainly one expects the temperature to increase if both neighbors are warmer, but also if the average is warmer; for example, if the point x_i is at 50 degrees while the left neighbor is at 45 degrees and the right neighbor is at 65 degrees, then the average is 55 degrees—5 degrees warmer than the home temperature.

In order to quantify this, we postulate the following dynamical law.

The time rate of change of temperature at the point x_i is proportional to the difference between the temperature at x_i and the average of the temperatures at the two neighboring points x_{i-1}, x_{i+1}

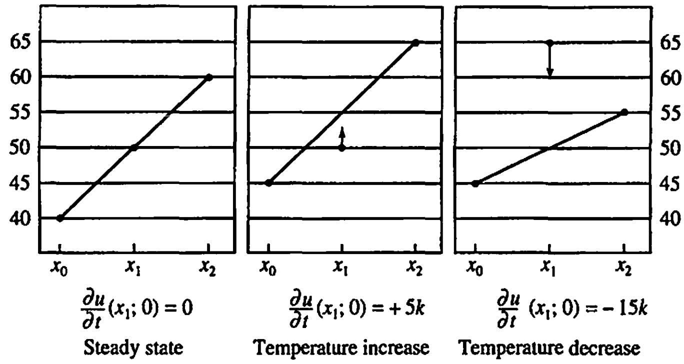


FIGURE 0.1.1 Three different configurations of heat flow dynamics

To translate this into a mathematical statement, we must introduce a constant of proportionality k, which will depend on the properties of the medium. If we have a "good conductor," then k will be large, whereas if we have a "bad conductor," then k will be small. The desired mathematical statement then reads

(0.1.2)

$$\boxed{\frac{\partial u}{\partial t}(x_i;t) = k\left(\frac{1}{2}[u(x_{i+1};t) + u(x_{i-1};t)] - u(x_i;t)\right), \qquad i = 2,\ldots,N-1}$$

Figure 0.1.1 presents three different configurations of heat flow dynamics, corresponding to local equilibrium (also called *steady state*), temperature increase, and temperature decrease.

The above mathematical model of heat flow can be expected to be rigorously valid for a finite system of equally spaced points $x_1 < x_2 < \cdots < x_N$. Equation (0.1.2) is a system of ordinary differential equations that can be solved by algebraic methods, if necessary. If we now consider these points as an approximation to a continuum of points, then we can expect this model to be valid as a first approximation when the spacing tends to zero. In order to obtain a partial differential equation we apply Taylor's theorem with remainder:

$$u(x_{i+1};t) - u(x_i;t) = (x_{i+1} - x_i)u_x(x_i;t) + \frac{1}{2}(x_{i+1} - x_i)^2 u_{xx}(x_i';t)$$
$$u(x_{i-1};t) - u(x_i;t) = (x_{i-1} - x_i)u_x(x_i;t) + \frac{1}{2}(x_{i-1} - x_i)^2 u_{xx}(x_i'';t)$$

where the points x_i', x_i'' satisfy $x_{i-1} \le x_i'' \le x_i \le x_i' \le x_{i+1}$. Recalling that the points are equally spaced, let $\Delta x = x_{i+1} - x_i$ be the common spacing, and substitute into (0.1.2) to obtain

$$\frac{\partial u}{\partial t}(x_i;t) = \frac{k(\Delta x)^2}{4}\left(u_{xx}(x_i';t) + u_{xx}(x_i'';t)\right) \tag{0.1.3}$$

The final simplification is to assert that, if the spacing is very small, then the values of the second partial derivative will vary very little from the nearby points x_i, x_i', x_i'', and thus we can replace the two values of the second partial derivatives by the value at the point x_i. Defining $K = k(\Delta x)^2/2$, we obtain the heat equation

$$\boxed{\frac{\partial u}{\partial t} = K u_{xx}} \tag{0.1.4}$$

The constant K is called the *diffusivity.*

With no further information, the heat equation (0.1.4) will have infinitely many solutions. In order to specify a solution of the heat equation, we consider various boundary conditions and initial conditions. Assuming that the rod occupies the interval $0 < x < L$ of the x-axis, we consider three types of boundary conditions at the endpoint $x = 0$:

$$\begin{aligned} \mathrm{I} &: u(0;t) = T_0 \\ \mathrm{II} &: u_x(0;t) = 0 \\ \mathrm{III} &: -u_x(0;t) = h(T_e - u(0;t)) \qquad \text{where } h > 0 \end{aligned}$$

Boundary condition I signifies that the temperature at the end $x = 0$ is held constant. In practice this could occur as the result of heating the end by means of some device. Boundary condition II signifies that there is no heat flow at the end $x = 0$. In practice this could occur by means of *insulation*, which prohibits the flow of heat at this end. Boundary condition III is sometimes called *Newton's law of cooling*: the negative of the partial derivative is interpreted as the *heat flux*, i.e., the rate of heat flow *out* of the end $x = 0$, and is required to be proportional to the difference between the outside temperature T_e and the endpoint temperature $u(0;t)$. If this difference is large, then we may expect heat to flow out of the rod at a rapid rate. If T_e is less than the endpoint temperature, then $u(0;t) > T_e$ and the rate will be negative, so that we may expect heat to flow *into* the rod from the exterior. The concept of flux will be discussed in more detail in Chapter 2, when we derive the three-dimensional heat equation.

Similarly, we can have each of the three boundary conditions present at the end $x = L$; in detail,

$$\begin{aligned} \mathrm{I} &: u(L;t) = T_0 \\ \mathrm{II} &: u_x(L;t) = 0 \\ \mathrm{III} &: u_x(L;t) = h(T_e - u(L;t)) \qquad \text{where } h > 0 \end{aligned}$$

The constants T_0, h, and T_e may be the same as for the endpoint $x = 0$ or may have different values. The interpretations are exactly the same as for the endpoint

$x = 0$, with one small exception: in the third boundary condition (III), the heat flux at the end $x = L$ is written without the minus sign, since this measures the rate of heat flow *out of* the end $x = L$. As before, we expect that if the external temperature T_e is much greater than the endpoint temperature $u(L;t)$, then the rate of heat flow out of the end will be large, whereas if the external temperature is less than the endpoint temperature, then the heat flow out will be negative.

A typical boundary-value problem for the heat equation will have one boundary condition for each end $x = 0$ and $x = L$. Considering all possible cases, we have nine different combinations, of which we list the first three below:

$$\begin{aligned} u(0;t) &= T_0, \quad u(L;t) = T_L \\ u(0;t) &= T_0, \quad u_x(L;t) = 0 \\ u(0;t) &= T_0, \quad u_x(L;t) = h(T_e - u(L;t)) \end{aligned}$$

The final piece of information used to specify the solution is the *inital data.* This is simply written

$$u(x;0) = f(x), \qquad 0 < x < L$$

This signifies that the temperature is known at time $t = 0$ and is given by the function $f(x), 0 < x < L$. Note that we do not insist that this agree with the values of the solution at the endpoints $x = 0$, $x = L$. Specification of boundary conditions and initial conditions is known as the *initial-boundary-value problem.* In Chapter 2 we will make a detailed study of this for the one-dimensional heat equation.

In the remainder of this subsection we will determine the *steady-state* solutions of the heat equation corresponding to the various boundary conditions. u is said to be a steady-state solution if $\partial u/\partial t = 0$. Referring to the heat equation (0.1.4), this is equivalent to the statement that $u_{xx} = 0$.

EXAMPLE 0.1.2. *Find the steady-state solution of the heat equation with the boundary conditions* $u(0;t) = T_1$, $u(L;t) = T_2$.

Solution. Since the solution is independent of time, we can write $u = U(x)$, with $U''(x) = 0$. The general solution of this is a linear function: $U(x) = Ax + B$. The boundary condition at $x = 0$ gives $B = T_1$, whereas the boundary condition at $x = L$ gives $AL + B = T_2$, $A = (T_2 - T_1)/L$. The steady-state solution is

$$U(x) = T_1 + \frac{T_2 - T_1}{L}x = T_2\frac{x}{L} + T_1\left(1 - \frac{x}{L}\right) \quad \bullet$$

EXAMPLE 0.1.3. *Find the steady-state solution of the heat equation with the boundary conditions* $u(0;t) = T_1$, $u_x(L;t) = h(T_e - u(L;t))$.

Solution. Since the solution is independent of time, we can write $u = U(x)$, with $U''(x) = 0$. The general solution of this is a linear function: $U(x) = Ax + B$. The boundary condition at $x = 0$ gives $B = T_1$, whereas the boundary condition at $x = L$ gives $A = h(T_e - AL - B)$, $A = -h(T_1 - T_e)/(1+hL)$ and the steady-state solution

$$U(x) = T_1 - \frac{hx}{1+hL}(T_1 - T_e) = T_e\frac{hx}{1+hL} + T_1\frac{1+h(L-x)}{1+hL} \quad \bullet$$

EXERCISES 0.1.4

1. Find the steady-state solution of the heat equation with the boundary conditions $u(0;t) = T_1$, $u_x(L;t) = 0$.
2. Find the steady-state solution of the heat equation with the boundary conditions $u_x(0;t) = h(T_0 - u(0;t)), u_x(L;t) = \Phi$, where h, T_0, Φ are positive constants.
3. Find the steady-state solution of the heat equation with the boundary conditions $-u_x(0;t) = h(T_0 - u(0;t)), u_x(L;t) = h(T_1 - u(L;t))$ where T_0, T_1, h are constants with $h > 0$.

0.1.5. Classification of second-order PDEs. It is impossible to formulate a general existence theorem that applies to all linear partial differential equations, even if we restrict attention to the important case of second-order equations. Instead, it is more natural to specify a solution through a set of boundary conditions or initial conditions related to the equation. For example, the solution of the heat equation $u_t = Ku_{xx}$ in the region $0 < x < L$, $0 < t < \infty$ may be specified uniquely in terms of the initial conditions at $t = 0$ and the boundary conditions at $x = 0$ and $x = L$. On the other hand, the solution of the wave equation $u_{tt} - c^2u_{xx} = 0$ in the region $0 < x < L$, $0 < t < \infty$ is uniquely obtained in terms of the boundary conditions at $x = 0$, $x = L$ and *two* initial conditions, pertaining to the solution $u(x;0)$ and its time derivative $\partial u/\partial t(x;0)$. In order to put this in a more general context, one may classify the second-order linear partial differential equation as follows:

(0.1.5)

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y)$$

If $4ac - b^2 > 0$, the PDE (0.1.5) is called *elliptic.*

If $4ac - b^2 = 0$, the PDE (0.1.5) is called *parabolic.*

If $4ac - b^2 < 0$, the PDE (0.1.5) is called *hyperbolic.*

For example, Laplace's equation and Poisson's equation are both elliptic, while the wave equation is hyperbolic. The heat equation is parabolic. General theorems about these classes of equations are stated and proved in more advanced texts and reference books. Here we indicate the types of boundary conditions that are natural for each of the three types of equations.

If the equation is elliptic, we may solve the *Dirichlet problem*, namely, in a region D to find a solution of $\mathcal{L}u = g$ that further satisfies the boundary condition that $u = \phi(x, y)$ on the boundary of D. For example, the physical problem of determining the electrostatic potential function $u(x, y)$ in the interior of the cylindrical region $x^2 + y^2 < R^2$ when the charge density $\rho(x, y)$ is specified and the boundary is required to be an equipotential surface leads to the elliptic boundary-value problem

$$\begin{aligned} u_{xx} + u_{yy} &= -\rho(x, y) & \qquad x^2 + y^2 < R^2 \\ u(x, y) &= C & \qquad x^2 + y^2 = R^2 \end{aligned}$$

If the equation is parabolic or hyperbolic, it is natural to solve the *Cauchy problem*, which amounts to specifying the solution and its time derivative on the line $t = 0$ as well as specifying the relevant boundary conditions. Here we indicate the Cauchy problem for the equation of the vibrating string, which will be derived in complete detail in Chapter 2:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 & \qquad t > 0,\ 0 < x < L \\ u(x; 0) &= f_1(x) & \qquad 0 < x < L \\ u_t(x; 0) &= f_2(x) & \qquad 0 < x < L \\ u(0; t) &= 0,\ u(L; t) = 0 & \qquad t > 0 \end{aligned}$$

The initial conditions f_1, f_2 represent the initial position and velocity of the vibrating string. The boundary conditions at $x = 0$ and $x = L$ signify that ends of the string are fixed for all time at the position $u = 0$.

EXERCISES 0.1.5

Classify each of the following second-order equations as elliptic, parabolic, or hyperbolic.

1. $u_{xx} + 3u_{xy} + u_{yy} + 2u_x - u_y = 0$
2. $u_{xx} + 3u_{xy} + 8u_{yy} + 2u_x - u_y = 0$
3. $u_{xx} - 2u_{xy} + u_{yy} + 2u_x - u_y = 0$
4. $u_{xx} + xu_{yy} = 0$

0.2. Separation of Variables

0.2.1. What is a separated solution? A fundamental technique for obtaining solutions of linear partial differential equations is the method of *separation of variables.* This means that we look for particular solutions in the form $u(x, y) = X(x)Y(y)$ and try to obtain ordinary differential equations for $X(x)$ and $Y(y)$. These equations will contain a parameter called the *separation constant.* The function $u(x, y)$ is called a *separated solution.* Then we can use the superposition principle to obtain more general solutions of a linear homogeneous PDE as sums of separated solutions.

0.2.2. Separated solutions of Laplace's equation. It is especially simple to obtain separated solutions for Laplace's equation, $u_{xx} + u_{yy} = 0$.

If we let $u(x,y) = X(x)Y(y)$ and substitute in Laplace's equation, we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Dividing by $X(x)Y(y)$ (assumed to be nonzero), we obtain

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

The first term depends only on x, while the second term depends only on y. The sum can equal zero only if both terms are constants that sum to zero. In order to express this in terms of a single parameter, we introduce the constant λ and obtain the system of two ordinary differential equations

$$\frac{X''(x)}{X(x)} = \lambda, \qquad \frac{Y''(y)}{Y(y)} = -\lambda$$

λ is the *separation constant.* These equations may be written in the more standard form

$$X''(x) - \lambda X(x) = 0 \tag{0.2.1}$$

$$Y''(y) + \lambda Y(y) = 0 \tag{0.2.2}$$

Both of these are second-order homogeneous linear ordinary differential equations, which may be solved in terms of exponential functions, trigonometric functions, or linear functions, depending on the sign of λ.[1] To proceed further, we consider separately the three cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

Case 1. If $\lambda > 0$, we write $\lambda = k^2$, where $k > 0$. The general solutions to (0.2.1) and (0.2.2) are

$$\begin{aligned} X(x) &= A_1 e^{kx} + A_2 e^{-kx} \\ Y(y) &= A_3 \cos ky + A_4 \sin ky \end{aligned}$$

where A_1, A_2, A_3, A_4 are arbitrary constants. These cannot be determined until we have imposed further conditions, which will be done later.

Case 2. If $\lambda = 0$, we have the equations $X'' = 0$, $Y'' = 0$, for which the general solutions to (0.2.1) and (0.2.2) are linear functions:

$$\begin{aligned} X(x) &= A_1 x + A_2 \\ Y(y) &= A_3 y + A_4 \end{aligned}$$

where A_1, A_2, A_3, A_4 are arbitrary constants.

[1]For a review of ordinary differential equations, consult Appendix A.1.

Case 3. If $\lambda < 0$, we write $\lambda = -l^2$, where $l > 0$; the general solutions of (0.2.1) and (0.2.2) are

$$\begin{aligned} X(x) &= A_1 \cos lx + A_2 \sin lx \\ Y(y) &= A_3 e^{ly} + A_4 e^{-ly} \end{aligned}$$

To summarize, we have found the following separated solutions of Laplace's equation:

$$u(x,y) = \begin{cases} (A_1 e^{kx} + A_2 e^{-kx})(A_3 \cos ky + A_4 \sin ky) & k > 0 \\ (A_1 x + A_2)(A_3 y + A_4) & \\ (A_1 \cos lx + A_2 \sin lx)(A_3 e^{ly} + A_4 e^{-ly}) & l > 0 \end{cases}$$

We can also write the separated solutions of Laplace's equation in terms of *hyperbolic functions*. These are defined by the formulas

$$\sinh a = \frac{1}{2}(e^a - e^{-a}), \qquad \cosh a = \frac{1}{2}(e^a + e^{-a})$$

From this it follows immediately that

$$e^a = \cosh a + \sinh a, \qquad e^{-a} = \cosh a - \sinh a$$

Using this notation, we can write the separated solutions of Laplace's equation in the equivalent form

$$u(x,y) = \begin{cases} (A_1 \sinh kx + A_2 \cosh kx)(A_3 \cos ky + A_4 \sin ky) & k > 0 \\ (A_1 x + A_2)(A_3 y + A_4) & \\ (A_1 \cos lx + A_2 \sin lx)(A_3 \sinh ly + A_4 \cosh ly) & l > 0 \end{cases}$$

We emphasize that the constants A_1, A_2, A_3, A_4 will change when we make this change of notation. But the *form* of the solution remains unchanged; put otherwise, the classes of separated solutions defined by the two sets of notations are identical.

To derive these, we assumed that $u(x,y) \neq 0$. Having now obtained the explicit forms, we can verify independently that in each case $u(x,y)$ satisfies Laplace's equation.

EXAMPLE 0.2.1. *Verify that the preceding separated solutions satisfy Laplace's equation.*

Solution. In case $\lambda > 0$, we have

$$u(x,y) = (A_1 e^{kx} + A_2 e^{-kx})(A_3 \cos ky + A_4 \sin ky)$$

so that

$$\begin{aligned} u_x &= (kA_1 e^{kx} - kA_2 e^{-kx})(A_3 \cos ky + A_4 \sin ky) \\ u_{xx} &= (k^2 A_1 e^{kx} + k^2 A_2 e^{-kx})(A_3 \cos ky + A_4 \sin ky) \\ u_y &= (A_1 e^{kx} + A_2 e^{-kx})(-kA_3 \sin ky + kA_4 \cos ky) \\ u_{yy} &= (A_1 e^{kx} + A_2 e^{-kx})(-k^2 A_3 \cos ky - k^2 A_4 \sin ky) \end{aligned}$$

The second and fourth terms are negatives of one another. Therefore $u_{xx} + u_{yy} = 0$, and we have verified Laplace's equation in case $\lambda > 0$.

In case $\lambda = 0$ we have

$$\begin{aligned} u_x &= A_1(A_3y + A_4), \qquad & u_{xx} = 0 \\ u_y &= (A_1x + A_2)A_3, \qquad & u_{yy} = 0 \end{aligned}$$

so that both of the partial derivatives u_{xx} and u_{yy} are zero and Laplace's equation is immediate in this case. The verification for $\lambda < 0$ is left to the exercises. •

EXERCISES 0.2.2

1. Verify that $u(x, y) = (A_1 \cos lx + A_2 \sin lx)(A_3e^{ly} + A_4e^{-ly})$ satisfies Laplace's equation, for any $l > 0$.
2. Suppose that $u(x, y)$ is a solution of Laplace's equation. If θ is a fixed real number, define the function $v(x, y) = u(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$. Show that $v(x, y)$ is a solution of Laplace's equation.
3. Apply the result of the previous exercise to the separated solutions of Laplace's equation of the form $u(x, y) = (A_1e^{kx} + A_2e^{-kx})(A_3 \cos ky + A_4 \sin ky)$, to obtain additional solutions of Laplace's equation. Are these new solutions separated?
4. From the definitions of the hyperbolic functions, prove the following properties:
   (a) $\sinh 0 = 0$, $\cosh 0 = 1$
   (b) $(d/dx)(\sinh x) = \cosh x$, $(d/dx)(\cosh x) = \sinh x$
   (c) $\cosh x \geq 1$ for all x
   (d) $\cosh x \geq \sinh x$ for all x
   (e) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
   (f) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

0.2.3. Real and complex separated solutions. In the previous subsection we found all of the separated solutions of Laplace's equation, in terms of trigonometric functions, exponential functions, and linear functions using a real separation constant.

In looking for separated solutions of a PDE, it is often convenient to allow the functions $X(x)$ and $Y(y)$ to be complex-valued, corresponding to a complex separation constant. The following proposition shows that the real and imaginary parts of *any* complex-valued solution will again satisfy the PDE.

PROPOSITION 0.2.1. *Let $u(x, y) = v_1(x, y) + iv_2(x, y)$ be a complex-valued solution of the linear PDE*

$$\mathcal{L}u = au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$$

where a, b, c, d, e, f, g are real-valued functions of (x, y). Then $v_1(x, y) = \operatorname{Re} u(x, y)$ satisfies the PDE $\mathcal{L}u = g$, and $v_2(x, y) = \operatorname{Im} u(x, y)$ satisfies the associated homogeneous PDE $\mathcal{L}u = 0$.

Proof. The operation of partial differentiation is linear; thus

$$\begin{aligned} u_x &= (v_1)_x + i(v_2)_x \\ u_{xx} &= (v_1)_{xx} + i(v_2)_{xx} \end{aligned}$$

with similar expressions for u_y, u_{yy}, and u_{xy}. Substituting these into the partial differential equation and separating the real and imaginary parts yields the result. •

We illustrate this technique with the example of Laplace's equation. Letting $u(x,y) = X(x)Y(y)$, consider a purely imaginary separation constant in the form $\lambda = 2ik^2$, where $k > 0$. This leads to the two ordinary differential equations

$$X''(x) - 2ik^2X(x) = 0 \tag{0.2.3}$$

$$Y''(y) + 2ik^2Y(y) = 0 \tag{0.2.4}$$

These can be solved in terms of the complex exponential function, using the observation that $[k(1+i)]^2 = 2ik^2$, $[k(1-i)]^2 = -2ik^2$. Thus

$$X(x) = A_1e^{k(1+i)x} + A_2e^{-k(1+i)x}, \qquad Y(y) = A_3e^{k(1-i)y} + A_4e^{-k(1-i)y}$$

Multiplying these, we obtain the complex separated solutions

$$u(x,y) = \begin{cases} e^{k(x+y)}e^{ik(x-y)} \\ e^{k(x-y)}e^{ik(x+y)} \\ e^{k(y-x)}e^{-ik(x+y)} \\ e^{-k(x+y)}e^{ik(y-x)} \end{cases}$$

When we take the real and imaginary parts, we obtain the following real-valued solutions of Laplace's equation:

$$u(x,y) = \begin{cases} e^{k(x+y)}\cos k(x-y), & e^{k(x+y)}\sin k(x-y) \\ e^{k(x-y)}\cos k(x+y), & e^{k(x-y)}\sin k(x+y) \\ e^{k(y-x)}\cos k(x+y), & e^{k(y-x)}\sin k(x+y) \\ e^{-k(x+y)}\cos k(y-x), & e^{-k(x+y)}\sin k(y-x) \end{cases}$$

When we consider more general linear PDEs, complex-valued separated solutions may always be found if the functions a, b, c, d, e, f that occur in the equation are independent of (x,y); in this case we speak of a *PDE with constant coefficients*, whose solutions may be written as exponential functions.

PROPOSITION 0.2.2. *Consider the linear homogeneous PDE*

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0$$

Suppose that a, b, c, d, e, f are real constants. Then there exist complex separated solutions of the form

$$u(x,y) = e^{\alpha x}e^{\beta y}$$

for appropriate choices of the complex numbers α, β.

Proof. We first note that the ordinary rules for differentiating $e^{\alpha x}$ are valid for complex-valued functions. For example, if $\alpha = a + ib$,

$$\begin{aligned}
\frac{d}{dx}(e^{\alpha x}) &= \frac{d}{dx}\left[e^{ax}(\cos bx + i\sin bx)\right] \\
&= ae^{ax}\cos bx - be^{ax}\sin bx \\
&\quad + i(ae^{ax}\sin bx + be^{ax}\cos bx) \\
&= e^{ax}(a+ib)(\cos bx + i\sin bx) \\
&= (a+ib)e^{(a+ib)x} \\
&= \alpha e^{\alpha x}
\end{aligned}$$

Similarly, $(d^2/dx^2)(e^{\alpha x}) = \alpha^2 e^{\alpha x}$, with similar expressions for (d/dy) and (d^2/dy^2). Applying this to $u(x,y) = e^{\alpha x}e^{\beta y}$, we have $u_x = \alpha u$, $u_{xx} = \alpha^2 u$, $u_y = \beta u$, $u_{yy} = \beta^2 u$, $u_{xy} = \alpha\beta u$. Substituting these into the PDE, we must have

$$(a\alpha^2 + b\alpha\beta + c\beta^2 + d\alpha + e\beta + f)e^{\alpha x}e^{\beta y} = 0$$

But $e^{\alpha x}e^{\beta y} \neq 0$; therefore we obtain a solution if and only if α, β satisfy the quadratic equation

$$\boxed{a\alpha^2 + b\alpha\beta + c\beta^2 + d\alpha + e\beta + f = 0} \qquad (0.2.5)$$

For a given value of β, we may solve this equation for α to obtain in general two roots α_1, α_2. Alternatively, we may fix α and solve for β to obtain in general two roots β_1, β_2. This proves the proposition. •

In the case of Laplace's equation, the quadratic equation (0.2.5) is $\alpha^2+\beta^2 = 0$. If α is real, then β must be purely imaginary; conversely if β is real, then α is purely imaginary. These two cases correspond to the separated solutions found in the previous subsection by solving (0.2.1) and (0.2.2). The solutions originating from (0.2.3) correspond to values of α for which α^2 is purely imaginary.

We now turn to some examples involving the heat equation, where complex separated solutions are useful.

EXAMPLE 0.2.2. *Find separated solutions of the PDE $u_{xx} - u_t = 0$ in the form $u(x,t) = e^{i\mu x}e^{\beta t}$, with μ real.*

Solution. Substituting $u(x,t) = e^{i\mu x}e^{\beta t}$ in the PDE yields the quadratic equation $-\mu^2 - \beta = 0$. Thus $\beta = -\mu^2$, and we have the separated solutions

$$\begin{aligned}
u(x,t) &= e^{i\mu x}e^{-\mu^2 t} \\
&= \cos\mu x\, e^{-\mu^2 t} + i(\sin\mu x\, e^{-\mu^2 t})
\end{aligned}$$

Taking the real and imaginary parts, we obtain the real-valued separated solutions

$$u(x;t) = \sin\mu x\, e^{-\mu^2 t}, \qquad u(x;t) = \cos\mu x\, e^{-\mu^2 t}$$

By taking linear combinations, we may write the general real-valued separated solution as

$$u(x;t) = (A_1 \sin \mu x + A_2 \cos \mu x)e^{-\mu^2 t}$$

where A_1, A_2 are arbitrary constants. •

In the above example the solutions tend to zero when the time t tends to infinity. In some problems we may wish to obtain a solution that oscillates in time, to represent a periodic disturbance.

EXAMPLE 0.2.3. *Find separated solutions of the PDE $u_{xx} - u_t = 0$ in the form $u(x,t) = e^{\alpha x}e^{\iota\omega t}$, where ω is real and positive.*

Solution. Substituting $u(x,t) = e^{\alpha x}e^{\iota\omega t}$ in the PDE $u_t - u_{xx} = 0$ yields the quadratic equation $\alpha^2 - i\omega = 0$. This equation has two solutions, which may be obtained as follows. Writing the complex number i in the polar form $i = e^{i\pi/2}$, we have the two square roots $i^{1/2} = \pm e^{i\pi/4} = \pm(1+i)/\sqrt{2}$. Therefore the solutions of the quadratic equation are $\alpha = \pm(1+i)\sqrt{\omega/2}$. The separated solutions are

$$u(x,t) = \begin{cases} \exp[x(1+i)\sqrt{\omega/2}]\exp(i\omega t) \\ \qquad\qquad = \exp(x\sqrt{\omega/2})\exp[i(\omega t + x\sqrt{\omega/2})] \\ \exp[-x(1+i)\sqrt{\omega/2}]\exp(i\omega t) \\ \qquad\qquad = \exp[-x\sqrt{\omega/2})\exp[i(\omega t - x\sqrt{\omega/2})] \end{cases}$$

Taking the real and imaginary parts, we have the real-valued solutions

$$u(x,t) = \begin{cases} e^{x\sqrt{\omega/2}}\cos(\omega t + x\sqrt{\omega/2}) \\ e^{x\sqrt{\omega/2}}\sin(\omega t + x\sqrt{\omega/2}) \\ e^{-x\sqrt{\omega/2}}\cos(\omega t - x\sqrt{\omega/2}) \\ e^{-x\sqrt{\omega/2}}\sin(\omega t - x\sqrt{\omega/2}) \end{cases}$$

These real-valued solutions are no longer in the separated form $X(x)T(t)$. But because they arise as the real and imaginary parts of complex separated solutions, we refer to them as *quasi-separated solutions.* •

If some of the coefficients a, b, c, d, e, f are not constant, we will no longer have separated solutions in the form of exponential functions. Even worse, the equation may not admit *any* nonconstant separated solutions, for example, $u_x + (x+y)u_y = 0$ (see the exercises). Nevertheless, various classes of equations can still be solved by the separation of variables. For example, for any equation of the form

$$a(x)u_{xx} + c(y)u_{yy} + d(x)u_x + e(y)u_y = 0$$

if we divide by $X(x)Y(y)$, we have

$$\left[a(x)\frac{X''(x)}{X(x)} + d(x)\frac{X'(x)}{X(x)}\right] + \left[c(y)\frac{Y''(y)}{Y(y)} + e(y)\frac{Y'(y)}{Y(y)}\right] = 0$$

The term in the first set of brackets depends only on x, while the term in the second set depends only on y; therefore both are constant and we have reduced the problem to ordinary differential equations. Introducing the separation constant λ, we have in detail

$$\begin{aligned} a(x)X''(x) + d(x)X'(x) + \lambda X(x) &= 0 \\ c(y)Y''(y) + e(y)Y'(y) - \lambda Y(y) &= 0 \end{aligned}$$

The following example gives a concrete illustration.

EXAMPLE 0.2.4. *Find all of the real-valued separated solutions of the PDE* $u_{xx} + y^2 u_{yy} + y u_y = 0$ *valid for* $y > 0$.

Solution. We let $u(x,y) = X(x)Y(y)$ and obtain the separated equations

$$X''(x) + \lambda X(x) = 0 \tag{0.2.6}$$

$$y^2 Y''(y) + y Y'(y) - \lambda Y(y) = 0 \tag{0.2.7}$$

Equation (0.2.6) has constant coefficients and was solved previously; equation (0.2.7) is a form of Euler's equidimensional equation, which can also be solved explicitly. We consider separately the cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

If $\lambda = k^2 > 0$, then the general solution of (0.2.6) is $X(x) = A_1 \cos kx + A_2 \sin kx$. Meanwhile (0.2.7) can be solved by a power $Y(y) = y^r$, where $r(r-1) + r - k^2 = 0$; thus $r = \pm k$ and the general solution $Y(y) = A_3 y^k + A_4 y^{-k}$.

If $\lambda = 0$, then the general solution of (0.2.6) is $X(x) = A_1 + A_2 x$, while (0.2.7) becomes $y^2 Y'' + y Y' = 0$, which has the general solution $Y(y) = A_3 + A_4 \log y$ valid for $y > 0$.

If $\lambda = -l^2 < 0$, then the general solution of (0.2.6) is $X(x) = A_1 e^{lx} + A_2 e^{-lx}$, while (0.2.7) becomes $y^2 Y'' + y Y' + l^2 Y = 0$, which has the general solution $Y(y) = A_3 \cos(l \log y) + A_4 \sin(l \log y)$.

Putting these together, we have the most general real-valued separated solution:

$$u(x,y) = \begin{cases} (A_1 \cos kx + A_2 \sin kx)(A_3 y^k + A_4 y^{-k}) & k > 0 \\ (A_1 + A_2 x)(A_3 + A_4 \log y) & \\ (A_1 e^{lx} + A_2 e^{-lx})(A_3 \cos(l \log y) + A_4 \sin(l \log y)) & l > 0 \end{cases} \qquad \bullet$$

EXERCISES 0.2.3

1. Find the separated equations satisfied by $X(x)$, $Y(y)$ for the following partial differential equations:
   (a) $u_{xx} - 2u_{yy} = 0$ (b) $u_{xx} + u_{yy} + 2u_x = 0$
   (c) $x^2 u_{xx} - 2y u_y = 0$ (d) $u_{xx} + u_x + u_y - u = 0$
2. Which of the following are solutions of Laplace's equation?
   (a) $u(x,y) = e^x \cos 2y$ (b) $u(x,y) = e^x \cos y + e^y \cos x$
   (c) $u(x,y) = e^x e^y$ (d) $u(x,y) = (3x+2)e^y$

In Exercises 3–7, find the separated solutions of the indicated equations.

3. $u_{xx} + 2u_x + u_{yy} = 0$
4. $u_{xx} + u_{yy} + 3u = 0$
5. $x^2 u_{xx} + x u_x + u_{yy} = 0$
6. $u_{xx} - u_{yy} + u = 0$
7. $u_{xx} + y u_y + u = 0$
8. This exercise provides an example of a homogeneous linear partial differential equation with no separated solutions other than $u(x, y) \equiv$ constant. Suppose that $u(x, y) = X(x)Y(y)$ is a solution of the equation $u_x + (x + y)u_y = 0$. Show that $X(x)$ and $Y(y)$ are both constant. [*Hint:* Show first that $X'(x)/X(x) + (x + y)(Y'(y)/Y(y)) = 0$ and deduce that $X'(x)/X(x) = cx+d$, $Y'(y)/Y(y) = -c$ for suitable constants c, d. By solving these ordinary differential equations, show that the PDE is satisfied if and only if $c = 0$, $d = 0$.]

0.2.4. Separated solutions with boundary conditions. In many problems we need separated solutions that satisfy certain additional conditions, which are suggested by the physics of the problem. They may be in the form of *boundary conditions* or *conditions of boundedness*. We shall now illustrate these by means of examples.

EXAMPLE 0.2.5. *Find the separated solutions of Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $0 < x < L$, $y > 0$ that satisfy the boundary conditions $u(0, y) = 0$, $u(L, y) = 0$, $u(x, 0) = 0$.*

Solution. From the discussion in subsection 0.2.2 we have the separated solutions of three types, depending on the separation constant.

$$u(x, y) = \begin{cases} (A_1 \sinh kx + A_2 \cosh kx)(A_3 \cos ky + A_4 \sin ky) & k > 0 \\ (A_1 x + A_2)(A_3 y + A_4) & \\ (A_1 \cos lx + A_2 \sin lx)(A_3 \sinh ly + A_4 \cosh ly) & l > 0 \end{cases}$$

In the first case, we must have $0 = u(0, y) = A_2(A_3 \cos ky + A_4 \sin ky)$, so $A_2 = 0$, while $0 = u(L, y) = A_1 \sinh kL(A_3 \cos ky + A_4 \sin ky)$ implies that $A_1 = 0$, so this case does not produce any separated solutions that satisfy the boundary conditions.

In the second case, we must have $0 = u(0, y) = A_2(A_3 y + A_4)$, so $A_2 = 0$, and $0 = u(L, y) = A_1 L(A_3 y + A_4)$, so $A_1 = 0$. Therefore this case does not produce any separated solutions that satisfy the boundary conditions.

In the third case, we must have $0 = u(0, y) = A_1(A_3 \sinh ly + A_4 \cosh ly)$, so that $A_1 = 0$; and $0 = u(L, y) = A_2 \sin Ll(A_3 \sinh ly + A_4 \cosh ly)$ has a nonzero solution if and only if $\sin Ll = 0$, which is satisfied if and only if $Ll = n\pi$ for some $n = 1, 2, 3, \ldots$. To satisfy the boundary condition $u(x, 0) = 0$, we must have $A_4 = 0$. Writing $A = A_2 A_3$, we have obtained the following separated solutions of Laplace's equation satisfying the boundary conditions:

$$u(x,y) = A\sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}, \qquad n = 1,2,\ldots \quad \bullet$$

The following example occurs repeatedly in the solution of the heat equation in Chapter 2.

EXAMPLE 0.2.6. *Find the separated solutions $u(x;t)$ of the heat equation $u_{xx} - u_t = 0$ in the region $0 < x < L$, $t > 0$ that satisfy the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$.*

Solution. In Example 0.2.2 we found the real-valued separated solutions

$$u(x;t) = (A_1 \sin\mu x + A_2\cos\mu x)e^{-\mu^2 t}$$

In order to satisfy the boundary condition at $x = 0$ we must have $0 = u(0;t) = A_2 e^{-\mu^2 t}$, which is satisfied if and only if $A_2 = 0$. In order to satisfy the boundary condition at $x = L$, we must have $0 = u(L;t) = A_1(\sin\mu L)\, e^{-\mu^2 t}$. This is satisfied if and only if $\mu L = n\pi$ for some $n = 1, 2, \ldots$. Therefore the separated solutions satisfying the boundary conditions are of the form

$$u(x;t) = A_1 \sin\frac{n\pi x}{L}e^{-(n\pi/L)^2 t}, \qquad n = 1,2,\ldots \quad \bullet$$

The next example occurs repeatedly in the discussion of the vibrating string in Chapter 2, Sec. 2.4.

EXAMPLE 0.2.7. *Find the separated solutions of the wave equation $u_{tt} - c^2 u_{xx} = 0$ that satisfy the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$.*

Solution. Assuming the separated form $u(x;t) = X(x)T(t)$, it follows that $X(x)T''(t) - c^2X''(x)T(t) = 0$. Thus $X''(x) + \lambda X(x) = 0$, $T''(t) + \lambda c^2 T(t) = 0$. The boundary conditions require $X(0) = 0$, $X(L) = 0$; thus $X(x) = A_3\sin(n\pi x/L)$, $T(t) = A_1\cos(n\pi ct/L) + A_2\sin(n\pi ct/L)$ for constants A_1, A_2, A_3. The required separated solutions are

$$u(x;t) = (A_1\cos(n\pi ct/L) + A_2\sin(n\pi ct/L))\sin(n\pi x/L) \qquad n = 1,2,\ldots \quad \bullet$$

In all of the preceding examples we used one or more boundary conditions to pick out certain values of the separation constant that satisfy the boundary conditions. This can also be carried out through conditions of *boundedness* as indicated in the following examples. Physically these represent a *stationary solution*, corresponding to a system that has been in existence over a very long period of time.

EXAMPLE 0.2.8. *Find the complex separated solutions $u(x;t)$ of the wave equation $u_{tt} - c^2u_{xx} = 0$, which are bounded in the form $|u(x;t)| \leq M$ for some constant M and all t, $-\infty < t < \infty$.*

Solution. Taking $u(x;t) = e^{ax+bt}$ and substituting in the wave equation, we have $b^2 - c^2a^2 = 0$; thus $b = \pm ca$. The separated solutions are of the form $u(x;t) = e^{ax}e^{cat}$, $e^{ax}e^{-cat}$. This solution is bounded for all t if and only if a is pure imaginary, $a = ik$ for k real. Thus the solutions are $u(x;t) = e^{ik(x+ct)}$, $e^{ik(x-ct)}$. The real (quasi-separated) solutions are $\cos k(x+ct)$, $\cos k(x-ct)$, $\sin k(x+ct)$, $\sin k(x-ct)$. •

The final example, concerning stationary solutions of the heat equation, will be developed in more detail in Chapter 2, Sec. 2.1, in connection with heat flow in the earth.

EXAMPLE 0.2.9. *Find the complex separated solutions $u(x;t)$ of the heat equation $u_t - u_{xx} = 0$, which are bounded in the form $|u(x;t)| \le M$ for some constant M and all t, $-\infty < t < \infty$.*

Solution. Taking $u(x;t) = e^{ax+bt}$ and substituting in the heat equation, we have $b - a^2 = 0$. In order that this solution be bounded for all $t, -\infty < t < \infty$, it is necessary that the constant b be purely imaginary; otherwise the solution would tend to $+\infty$ for large $|t|$ if b had a nonzero real part. Hence we set $b = i\omega$, where ω is real. Assuming $\omega > 0$, the equation $a^2 = i\omega$ has two solutions,

$$a = \sqrt{\frac{\omega}{2}}(1+i), \qquad a = -\sqrt{\frac{\omega}{2}}(1+i)$$

leading to the separated solution

$$u(x;t) = e^{i\omega t}\left(A_1 e^{\sqrt{\omega/2}(1+i)x} + A_2 e^{-\sqrt{\omega/2}(1+i)x}\right)$$

If $\omega < 0$, then the equation $a^2 = i\omega$ has two solutions,

$$a = \sqrt{\frac{|\omega|}{2}}(1-i), \qquad a = -\sqrt{\frac{|\omega|}{2}}(1-i)$$

leading to the separated solution

$$u(x;t) = e^{i\omega t}\left(A_1 e^{\sqrt{|\omega|/2}(1-i)x} + A_2 e^{-\sqrt{|\omega|/2}(1-i)x}\right) \quad \bullet$$

The alert reader will note that these separated solutions are closely related to those found in Example 0.2.3, where we stipulated in advance that ω be real and positive. Now we have shown that the reality of ω can be deduced from the qualitative condition of boundedness of the solution for all time.

EXERCISES 0.2.4

1. Find the separated solutions $u(x,y)$ of Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $0 < x < L$, $y > 0$ that satisfy the boundary conditions $u_x(0,y) = 0$, $u_x(L,y) = 0$, $u(x,0) = 0$.

2. Find the separated solutions $u(x, y)$ of Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $0 < x < L$, $y > 0$, that satisfy the boundary conditions $u(0, y) = 0$, $u(L, y) = 0$ and the boundedness condition $|u(x, y)| \leq M$ for $y > 0$, where M is a constant independent of (x, y).
3. Find the separated solutions $u(x; t)$ of the heat equation $u_t - u_{xx} = 0$ in the region $0 < x < L$, $t > 0$, that satisfy the boundary conditions $u(0; t) = 0$, $u(L; t) = 0$.
4. Find the separated solutions $u(x; t)$ of the heat equation $u_t - u_{xx} = 0$ in the region $0 < x < L$, $t > 0$, that satisfy the boundary conditions $u_x(0; t) = 0$, $u_x(L; t) = 0$.
5. Find the separated solutions $u(x; t)$ of the heat equation $u_t - u_{xx} = 0$ in the region $0 < x < L$, $t > 0$ that satisfy the boundary conditions $u(0; t) = 0$, $u_x(L; t) = 0$.

0.3. Orthogonal Functions

Separated solutions of linear partial differential equations with suitable boundary conditions lead to systems of *orthogonal functions*, which are introduced in this section. The most important system of orthogonal functions gives rise to the trigonometric Fourier series, which will be discussed in Chapter 1, including the more general *Sturm-Liouville eigenvalue problem.* In order to formulate the property of orthogonality, we first introduce the general notion of *inner product.*

0.3.1. Inner product space of functions. The notions of dot product, distance, orthogonality, and projection, which are familiar for vectors in three dimensions, can also be formulated for real-valued functions on an interval $a \leq x \leq b$. The basic notion is the *inner product* of two functions $\varphi(x)$, $\psi(x)$ on the interval $a \leq x \leq b$. This is defined by the integral

$$\boxed{\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\psi(x)\, dx} \tag{0.3.1}$$

For example, on the interval $0 \leq x \leq 1$, we have $\langle x, e^{x^2} \rangle = \int_0^1 x e^{x^2} dx = \frac{1}{2}(e-1) = 0.86$, to two decimal places.

The inner product defined by (0.3.1) has many properties in common with the ordinary dot product of two vectors in three-dimensional space, defined by $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$. The analogy between the inner product and the three-dimensional dot product is intuitive if we think of the integral as a "continuous sum" of the pointwise products $\varphi(x)\psi(x)$, a generalization of the three-dimensional dot product formula.

The inner product is *linear* and *homogeneous* in both arguments. This means that, for any functions φ_1, φ_2, ψ_1, ψ_2 and any real number a,

$$\begin{aligned}
\langle\varphi_1,\psi_1+\psi_2\rangle &= \langle\varphi_1,\psi_1\rangle+\langle\varphi_1,\psi_2\rangle\\
\langle\varphi_1+\varphi_2,\psi_1\rangle &= \langle\varphi_1,\psi_1\rangle+\langle\varphi_2,\psi_1\rangle\\
\langle a\varphi_1,\psi_1\rangle &= a\langle\varphi_1,\psi_1\rangle\\
\langle\varphi_1,a\psi_1\rangle &= a\langle\varphi_1,\psi_1\rangle
\end{aligned}$$

The proofs of these properties are left as exercises.

Definition Two functions φ, ψ are *orthogonal* on the interval $a \le x \le b$ if and only if $\langle\varphi,\psi\rangle = 0$.

This definition requires some comment. It is formulated as a generalization of the notion of perpendicularity for vectors in three-dimensional space, which is expressed as the equation $\mathbf{v}\cdot\mathbf{w}=0$. In working with functions, it is difficult to visualize the notion of orthogonality, as we are accustomed to for vectors in two- and three-dimensional space. In particular, there is no suggestion that the graphs of the two orthogonal functions intersect at 90 degrees.

A few examples may help to illustrate these concepts.

EXAMPLE 0.3.1. *Show that the functions* $\phi(x) = \sin x$, $\psi(x) = \cos x$ *are orthogonal on the interval* $0 \le x \le \pi$ *but are not orthogonal on the interval* $0 \le x \le \pi/2$.

Solution. The inner product on the interval $0 \le x \le \pi$ is computed as the integral

$$\int_0^\pi \sin x\,\cos x\,dx = \frac{1}{2}(\sin x)^2\Big|_0^\pi = 0$$

If we do the same computation on the interval $0 \le x \le \pi/2$, we obtain

$$\int_0^{\pi/2} \sin x\,\cos x\,dx = \frac{1}{2}(\sin x)^2\Big|_0^{\pi/2} = \frac{1}{2}$$

Therefore we have orthogonality in the first case but not in the second case. •

For more than two functions, we say that $(\varphi_1,\ldots,\varphi_N)$ are orthogonal if $\langle\varphi_i,\varphi_j\rangle = 0$ for $i \ne j$. This is illustrated by the next example.

EXAMPLE 0.3.2. *Show that the set of functions* $\sin x, \sin 2x, \ldots, \sin Nx$ *is orthogonal on the interval* $0 \le x \le \pi$ *for any* $N \ge 2$.

Solution. The inner product on the interval $0 \le x \le \pi$ is computed as the integral

$$\int_0^\pi \sin mx \, \sin nx \, dx$$

We use the trigonometric identity

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

If $m \neq n$, the integral of each cosine function is a sine function, which vanishes at the endpoints $x = 0$, $x = \pi$. Therefore each of the integrals is zero, and we have proved orthogonality. •

The *norm* of a function is the nonnegative number $||\varphi||$ that satisfies

$$||\varphi||^2 = \langle \varphi, \varphi \rangle$$

For example, on the interval $0 \le x \le \pi$,

$$||\sin x||^2 = \int_0^\pi \sin^2 x \, dx = \int_0^\pi \frac{1}{2}(1 - \cos 2x) dx = \frac{1}{2}\pi$$

The *distance* between φ and ψ is defined by $d(\varphi, \psi) = ||\varphi - \psi||$. For example, the distance between $\sin x$ and $\cos x$ on the interval $0 \le x \le \pi$ is obtained from

$$[d(\sin x, \cos x)]^2 = \int_0^\pi (\sin x - \cos x)^2 \, dx = \int_0^\pi (\sin^2 x + \cos^2 x) \, dx = \pi$$

so that the distance is given by $d = \sqrt{\pi} \sim 1.77$ to two decimals. Since these two functions are orthogonal, one may think of a "right triangle" in the space of functions, for which we have computed the hypotenuse.

In order to formulate the notion of *angle* for functions on an interval, we recall that for vectors in three-dimensional space we have the dot product formula $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| \, ||\mathbf{w}|| \cos\theta$, where θ is the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$ and $||\mathbf{v}||$, $||\mathbf{w}||$ are the lengths of the respective vectors. Hence the cosine of the angle between the two vectors may be computed as the ratio of the dot product to the product of the lengths. In order to extend this to functions on an interval, we need to know that the corresponding ratio is not greater than 1 in absolute value. This is known as the *Schwarz inequality*.

PROPOSITION 0.3.1. *Suppose that $\varphi(x)$, $\psi(x)$ are nonzero functions defined on an interval $a \le x \le b$. Then*

$$\boxed{\langle \varphi, \psi \rangle^2 \le ||\varphi||^2 ||\psi||^2} \tag{0.3.2}$$

Proof. By the linearity and homogeneity of the inner product, we have, for any real number t,

$$\begin{aligned} D(t) := \|\varphi - t\psi\|^2 &= \|\varphi\|^2 - 2t\langle\varphi,\psi\rangle + t^2\|\psi\|^2 \\ &= \|\psi\|^2\left(t^2 - 2t\frac{\langle\varphi,\psi\rangle}{\|\psi\|^2} + \frac{\langle\varphi,\psi\rangle^2}{\|\psi\|^4}\right) \\ &\quad + \left(\|\varphi\|^2 - \frac{\langle\varphi,\psi\rangle^2}{\|\psi\|^2}\right) \\ &= \|\psi\|^2\left(t - \frac{\langle\varphi,\psi\rangle}{\|\psi\|^2}\right)^2 + \left(\|\varphi\|^2 - \frac{\langle\varphi,\psi\rangle^2}{\|\psi\|^2}\right) \end{aligned}$$

From these transformations we see that this quadratic function of t is nonnegative and has a global minimum at $t = t_0$, where $t_0 = \langle\varphi,\psi\rangle/\|\psi\|^2$; at this point the value of the function is nonnegative and given explicitly by

$$D(t_0) = \left(\|\varphi\|^2 - \frac{\langle\varphi,\psi\rangle^2}{\|\psi\|^2}\right) \geq 0$$

which completes the proof of the Schwarz inequality. •

In case the equality sign holds in equation (0.3.2), we expect that the functions $\varphi(x)$, $\psi(x)$ will be proportional to one another, analogous to the case of three-dimensional vectors that are colinear. This is rigorously true if both functions $\varphi(x)$, $\psi(x)$ are *continuous*: from the above computations, the integral of the nonnegative continuous function $|\varphi(x) - t_0\psi(x)|^2$ is equal to zero. But this means that the function must be identically zero, so that we conclude $\varphi(x) - t_0\psi(x) = 0$ for all x, $a \leq x \leq b$; thus we have established the desired proportionality, with the proportionality constant t_0. If one of the functions fails to be continuous, we cannot conclude that the integrand is zero everywhere, but only *almost everywhere*[2] (for example, a finite set).

0.3.2. Projection of a function onto an orthogonal set. We now discuss minimizing properties of orthogonal functions. This will motivate the definition of *Fourier coefficients* in a general setting. Let $(\varphi_1, \ldots, \varphi_N)$ be a set of orthogonal functions with $\|\varphi_i\| \neq 0$ for $1 \leq i \leq N$. If f is an arbitrary function, we compute the minimum of

$$D(c_1, \ldots, c_N) = \|f - (c_1\varphi_1 + \cdots + c_N\varphi_N)\|^2$$

where $(c_1, \ldots, c_N)$ range over all real values. In other words, we are trying to find the best "mean square approximation" of the given function $f(x), a \leq x \leq b$, by means of linear combinations of the members of the orthogonal set.

[2]This means that the set of exceptional values can be included in a union of intervals whose total length is arbitrarily small.

PROPOSITION 0.3.2. *The minimization problem has the following properties:*

- *The minimum is attained uniquely when*

$$c_i = \hat{c}_i := \frac{\langle f, \varphi_i \rangle}{||\varphi_i||^2}, \qquad 1 \le i \le N$$

- *The minimum distance is given by*

$$d^2_{\min} = ||f||^2 - \sum_{i=1}^{N} \frac{\langle f, \varphi_i \rangle^2}{||\varphi_i||^2}$$

- *The Fourier coefficients $\hat{c}_1, \ldots, \hat{c}_N$ satisfy Bessel's inequality*

$$\hat{c}_1^2 ||\varphi_1||^2 + \cdots + \hat{c}_N^2 ||\varphi_N||^2 \le ||f||^2$$

The function $\hat{c}_1\varphi_1 + \cdots + \hat{c}_N\varphi_N$ is called the **projection** *of f onto the orthogonal set $(\varphi_1, \ldots, \varphi_N)$; $\hat{c}_i$ is called the* ith **Fourier coefficient** *of f.*

Proof. The proof of these facts can be done by rewriting the formula for D. We use the linearity and homogeneity of the inner product to write

$$\begin{aligned}
D(c_1, \ldots, c_N) &= ||f||^2 - 2\sum_{i=1}^{N} c_i \langle f, \varphi_i \rangle + \sum_{i=1}^{N} c_i^2 ||\varphi_i||^2 \\
&= \sum_{i=1}^{N} ||\varphi_i||^2 \left(c_i^2 - 2c_i \frac{\langle f, \varphi_i \rangle}{||\varphi_i||^2} + \frac{\langle f, \varphi_i \rangle^2}{||\varphi_i||^4} \right) \\
&\quad + ||f||^2 - \sum_{i=1}^{N} \frac{\langle f, \varphi_i \rangle^2}{||\varphi_i||^2} \\
&= \sum_{i=1}^{N} ||\varphi_i||^2 \left(c_i - \frac{\langle f, \varphi_i \rangle}{||\varphi_i||^2} \right)^2 + ||f||^2 - \sum_{i=1}^{N} \frac{\langle f, \varphi_i \rangle^2}{||\varphi_i||^2} \\
&\ge ||f||^2 - \sum_{i=1}^{N} \frac{\langle f, \varphi_i \rangle^2}{||\varphi_i||^2}
\end{aligned}$$

Clearly, the minimum is achieved when $c_i = \hat{c}_i := \langle f, \varphi_i \rangle / ||\varphi_i||^2$, as required. The value of the minimum is

$$D(\hat{c}_1, \ldots, \hat{c}_N) = ||f||^2 - \sum_{i=1}^{N} \frac{\langle f, \varphi_i \rangle^2}{||\varphi_i||^2} = ||f||^2 - \sum_{i=1}^{N} \hat{c}_i^2 ||\varphi_i||^2$$

as required. Since this is nonnegative, Bessel's inequality is merely the statement that $D(\hat{c}_1, \ldots, \hat{c}_N) \ge 0$. •

As a first example, we consider the orthogonal set consisting of the three functions $\{\sin x, \sin 2x, \sin 3x\}$ on the interval $0 \le x \le \pi$.

EXAMPLE 0.3.3. *Find the projection of the function $f(x) = 1$ onto the orthogonal set $\{\sin x, \sin 2x, \sin 3x\}$ on the interval $0 \le x \le \pi$ and compute $d_{\min}$.*

Solution. We first note that the norms are given by

$$\|\varphi_m\|^2 = \int_0^\pi \sin^2 mx\, dx = \frac{1}{2}\int_0^\pi (1 - \cos 2mx)\, dx = \frac{\pi}{2}$$

From Proposition 0.3.2, the Fourier coefficients are

$$\begin{aligned}
\hat{c}_1 &= \frac{\int_0^\pi \sin x\, dx}{\int_0^\pi \sin^2 x\, dx} = \frac{\cos x|_{x=\pi}^{x=0}}{\pi/2} = \frac{4}{\pi} \\
\hat{c}_2 &= \frac{\int_0^\pi \sin 2x\, dx}{\int_0^\pi \sin^2 2x\, dx} = \frac{1}{2}\frac{\cos 2x|_{x=\pi}^{x=0}}{\pi/2} = 0 \\
\hat{c}_3 &= \frac{\int_0^\pi \sin 3x\, dx}{\int_0^\pi \sin^2 3x\, dx} = \frac{1}{3}\frac{\cos 3x|_{x=\pi}^{x=0}}{\pi/2} = \frac{4}{3\pi}
\end{aligned}$$

The projection is the function

$$s(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x$$

The minimum distance is obtained from

$$\begin{aligned}
d_{\min}^2 &= \|f\|^2 - \sum_{i=1}^{3} \hat{c}_i^2 \|\varphi_i\|^2 \\
&= \pi - \left(\frac{4}{\pi}\right)^2\left(\frac{\pi}{2}\right) - \left(\frac{4}{3\pi}\right)^2\left(\frac{\pi}{2}\right) \\
&= 3.14 - (1.27)^2(1.57) - (0.42)^2(1.57) \\
&= 0.33
\end{aligned}$$

to two decimal places. •

In the next example we consider an orthogonal set of three polynomial functions on the interval $-1 \le x \le 1$. This is closely related to the *Legendre polynomial expansion*, which will be considered in Chapter 4.

EXAMPLE 0.3.4. *Find the projection of the function $f(x) = \cos(\pi x/2)$ on the orthogonal set $(1, x, x^2 - \frac{1}{3})$ on the interval $-1 \le x \le 1$, and compute $d_{\min}$.*

Solution. The solution may be written in the form

$$s(x) = \hat{c}_0 + \hat{c}_1 x + \hat{c}_2\left(x^2 - \frac{1}{3}\right)$$

where the Fourier coefficients $\hat{c}_0$, $\hat{c}_1$, $\hat{c}_2$ are computed from the equations

$$\begin{aligned}
\hat{c}_0 \int_{-1}^{1} dx &= \int_{-1}^{1} \cos\frac{\pi x}{2} dx \\
\hat{c}_1 \int_{-1}^{1} x^2\, dx &= \int_{-1}^{1} x\cos\frac{\pi x}{2}\, dx \\
\hat{c}_2 \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx &= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right) \cos\frac{\pi x}{2}\, dx
\end{aligned}$$

The first of these is straightforward since

$$\int_{-1}^{1} \cos\frac{\pi x}{2}\, dx = \frac{2}{\pi}\sin\frac{\pi x}{2}\bigg|_{x=-1}^{x=1} = \frac{4}{\pi}; \quad \text{thus } \hat{c}_0 = \frac{2}{\pi}$$

The next is also easy since the function $x\cos\pi x/2$ is odd; thus $\hat{c}_1 = 0$. To perform the final integral, we write

$$\begin{aligned}
\int_{-1}^{1} x^2\cos\frac{\pi x}{2}\, dx &= \frac{2}{\pi}\int_{-1}^{1} x^2 d\left(\sin\frac{\pi x}{2}\right) \\
&= \frac{2x^2}{\pi}\sin\frac{\pi x}{2}\bigg|_{-1}^{1} - \frac{4}{\pi}\int_{-1}^{1} x\sin\frac{\pi x}{2}\, dx \\
&= \frac{4}{\pi} + \frac{8}{\pi^2}\int_{-1}^{1} x\, d\left(\cos\frac{\pi x}{2}\right) \\
&= \frac{4}{\pi} + \frac{8}{\pi^2}\left(x\cos\frac{\pi x}{2}\bigg|_{-1}^{1} - \int_{-1}^{1}\cos\frac{\pi x}{2}\, dx\right) \\
&= \frac{4}{\pi} - \frac{32}{\pi^3}
\end{aligned}$$

Combining this with the previous integral, we have

$$\begin{aligned}
\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)\cos\frac{\pi x}{2}\, dx &= \frac{4}{\pi} - \frac{32}{\pi^3} - \frac{4}{3\pi} \\
&= \frac{8\pi^2 - 96}{3\pi^3}
\end{aligned}$$

But $\int_{-1}^{1}(x^2 - \frac{1}{3})^2 dx = \int_{-1}^{1}(x^4 - \frac{2}{3}x^2 + \frac{1}{9})dx = \frac{2}{5} - \left(\frac{2}{3}\right)^2 + \frac{2}{9} = \frac{8}{45}$. Therefore $\hat{c}_2 = \frac{45}{8}(8\pi^2 - 96)/3\pi^3 = 15(\pi^2 - 12)/\pi^3$. Thus the required orthogonal projection is

$$s(x) = \frac{2}{\pi} + \frac{15(\pi^2 - 12)}{\pi^3}\left(x^2 - \frac{1}{3}\right)$$

To compute $d_{\min}$, we have, to four decimals,

$$\begin{aligned}
\hat{c}_0^2 &= \left(\frac{2}{\pi}\right)^2 = 0.4053 \\
\|1\|^2 &= \int_{-1}^{1} dx = 2 \\
\hat{c}_2^2 &= \left[\frac{15(\pi^2-12)}{\pi^3}\right]^2 = 1.0622 \\
\left\|x^2 - \frac{1}{3}\right\|^2 &= \int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2 dx = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = 0.1778 \\
\left\|\cos\frac{\pi x}{2}\right\|^2 &= \int_{-1}^{1} \cos^2\frac{\pi x}{2} dx = \frac{1}{2}\int_{-1}^{1}(1+\cos\pi x)dx = 1
\end{aligned}$$

Thus, to four decimals,

$$d_{\min}^2 = 1 - (0.4053)(2) - (1.0622)(0.1778) = 0.0004$$

and, to two decimals, $d_{\min} = 0.02$. •

It is instructive to compare the orthogonal projection with the corresponding values of $\cos(\pi x/2)$ at some representative points. For example, to four decimal places of accuracy, we have

$$\begin{aligned}
s(0) &= (0.6366) + \tfrac{1}{3}(1.0306) = 0.9801 \\
s(1) &= (0.6366) - \tfrac{2}{3}(1.0306) = 0.0505 \\
s\left(\frac{1}{2}\right) &= (0.6366) + \tfrac{1}{12}(1.0306) = 0.7225 \\
s\left(\frac{1}{3}\right) &= (0.6366) + \tfrac{2}{9}(1.0306) = 0.8656 \\
s\left(\frac{2}{3}\right) &= (0.6366) - \tfrac{1}{9}(1.0306) = 0.5221
\end{aligned}$$

On the other hand, the corresponding values of $\cos(\pi x/2)$ are 1, 0, 0.7071, 0.8667, 0.5000.

0.3.3. Orthonormal sets of functions. The formulas for the Fourier coefficients and the minimum distance become especially simple when the functions $(\varphi_1, \ldots, \varphi_N)$ are *orthonormal*. This means that $\langle\varphi_i, \varphi_j\rangle = 0$ for $i \neq j$ and $\langle\varphi_i, \varphi_i\rangle = 1$, $1 \le i \le N$. Thus we have for orthonormal functions

$$\hat{c}_i = \langle f, \varphi_i\rangle \qquad 1 \le i \le N \tag{0.3.3}$$

$$d_{\min}^2 = D(\hat{c}_1, \ldots, \hat{c}_N) = \|f\|^2 - (\hat{c}_1^2 + \cdots + \hat{c}_N^2) \tag{0.3.4}$$

If $(\varphi_1, \ldots, \varphi_N)$ is an orthogonal set of functions, we obtain an orthonormal set by replacing φ_i by $\varphi_i/\|\varphi_i\|$, $1 \le i \le N$.

EXAMPLE 0.3.5. *Let $\varphi_1 = 1$, $\varphi_2 = \sin x$, $\varphi_3 = \cos x$ for $-\pi < x < \pi$. Verify that this is an orthogonal set and find the corresponding orthonormal set.*

Solution. Direct computation reveals that each of the inner products $\langle \varphi_i, \varphi_j \rangle$ is zero for $i \neq j$. To find the orthonormal set, we compute

$$\begin{aligned}
\|\varphi_1\|^2 &= \int_{-\pi}^{\pi} dx = 2\pi \\
\|\varphi_2\|^2 &= \int_{-\pi}^{\pi} \sin^2 x\, dx = \pi \\
\|\varphi_3\|^2 &= \int_{-\pi}^{\pi} \cos^2 x\, dx = \pi
\end{aligned}$$

The orthonormal set is $(1/\sqrt{2\pi}, (\sin x)/\sqrt{\pi}, (\cos x)/\sqrt{\pi})$. •

In many problems we are given an *infinite* orthonormal set

$$(\varphi_n)_{n \geq 1} = (\varphi_1, \varphi_2, \dots)$$

To study such a set, we apply the above procedure to the finite orthonormal set $(\varphi_1, \dots, \varphi_n)$. The Fourier coefficients are

$$\hat{c}_i = \langle f, \varphi_i \rangle, \qquad 1 \leq i \leq N$$

which don't depend on N. Furthermore we have Bessel's inequality: for each N

$$\sum_{i=1}^{N} \hat{c}_i^2 \leq \|f\|^2 \qquad N = 1, 2, \dots$$

This is valid for every $N = 1, 2, \dots$; hence the infinite series $\sum_{i=1}^{\infty} \hat{c}_i^2$ converges and we have

$$\boxed{\sum_{i=1}^{\infty} \hat{c}_i^2 \leq \|f\|^2} \tag{0.3.5}$$

This is formulated as follows.

PROPOSITION 0.3.3. *Suppose that $(\varphi_n)_{n \geq 1} = (\varphi_1, \varphi_2, \dots)$ is an infinite orthonormal set of functions and that f is a function for which $\int_a^b |f(x)|^2\, dx < \infty$. Then the series of sums of squares of Fourier coefficients converges and satisfies the Bessel inequality (0.3.5).*

0.3.4. Parseval's equality, completeness, and mean square convergence. If we have an infinite orthonormal set, it may happen that Bessel's inequality (0.3.5) is an equality, namely

$$\boxed{\sum_{i=1}^{\infty} \hat{c}_i^2 = \|f\|^2} \tag{0.3.6}$$

This is called *Parseval's equality.* We will show that Parseval's equality is equivalent to the *mean square convergence* of the series $\sum_{i=1}^{\infty} \hat{c}_i\varphi_i$, which is defined by the limiting statement

$$\lim_{N\to\infty} \left\| f - \sum_{i=1}^{N} \hat{c}_i\varphi_i \right\|^2 = 0 \tag{0.3.7}$$

The formal statement of equivalence follows.

PROPOSITION 0.3.4. *Let $(\varphi_n)_{n\geq 1}$ be an orthonormal set and f a function with $\int_a^b f(x)^2\,dx < \infty$. Parseval's equality is true if and only if we have mean square convergence of the series $\sum_{i=1}^{\infty} \hat{c}_i\varphi_i$.*

Proof. Let $\hat{c}_i = \langle f, \varphi_i\rangle$ be the ith Fourier coefficient of f. Then by expanding the inner product and using orthonormality on the left side, we have

$$\begin{aligned} \left\| f - \sum_{i=1}^{N} \hat{c}_i\varphi_i \right\|^2 &= \|f\|^2 - 2\sum_{i=1}^{N} \hat{c}_i\langle f, \varphi_i\rangle + \sum_{i=1}^{N} \hat{c}_i^2 \\ &= \|f\|^2 - \sum_{i=1}^{N} \hat{c}_i^2 \end{aligned}$$

Letting $N \to \infty$, we see that the right side tends to zero if and only if Parseval's equality is valid. The left side tends to zero (by definition) if and only if we have mean square convergence. Therefore the proposition is proved. •

One may note that Parseval's equality is not true for an arbitrary function. For example, the set of functions $\pi^{-1/2}(\sin nx, \cos nx)_{n\geq 1}$ is an orthonormal set for $-\pi \leq x \leq \pi$. The function $f(x) = 1$ has all Fourier coefficients zero; indeed, $\int_{-\pi}^{\pi} \sin nx\,dx = 0 = \int_{-\pi}^{\pi} \cos nx\,dx, n \geq 1$. Yet $\|f\|^2 = \int_{-\pi}^{\pi} 1\,dx = 2\pi$. In this case Bessel's inequality is the statement that $0 = \sum_{i=1}^{\infty} \hat{c}_i^2 < \|f\|^2 = 2\pi$.

If Parseval's equality holds for all functions f with $\int_a^b f(x)^2\,dx < \infty$, then we say that the orthonormal set is *complete* on the interval $a \leq x \leq b$. For example, in Chapter 1 it will be shown that the trigonometric system consisting of $\{1/\sqrt{2\pi}, (\sin nx)/\sqrt{\pi}, (\cos nx)/\sqrt{\pi}\}_{n\geq 1}$ is complete on the interval $-\pi \leq x \leq \pi$.

0.3.5. Weighted inner product. In many problems we are required to deal with a weighted inner product with respect to a positive *weight function* $\rho(x), a \leq x \leq b$. This is defined by the integral

$$\langle \varphi, \psi\rangle_\rho = \int_a^b \varphi(x)\psi(x)\rho(x)\,dx$$

This has the same properties of linearity and homogeneity as the ordinary inner product. We say that two functions φ, ψ are orthogonal with respect to the weight function $\rho(x), a \leq x \leq b$, if $\langle \varphi, \psi\rangle_\rho = 0$.

Weighted orthogonality arises when we make a change of variable by means of an increasing differentiable function $x = h(y)$. The ordinary inner product is transformed as follows:

$$\int_a^b \varphi(x)\psi(x)\,dx = \int_c^d \varphi(h(y))\psi(h(y))h'(y)\,dy$$

Therefore we see that if $\varphi(x), \psi(x)$ are orthogonal on the interval $a \le x \le b$, then the functions $\varphi(h(y)), \psi(h(y))$ are orthogonal with respect to the weight function $h'(y)$ on the interval $c \le y \le d$, where $a = h(c)$, $b = h(d)$.

EXAMPLE 0.3.6. *Given the orthogonal functions $P_1(x) = x$, $P_2(x) = 3x^2 - 1$ on the interval $-1 \le x \le 1$, find the weighted orthogonality relation on the interval $0 \le y \le \pi$ under the transformation $x = -\cos y$.*

Solution. We have the transformed functions $P_1(h(y)) = -\cos y$, $P_2(y) = 3\cos^2 y - 1$, with the weight function $\rho(y) = h'(y) = \sin y$. •

0.3.6. Gram-Schmidt orthogonalization. When we deal with separated solutions of boundary-value problems in PDEs, the property of orthogonality is often immediately verified. This will be discussed in more detail in the following chapters. Nevertheless, it is interesting to know how we may manufacture orthogonal sets of functions from arbitrary sets of functions, by the so-called *Gram-Schmidt procedure.*[3] Suppose that $(\varphi_1, \ldots, \varphi_n)$ is a given set of functions, not necessarily orthogonal. Instead we suppose linear independence, i.e., that there are no relations of the form $c_1\varphi_1 + \cdots + c_n\varphi_n = 0$ among the $(\varphi_1, \ldots, \varphi_n)$, other than the trivial relation where $c_1 = 0, \ldots, c_n = 0$. In particular, $\|\varphi_i\| \ne 0$ for $1 \le i \le n$. Then we define

$$\begin{aligned}
\psi_1 &= \varphi_1 \\
\psi_2 &= \varphi_2 - \frac{\langle \varphi_2, \psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle}\psi_1 \\
\psi_3 &= \varphi_3 - \frac{\langle \varphi_3, \psi_2 \rangle}{\langle \psi_2, \psi_2 \rangle}\psi_2 - \frac{\langle \varphi_3, \psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle}\psi_1 \\
&\vdots \\
\psi_n &= \varphi_n - \sum_{i=1}^{n-1} \frac{\langle \varphi_n, \psi_i \rangle}{\langle \psi_i, \psi_i \rangle}\psi_i
\end{aligned}$$

The functions $(\psi_1, \ldots, \psi_n)$ are orthogonal. These formulas may seem less mysterious if we note that in the ith formula we are subtracting from φ_i its projection onto the orthogonal set $\psi_1, \ldots, \psi_{i-1}$.

[3]This material is not used in the subsequent chapters.

The sets $(\varphi_1, \dots, \varphi_n)$ and $(\psi_1, \dots, \psi_n)$ have the same *linear span*; i.e., any function of the form $f = c_1\varphi_1 + \cdots + c_n\varphi_n$ can be written in the form $d_1\psi_1 + \cdots + d_n\psi_n$ for appropriate $(d_1, \dots, d_n)$, and the converse is also true.

EXAMPLE 0.3.7. *Let $\varphi_1 = 1$, $\varphi_2 = x$, $\varphi_3 = x^2$ for $0 \le x \le 1$. Apply the Gram-Schmidt procedure to find the orthogonal functions ψ_1, ψ_2, ψ_3.*

Solution. We have $\psi_1 = \varphi_1 = 1$, $\langle \varphi_2, \psi_1 \rangle = \int_0^1 x\,dx = \frac{1}{2}$, $\langle \psi_1, \psi_1 \rangle = 1$. Thus $\psi_2 = x - \frac{1}{2}$. The remaining inner products are

$$\begin{aligned}
\langle \varphi_3, \psi_2 \rangle &= \int_0^1 x^2 \left(x - \frac{1}{2}\right) dx = \frac{1}{4} - \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{12} \\
\langle \psi_2, \psi_2 \rangle &= \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12} \\
\langle \varphi_3, \psi_1 \rangle &= \int_0^1 x^2 dx = \frac{1}{3}
\end{aligned}$$

Thus $\psi_3 = x^2 - (x - \frac{1}{2}) - \frac{1}{3} = x^2 - x + \frac{1}{6}$. The orthogonal functions are 1, $x - \frac{1}{2}$, $x^2 - x + \frac{1}{6}$, $0 \le x \le 1$. •

0.3.7. Complex inner product. In dealing with complex-valued functions, it is necessary to modify the definition of inner product and orthogonality. The guiding principle is that the norm of a function should be a nonnegative number. With this in mind, we define the complex inner product and norm on the interval $a < x < b$ as

$$\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\bar{\psi}(x)\,dx \tag{0.3.8}$$

$$\|\varphi\| = \sqrt{\langle \varphi, \varphi \rangle} \ge 0 \tag{0.3.9}$$

where the bar denotes the complex conjugate of a function, defined by $\bar{\psi}(x) = f(x) - ig(x)$ when $\psi(x) = f(x) + ig(x)$. *Orthogonality* of complex-valued functions is defined by the requirement that the complex inner product be zero: $\langle \varphi, \psi \rangle = 0$.

The properties of linearity and homogeneity of the complex inner product are almost identical to those of the real inner product, with the exception that we have $\langle \varphi, a\psi \rangle = \bar{a}\langle \varphi, \psi \rangle$ for any complex constant. We record here the appropriate statement of Schwarz's inequality.

PROPOSITION 0.3.5. *Suppose that $\varphi(x)$ and $\psi(x)$, $a < x < b$, are complex-valued functions. Then $|\langle \varphi, \psi \rangle| \le \|\varphi\|\,\|\psi\|$. If equality holds and both functions are continuous, then the functions are proportional: $C_1\varphi(x) = C_2\psi(x)$ for some complex constants C_1, C_2.*

The proof is suggested as an optional exercise.

EXERCISES 0.3

1. Let $\varphi_1 = 1$, $\varphi_2 = x$, $\varphi_3 = x^2$ on the interval $0 \le x \le 1$. Find the following inner products:
   (a) $\langle \varphi_1, \varphi_2 \rangle$ (b) $\langle \varphi_1, \varphi_3 \rangle$
   (c) $\|\varphi_1 - \varphi_2\|^2$ (d) $\|\varphi_1 + 3\varphi_2\|^2$
2. Which of the following pairs of functions are orthogonal on the interval $0 \le x \le 1$?
$$\varphi_1 = \sin 2\pi x \qquad \varphi_2 = x \qquad \varphi_3 = \cos 2\pi x \qquad \varphi_4 = 1$$
3. Let $\bar{f} = \hat{c}_1\varphi_1 + \cdots + \hat{c}_n\varphi_n$ be the projection of f on the orthogonal set $(\varphi_1, \ldots, \varphi_n)$. Show that $f - \bar{f}$ is orthogonal to each of the functions $(\varphi_1, \ldots, \varphi_n)$.
4. Find the projection of the function $\sin \pi x$ on the orthogonal set $(1, x - \frac{1}{2})$ on the interval $0 \le x \le 1$ and compute the minimum distance $d_{\min}$.
5. Find the projection of the function $f(x) = \cos^2 x$ on the orthogonal set $(1, \cos x, \cos 2x)$ on the interval $-\pi \le x \le \pi$.
6. Let $\varphi_1(x) = 1$, $\varphi_2(x) = x/|x|$, $\varphi_3(x) = x^2 - \frac{1}{3}$ for $-1 \le x \le 1$.
   (a) Show that $(\varphi_1, \varphi_2, \varphi_3)$ form an orthogonal set.
   (b) Find the projection of $f(x) = x$ on this orthogonal set and compute the minimum distance $d_{\min}$.
7. Let $(\varphi_1, \varphi_2, \varphi_3)$ be an orthonormal set of functions on the interval $-1 \le x \le 1$, and let f be any function of the form $f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x)$.
   (a) Show that $\|f\|^2 = a_1^2 + a_2^2 + a_3^2$.
   (b) Show that $\langle f, \varphi_1 \rangle = a_1$, $\langle f, \varphi_2 \rangle = a_2$, $\langle f, \varphi_3 \rangle = a_3$.
8. Let $(\varphi_1, \varphi_2, \varphi_3)$ be an orthonormal set of functions on the interval $-1 \le x \le 1$, and let $f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x)$, $g(x) = b_1\varphi_1(x) + b_2\varphi_2(x) + b_3\varphi_3(x)$.
   (a) Show that $\langle f, g \rangle = a_1b_1 + a_2b_2 + a_3b_3$.
   (b) Discuss the relation with the three-dimensional dot product formula.
9. Define the angle between two nonzero functions φ, ψ by the formula $\cos\theta = \langle \varphi, \psi \rangle / \|\varphi\| \|\psi\|$, $0 \le \theta \le \pi$.
   (a) If φ and ψ are orthogonal, show that $\theta = \pi/2$.
   (b) If φ and ψ are proportional, show that $\theta = 0$ or $\theta = \pi$.
   (c) If $\theta = 0$ or π, does it follow that φ and ψ are necessarily proportional? (*Hint:* Compute $\|\varphi - c\psi\|^2$ and write it as a perfect square.)
   (d) Compute θ if $\varphi(x) = 1$, $\psi(x) = x$ for $0 \le x \le 1$.
10. (a) Apply the Gram-Schmidt procedure to obtain orthogonal functions beginning with the functions $\varphi_1 = 1$, $\varphi_2 = x$, $\varphi_3 = x^2$ for $-1 \le x \le 1$.
    (b) Find the orthonormal set corresponding to the orthogonal set found in part (a).

11. Prove that the inner product defined by (0.3.1) satisfies $\langle \varphi_1, \psi_1 + \psi_2 \rangle = \langle \varphi_1, \psi_1 \rangle + \langle \varphi_1, \psi_2 \rangle$.
12. Prove that the inner product defined by (0.3.1) satisfies $\langle \varphi_1 + \varphi_2, \psi_1 \rangle = \langle \varphi_1, \psi_1 \rangle + \langle \varphi_2, \psi_1 \rangle$.
13. Prove that the inner product defined by (0.3.1) satisfies $\langle a\varphi_1, \psi_1 \rangle = a\langle \varphi_1, \psi_1 \rangle$.
14. Prove that the inner product defined by (0.3.1) satisfies $\langle \varphi_1, a\psi_1 \rangle = a\langle \varphi_1, \psi_1 \rangle$.
15. Prove the complex form of Schwarz's inequality. [*Hint:* Examine the non-negative quadratic polynomial $G(t,s) = \|t\psi - s\varphi e^{-i\theta}\|^2$, where the inner product has the polar form $\langle \varphi, \psi \rangle = Re^{i\theta}$. Check that the discriminant $= R^2 = \|\varphi\|^2\|\psi\|^2 - \langle \varphi, \psi \rangle^2 \geq 0$.]

CHAPTER 1

FOURIER SERIES

INTRODUCTION

Many of the classical partial differential equations with boundary conditions have separated solutions that involve sums of trigonometric functions. This leads to the theory of Fourier series, which is developed here in its own right. This chapter explores the basic properties of Fourier series, including a discussion of convergence and the closely related Sturm-Liouville eigenvalue problem. Basic definitions and examples are given in Sec. 1.1; the next two sections treat more theoretical material and can be omitted without loss of continuity. The basic material resumes in Sec. 1.4 with Parseval's theorem and its applications. The complex Fourier series in Sec. 1.5 are not used until the discussion of Fourier transforms in Chapter 5, but the Sturm-Liouville theory of Sec. 1.6 is used immediately in Chapter 2.

1.1. Definitions and Examples

A *trigonometric series* is a function of the form

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \tag{1.1.1}$$

where A_0, A_1, $B_1, \ldots$ are constants. This is a series of sines and cosines whose frequencies are multiples of a basic angular frequency π/L and whose amplitudes are arbitrary. In this chapter we will explore the possibility of expanding a large class of functions $f(x), -L < x < L$, as trigonometric series. We first prove directly that this set of functions is orthogonal on the interval $-L < x < L$.

1.1.1. Orthogonality relations. In the following discussion the indices m, n assume the values $0, 1, 2, \ldots$.

PROPOSITION 1.1.1. *We have the orthogonality relations*

$$\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases} \tag{1.1.2}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 0 & n = m = 0 \end{cases} \tag{1.1.3}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0 \qquad \text{all} \quad m, n \tag{1.1.4}$$

Proof. We use the trigonometric identities

$$\begin{aligned} \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \\ \sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \\ \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)] \end{aligned} \tag{1.1.5}$$

Thus to prove (1.1.2), we have, for $n \neq m$,

$$\begin{aligned} \int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx &= \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(n-m)\pi x}{L} + \cos\frac{(n+m)\pi x}{L}\right]dx \\ &= \frac{L}{2\pi}\left[\frac{\sin(n-m)\pi x/L)}{n-m}\bigg|_{-L}^{L} + \frac{\sin(n+m)\pi x/L)}{n+m}\bigg|_{-L}^{L}\right] \\ &= 0 \end{aligned}$$

If $n = m \neq 0$, we have

$$\begin{aligned} \int_{-L}^{L}\cos^2\frac{n\pi x}{L}\,dx &= \frac{1}{2}\int_{-L}^{L}\left(1+\cos\frac{2n\pi x}{L}\right)dx \\ &= \frac{1}{2}\left(2L + \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\bigg|_{-L}^{L}\right) \\ &= L \end{aligned}$$

Finally, if $n = m = 0$, the integral is $2L$. This completes the proof of (1.1.2). The proofs of (1.1.3) and (1.1.4) are left as exercises. •

Having established the orthogonality and performed the computation of these integrals, we can now define the Fourier series of a function $f(x)$, $-L < x < L$.

1.1.2. Definition of Fourier coefficients. In order to define the Fourier series of a function, it suffices to define the Fourier coefficients A_n, B_n, which is done as follows.

Definition Let $f(x)$, $-L < x < L$, be a real-valued function. The *Fourier series* of f is the trigonometric series (1.1.1) where (A_n, B_n) are defined by

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \tag{1.1.6}$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \ldots \tag{1.1.7}$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \ldots \tag{1.1.8}$$

These definitions were suggested in Chapter 0, where we showed that for any orthogonal set $(\varphi_1, \ldots, \varphi_N)$, the minimum of $\|f - \sum_{n=1}^{N} c_n\varphi_n\|^2$ is determined by choosing $(c_1, \ldots, c_N)$ as the Fourier coefficients $\langle f, \varphi_n\rangle / \langle \varphi_n, \varphi_n\rangle, 1 \le n \le N$.

1.1.3. Even functions and odd functions. In order to simplify the computation of Fourier series of many functions encountered in practice, we often exploit symmetry arguments. A function $f(x)$, $-L < x < L$, is *even* if $f(-x) = f(x)$, $-L < x < L$. A function $f(x)$, $-L < x < L$, is *odd* if $f(-x) = -f(x)$, $-L < x < L$. For example, $f(x) = x$, $f(x) = x^3$, and $f(x) = \sin x$ are odd functions, whereas $f(x) = x^2$, $f(x) = x^4$, and $f(x) = \cos x$ are even functions. Of course, many functions are neither even nor odd, for example, $f(x) = x + x^2$. The product of two even functions is an even function, the product of an odd function and an even function is an odd function, and the product of two odd functions is an even function. These properties result from the multiplication facts $(+1)(+1) = +1$, $(-1)(+1) = -1$, and $(-1)(-1) = +1$. If $f(x)$, $-L < x < L$, is an odd function, the integral $\int_{-L}^{L} f(x)dx = 0$. This may be seen in detail by writing

$$\begin{aligned}
\int_{-L}^{0} f(x)dx &= -\int_{L}^{0} f(-t)dt \quad (x = -t, dx = -dt) \\
&= \int_{0}^{L} f(-t)dt \qquad \left(\int_0^L = -\int_L^0\right) \\
&= -\int_{0}^{L} f(t)dt \qquad \text{(oddness)}
\end{aligned}$$

But t is a dummy variable of integration; thus

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx = -\int_{0}^{L} f(x)dx + \int_{0}^{L} f(x)dx = 0$$

In a similar fashion it may be shown that if $f(x)$, $-L < x < L$, is an even function, then $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$.

PROPOSITION 1.1.2. *If $f(x)$, $-L < x < L$, is an even function, then $B_n = 0$, $n = 1, 2, \ldots$. If $f(x)$, $-L < x < L$, is an odd function, then $A_n = 0$, $n = 0, 1, 2, \ldots$.*

Proof. To prove these facts, we first note that $\sin(n\pi x/L)$ is an odd function and $\cos(n\pi x/L)$ is an even function since $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$. Now, if $f(x)$, $-L < x < L$, is an even function, the product $f(x)\sin(n\pi x/L)$ is an odd function and we have $B_n = 0$. If $f(x)$, $-L < x < L$, is an odd function, the product $f(x)\cos(n\pi x/L)$ is an odd function and we have $A_n = 0$. •

EXAMPLE 1.1.1. *Compute the Fourier series of $f(x) = x$, $-L < x < L$.*

Solution. $f(x)$, $-L < x < L$, is an odd function; therefore $A_n = 0$. To compute B_n, we note that $f(x)\sin(n\pi x/L)$ is an even function; thus

$$\begin{aligned} B_n &= \frac{1}{L}\int_{-L}^{L} x \sin\frac{n\pi x}{L}\,dx \\ &= \frac{2}{L}\int_{0}^{L} x \sin\frac{n\pi x}{L}\,dx \end{aligned}$$

We integrate by parts with $u = x$, $dv = \sin(n\pi x/L)\,dx$. Thus

$$B_n = \frac{2}{L}\left(-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\bigg|_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\,dx\right)$$

The last integral is zero, and we have $B_n = -(2L/n\pi)\cos n\pi = (2L/n\pi)(-1)^{n+1}$. Therefore the Fourier series of $f(x) = x$, $-L < x < L$, is

$$\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L} \quad \bullet$$

EXAMPLE 1.1.2. *Compute the Fourier series of $f(x) = |x|$, $-L < x < L$.*

Solution. $f(x)$, $-L < x < L$, is an even function; therefore $B_n = 0$. To compute A_n, we note that the product $f(x)\cos(n\pi x/L)$ is an even function; thus, for $n \neq 0$,

$$A_n = \frac{1}{L}\int_{-L}^{L} |x| \cos\frac{n\pi x}{L}\,dx \tag{1.1.9}$$

$$= \frac{2}{L}\int_{0}^{L} x \cos\frac{n\pi x}{L}\,dx \tag{1.1.10}$$

We integrate by parts with $u = x$, $dv = \cos(n\pi x/L)dx$. Thus

$$A_n = \frac{2}{L}\left(x\frac{L}{n\pi}\sin\frac{n\pi x}{L}\bigg|_0^L - \frac{L}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\,dx\right)$$

The first term is zero at both endpoints $x = 0$, $x = L$, while the integral can be evaluated as $\int_0^L \sin(n\pi x/L)\,dx = (L/n\pi)[1 - (-1)^n]$. Thus we have $A_n = -(2L/n^2\pi^2)[1-(-1)^n]$ for $n \neq 0$. For $n = 0$, we have $A_0 = (1/L)\int_0^L x\,dx = L/2$. Therefore the Fourier series of $f(x) = |x|$, $-L < x < L$, is

$$\frac{L}{2} - \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos\frac{n\pi x}{L}$$

This may also be written as

$$(L/2) - (4L/\pi^2)\sum_{m=1}^{\infty}\cos[(2m-1)\pi x/L]/(2m-1)^2$$

by writing $n = 2m-1$ and noting that $1-(-1)^n = 0$ if n is even and $1-(-1)^n = 2$ if n is odd. •

It will be shown in Sec. 1.2 that these Fourier series are convergent and that the equation

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)$$

is valid for $-L < x < L$. We illustrate this graphically for the preceding two examples. To do this, we define the *partial sum of order N* of a trigonometric series as the function

$$f_N(x) = A_0 + \sum_{n=1}^{N}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)$$

In Figs. 1.1.1 and 1.1.2 we give the partial sums for the Fourier series of the preceding two examples.

The method of these two examples may be extended to compute the Fourier series of any polynomial $f(x) = c_0 + c_1x + \cdots + c_kx^k$. To do this, it is sufficient to handle each term separately and integrate by parts. Thus we have the reduction formulas

$$\begin{aligned}\int_{-L}^{L} x^k\sin\frac{n\pi x}{L}\,dx &= -\frac{Lx^k}{n\pi}\cos\frac{n\pi x}{L}\bigg|_{-L}^{L} + \frac{Lk}{n\pi}\int_{-L}^{L}x^{k-1}\cos\frac{n\pi x}{L}\,dx\\ \int_{-L}^{L} x^k\cos\frac{n\pi x}{L}\,dx &= \frac{Lx^k}{n\pi}\sin\frac{n\pi x}{L}\bigg|_{-L}^{L} - \frac{Lk}{n\pi}\int_{-L}^{L}x^{k-1}\sin\frac{n\pi x}{L}\,dx\end{aligned}$$

Proceeding inductively, we can compute the necessary integrals.

If a function $f(x)$, $-L < x < L$, can be written as a finite trigonometric sum, then its Fourier series is that trigonometric sum. For example, the Fourier

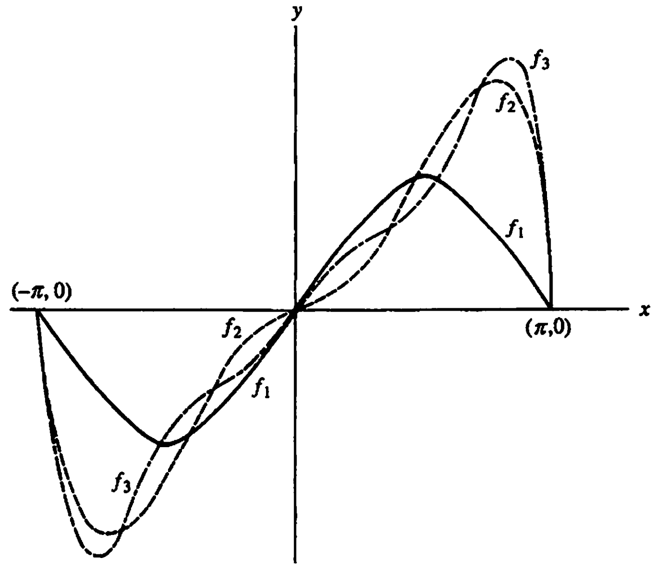


FIGURE 1.1.1 Graphs of the partial sums $f_N(x)$ for $N = 1, 2, 3$ of the Fourier series of $f(x) = x$, $-\pi < x < \pi$.

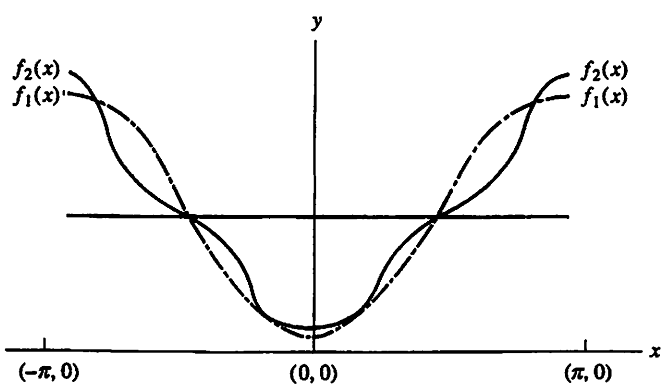


FIGURE 1.1.2 Graphs of the partial sums $f_N(x)$ for $N = 0, 1, 2$ of the Fourier series of $f(x) = |x|$, $-\pi < x < \pi$.

series of $f(x) = \sin^2 x$, $-\pi < x < \pi$, can be obtained by observing that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$; thus $B_n = 0$ for all n, while $A_0 = \frac{1}{2}$, $A_2 = -\frac{1}{2}$, and $A_n = 0$ for $n = 1, 3, 4, 5, \ldots$. It is not necessary to perform any integrations to find the Fourier series in this case.

1.1.4. Periodic functions. We now discuss Fourier series in the context of periodic functions.

Definition A function $f(x)$, $-\infty < x < \infty$, is *2L-periodic* if

$$f(x+2L) = f(x) \qquad -\infty < x < \infty$$

For example, $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ are $2L$-periodic for $n = 1, 2, \ldots$ since

$$\begin{aligned} \sin\frac{n\pi}{L}(x+2L) &= \sin\left(\frac{n\pi x}{L} + 2n\pi\right) = \sin\frac{n\pi x}{L} \\ \cos\frac{n\pi}{L}(x+2L) &= \cos\left(\frac{n\pi x}{L} + 2n\pi\right) = \cos\frac{n\pi x}{L} \end{aligned}$$

The sum, difference, or product of any two $2L$-periodic functions is again $2L$-periodic. Therefore any convergent trigonometric series defines a $2L$-periodic function $f(x)$, $-\infty < x < \infty$. Conversely, we can speak of the Fourier series of a $2L$-periodic function $f(x)$, $-\infty < x < \infty$, by restricting x to $-L < x < L$ and computing the Fourier series as we have just done.

EXAMPLE 1.1.3. *Compute the Fourier series of the 2L-periodic function* $f(x) = -1$ *if* $(2n-1)L < x < 2nL$, $f(x) = 1$ *if* $2nL < x < (2n+1)L$, $n = 0, \pm 1, \pm 2, \ldots$.

Solution. f is an odd function, and thus $A_n = 0$, $B_n = (2/L)\int_0^L \sin n\pi x/L\, dx = (2/L)(L/n\pi)[1 - (-1)^n]$. The Fourier series is $(2/\pi)\sum_{n=1}^{\infty}[1 - (-1)^n] \times \sin(n\pi x/L)/n$. •

1.1.5. Implementation with Mathematica. Let us redo Example 1.1.1 using Mathematica. The Fourier series of $f(x) = x, -\pi < x < \pi$, was found to be

$$2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin kx}{k}$$

We first define a function of two variables,

```
F[x_,n_]:=2 Sum[(((-1)^(k+1))/k) Sin[k x], {k,1,n}]
```

and a plot-valued function by

```
F[n_]:=Plot[F[x,n],{x,-Pi,Pi}]
```

By typing **Enter**, we record the values of these functions. The correct input can be verified by typing **?F**. To verify the first three terms of the series, move the cursor to a new cell and type **F[x,3]** followed by **Enter**. Mathematica should respond with

```
                  Sin [2x]  Sin[3x]
Out[2] =2(Sin[x]- ------- + -------)
                     2         3
```

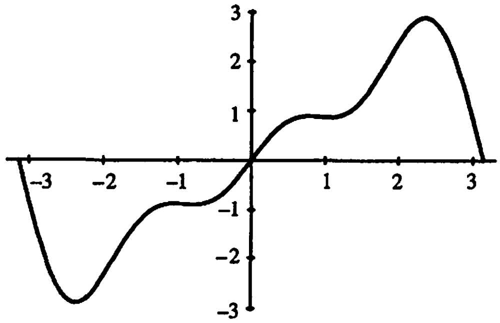


FIGURE 1.1.3 A three-term Fourier series.

To graph the function $F[x, 3]$, type **F[3]** instead of **F[x,3]**, and the result is as shown in Fig. 1.1.3.

Mathematica can also be used to compute the Fourier coefficients of a piecewise smooth function $f(x), -L < x < L$. To do this, we make the following commands:

```
A0[L_,f_]:=(1/(2Pi)) Integrate[f[x], {x,-L,L}]
A[n_,L_,f_]:=(1/(Pi)) Integrate[f[x] Cos[n x], {x,-L,L}]
B[n_,L_,f_]:=(1/(Pi)) Integrate[f[x] Sin[n x], {x,-L,L}]
```

Then we can define a function $f(x)$ in Mathematica and use the above definitions to compute the Fourier coefficients. For example, consider $f(x) = e^x, -L < x < L$. To enter this, we type

```
f[x_]:=E^x
```

and then type

```
A[n,L,f]
```

which produces the output

```
              n    L
          (-1)    E
Out[2]= ----------
                2  2
              Pi  n
        L(1+ ------)
                 2
                L
```

1.1.6. Fourier sine and cosine series. Suppose we are given a function $f(x), 0 < x < L$, and we desire a Fourier series representation. To get this, we extend f to the interval $-L < x < L$ and then compute the Fourier coefficients.

There are two natural ways of doing this, giving rise to the Fourier sine series and the Fourier cosine series.

One way of extending f is to define a new function f_O by

$$f_O(x) = \begin{cases} f(x) & 0 < x < L \\ -f(-x) & -L < x < 0 \\ 0 & x = 0 \end{cases} \tag{1.1.11}$$

f_O is called the *odd extension* of f to $(-L, L)$. It is an odd function, and therefore its Fourier coefficients are given as follows:

$$A_n = 0 \qquad n = 0, 1, \ldots \tag{1.1.12}$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f_O(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \tag{1.1.13}$$

Therefore we have the *Fourier sine series*

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad 0 \le x \le L$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Another way of extending f to the interval $(-L, L)$ is to define

$$f_E(x) = \begin{cases} f(x) & 0 < x < L \\ f(-x) & -L < x < 0 \\ 0 & x = 0 \end{cases} \tag{1.1.14}$$

f_E is called the *even extension* of f to $(-L, L)$. It is an even function defined on the interval $(-L, L)$. [Of course, we could define $f_E(0) = \lim_{x\to 0} f(x)$, if this limit exists. The definition $f_E(0) = 0$ is completely arbitrary.] The Fourier coefficients of f_E are as follows:

$$B_n = 0 \qquad n = 1, 2, \ldots \tag{1.1.15}$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f_E(x)dx = \frac{1}{L}\int_0^L f(x)dx \tag{1.1.16}$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f_E(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \tag{1.1.17}$$

Therefore we have the *Fourier cosine series*

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \qquad 0 \le x \le L$$

where

$$A_0 = \frac{1}{L}\int_0^L f(x)dx$$

and

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \ldots$$

EXAMPLE 1.1.4. *Compute the Fourier sine series of $f(x) = 1$, $0 < x < L$.*

Solution. We have

$$B_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx = -\frac{2L}{Ln\pi}\cos\frac{n\pi x}{L}\Big|_0^L = \frac{2}{n\pi}[1-(-1)^n]$$

The Fourier sine series is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{n\pi x}{L} \quad \bullet$$

We now give an alternative method for computing the Fourier sine series of certain functions that satisfy *boundary conditions.* Let $f(x)$, $0 \le x \le L$, be a function with $f(0) = 0$, $f(L) = 0$, and $f''(x)$ continuous for $0 \le x \le L$. Then

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \\ &= \frac{2}{n\pi}f(x)\cos\frac{n\pi x}{L}\Big|_L^0 + \frac{2}{n\pi}\int_0^L f'(x)\cos\frac{n\pi x}{L}\,dx \end{aligned} \tag{1.1.18}$$

The first term is zero, and the second term can be integrated again by parts, with the result

$$B_n = -\left(\frac{L}{n\pi}\right)^2\frac{2}{L}\int_0^L f''(x)\sin\frac{n\pi x}{L}\,dx$$

Therefore the Fourier sine series of $f(x)$, $0 < x < L$, is obtained from the Fourier sine series of $f''(x)$, $0 < x < L$, by multiplication of the nth term of the series by $-(L/n\pi)^2$.

EXAMPLE 1.1.5. *Find the Fourier sine series of $f(x) = x^3 - L^2x$, $0 < x < L$.*

Solution. The function satisfies $f(0) = 0$, $f(L) = 0$ with $f''(x) = 6x$. The Fourier sine series of $6x$ is $(12L/\pi)\sum_1^\infty(-1)^{n+1}\sin(n\pi x/L)/n$. Therefore the Fourier sine series of $f(x)$ is $(12L^3/\pi^3)\sum_1^\infty(-1)^n\sin(n\pi x/L)/n^3$. •

EXERCISES 1.1

In Exercises 1 to 10, compute the Fourier series of the indicated functions.

1. $f(x) = x^2$, $-L < x < L$
2. $f(x) = x^3$, $-L < x < L$
3. $f(x) = |x|^3$, $-L < x < L$
4. $f(x) = e^x$, $-L < x < L$
5. $f(x) = \sin^2 2x$, $-\pi < x < \pi$

6. $f(x) = \cos^3 x,\ -\pi < x < \pi$
7. $f(x) = 0$ if $-L < x < 0$ and $f(x) = 1$ if $0 \le x < L$
8. $f(x) = 0$ if $-L < x < 0$ and $f(x) = x$ if $0 \le x < L$
9. $f(x) = 0$ if $-\pi < x < 0$ and $f(x) = \sin x$ if $0 \le x < \pi$
10. $f(x) = \sinh x = \frac{1}{2}(e^x - e^{-x}),\ -\pi < x < \pi$
11. Prove the orthogonality relations (1.1.3). [*Hint:* Use the trigonometric identities (1.1.5).]
12. Prove the orthogonality relations (1.1.4). [*Hint:* Use the trigonometric identities (1.1.5).]
13. Prove the following facts about even and odd functions:
    (a) The product of two even functions is even.
    (b) The product of two odd functions is even.
    (c) The product of an even function and an odd function is odd.
    (d) Which of statements (a), (b), (c) remains true if the word "product" is replaced by "sum"?
14. Let f be an arbitrary function. Show that there is an odd function f_1 and an even function f_2 such that $f = f_1 + f_2$.
15. Which of the following functions are even, odd, or neither?
    (a) $f(x) = x^3 - 3x$ (b) $f(x) = x^2 + 4$
    (c) $f(x) = \cos 3x$ (d) $f(x) = x^3 - 3x^2$
    (e) $f(x) = \sin x - 3x^5$ (f) $f(x) = |x| \sin x$
    (g) $f(x) = x^2 - \cos x$ (h) $f(x) = \cos^3 x$
16. Find the Fourier sine series for the following functions:
    (a) $f(x) = x,\ 0 \le x \le L$ (b) $f(x) = x^2,\ 0 \le x \le L$
    (c) $f(x) = e^x,\ 0 \le x \le L$ (d) $f(x) = x^3,\ 0 \le x \le L$
    (e) $f(x) = \sin x,\ 0 \le x \le L$ (f) $f(x) = \cos x,\ 0 \le x \le L$
17. Find the Fourier cosine series for the functions in Exercise 16.
18. Let $f(x),\ -L < x < L$, be an odd function that satisfies the symmetry condition

$$f(L - x) = f(x)$$

Show that

$$\begin{aligned} A_n &= 0 \quad \text{for all } n \\ B_n &= 0 \quad \text{for all even } n \end{aligned}$$

19. Let $f(x),\ -L < x < L$, be an odd function that satisfies the symmetry condition

$$f(L - x) = -f(x)$$

Show that

$$\begin{aligned} A_n &= 0 \quad \text{for all } n \\ B_n &= 0 \quad \text{for all odd } n \end{aligned}$$

20. A function $f(x), 0 < x < \pi/2$, is to be expanded into a Fourier series

$$f(x) = A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$$

By extending f to $-\pi < x < \pi$ in four different ways, give four different prescriptions for finding the Fourier coefficients $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$. (*Hint:* There are two choices for extending f to $0 < x < \pi$ and two more choices for further extending f to $-\pi < x < \pi$.)
21. Illustrate the expansions of Exercise 20 with $f(x) = 1$, $0 < x < \pi/2$. Find the four different Fourier series.

For each of the functions in Exercises 22 to 29 state whether or not it is periodic and find the smallest period.

22. $f(x) = \sin \pi x$
23. $f(x) = \sin 2x + \sin 3x$
24. $f(x) = \sin 4x + \cos 6x$
25. $f(x) = \sin x + \sin \pi x$
26. $f(x) = x - [x]$ ($[x] =$ integer part of x)
27. $f(x) = \tan x$
28. $f(x) = \sum_{n=1}^{\infty} (-1)^n x^{2n}/(2n)!$
29. $f(x) = \sin x^2$
30. Compute the Fourier sine series of $f(x) = x^2 - Lx$, $0 < x < L$.
31. Compute the Fourier sine series of $f(x) = x^4 - 2Lx^3 + L^3x$, $0 < x < L$.
32. Let $f(x)$, $-L < x < L$, be an even function. Show that

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

33. Show that the derivative of an even function is an odd function.
34. Show that the derivative of an odd function is an even function.

1.2. Convergence of Fourier Series[1]

In this section we discuss the validity of the equation

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L}\right)$$

where (A_n, B_n) are the Fourier coefficients of the function $f(x)$, $-L < x < L$. For simplicity in writing, we take $L = \pi$ in the exposition; all results obtained can be transformed to the interval $-L < x < L$ by the change of variable $x' = \pi x/L$.

[1]This section treats theoretical material and can be omitted without loss of continuity.

1.2.1. Piecewise smooth functions. Recall that a function f is *continuous* at x if $\lim_{y\to x} f(y) = f(x)$. Not all Fourier series converge, even if we impose the restriction that their functions are continuous. In fact, there exist continuous functions on $[-\pi, \pi]$ whose Fourier series diverge at an infinite number of points! We therefore need to focus our attention on another class of functions, the so-called piecewise smooth functions. We first define the concept of a piecewise continuous function.

Definition A function $f(x)$, $a < x < b$, is *piecewise continuous* if there is a finite set of points $a = x_0 < x_1 < \cdots < x_p < x_{p+1} = b$ such that

$$f \text{ is continuous at } x \neq x_i, \quad i = 1, \ldots, p \tag{1.2.1}$$

$$\lim_{\epsilon\to 0} f(x_i + \epsilon) \text{ exists} \quad i = 0, \ldots, p \tag{1.2.2}$$

$$\lim_{\epsilon\to 0} f(x_i - \epsilon) \text{ exists} \quad i = 1, \ldots, p+1 \tag{1.2.3}$$

The limit (1.2.2) is denoted $f(x_i+0)$ and is called the *right-hand limit.* Likewise, the limit (1.2.3) is denoted $f(x_i - 0)$ and is called the *left-hand limit.* These are supposed to be finite.

Definition A function $f(x)$, $a < x < b$, is said to be *piecewise smooth* if f and all of its derivatives are piecewise continuous.

Of course, we assume that the subdivision points $x_0 < x_1 < \cdots < x_{p+1}$ are the same for f and all of its derivatives. With this definition, the derivative of a piecewise smooth function is again piecewise smooth.

If $f(x)$, $a < x < b$, is piecewise smooth, then $f'(x)$ exists except for $x = x_1, \ldots, x_p$. This is the *piecewise derivative* of f. Many of the usual operations with ordinary derivatives are valid for piecewise derivatives; the sum, difference, and product rules are valid except at the subdivision points $(x_1, \ldots, x_p)$. The quotient rule is also valid unless the denominator is zero. The fundamental theorem of calculus must be modified for piecewise smooth functions to the form

$$f(b-0) - f(a+0) = \int_a^b f'(x)dx + \sum_{i=1}^{p}[f(x_i+0) - f(x_i-0)]$$

Indeed, on each interval (x_i, x_{i+1}) we may apply the ordinary fundamental theorem of calculus in the form

$$f(x_{i+1}-0) - f(x_i+0) = \int_{x_i}^{x_{i+1}} f'(x)dx$$

Adding these equations for $i = 0, 1, \ldots, p$ gives the result.

If the piecewise smooth function $f(x)$, $a < x < b$, is also *continuous*, then the fundamental theorem of calculus may be applied in its usual form,

$$f(b-0) - f(a+0) = \int_a^b f'(x)dx$$

With these rules in mind, we may operate freely with piecewise smooth functions.

EXAMPLE 1.2.1.

$$f(x) = |x| \qquad -\pi < x < \pi$$

We take $x_0 = -\pi$, $x_1 = 0$, $x_2 = \pi$. Here f is continuous on the entire interval. f' is piecewise continuous, with $f'(0+0) = 1$, $f'(0-0) = -1$. All higher derivatives are zero; hence $f(x), -\pi < x < \pi$, is piecewise smooth.

EXAMPLE 1.2.2.

$$f(x) = \begin{cases} x^2 & -\pi < x < 0 \\ x^2 + 1 & 0 \le x < \pi \end{cases}$$

In this example f is continuous, with the exception of the point $x = 0$, where we have $f(0+0) = 1$ and $f(0-0) = 0$. All higher derivatives are piecewise continuous on $(-\pi, \pi)$, so $f(x), -\pi < x < \pi$, is piecewise smooth.

EXAMPLE 1.2.3.

$$f(x) = x|x| \qquad -\pi < x < \pi$$

In this case f and f' are continuous. f'' is continuous everywhere except at $x = 0$, where we have $f''(0+0) = 2$ and $f''(0-0) = -2$. All higher derivatives are zero; thus $f(x), -\pi < x < \pi$, is piecewise smooth.

EXAMPLE 1.2.4.

$$f(x) = x^2 \sin\frac{1}{x} \qquad -\pi < x < \pi$$

f is continuous on $(-\pi, \pi)$. f' is continuous on $(-\pi, \pi)$ with the exception of the point $x = 0$. However, $f'(0+0)$ and $f'(0-0)$ do not exist, so $f(x), -\pi < x < \pi$, is piecewise continuous but *is not piecewise smooth.*

EXAMPLE 1.2.5.

$$f(x) = \frac{1}{x^2 - \pi^2} \qquad -\pi < x < \pi$$

In this case $f(x), -\pi < x < \pi$, is continuous, but it is not piecewise continuous since $f(-\pi+0)$ and $f(\pi-0)$ are not finite. In particular, $f(x), -\pi < x < \pi$, is not piecewise smooth.

When working with piecewise smooth functions, we may omit the definition of $f(x)$ at the subdivision points $x_0, x_1, \ldots, x_{p+1}$. This causes no difficulty in the discussion of Fourier series, since the Fourier coefficients A_n, B_n are defined as integrals, which are insensitive to the value of $f(x)$ at a finite number of points. More precisely, if $f_1(x) = f_2(x)$, except for $x = x_0, x_1, \ldots, x_{p+1}$, then $\int_a^b f_1(x)dx = \int_a^b f_2(x)dx$. Therefore we see that the *Fourier coefficients do not depend on any of the numbers* $f(x_0), \ldots, f(x_{p+1})$.

Suppose $f(x), -\pi < x < \pi$, is piecewise smooth. We define the *2π-periodic extension* of f by setting

$$f(x + 2n\pi) = f(x) \qquad \text{where } x \in (-\pi, \pi)$$

and n is an integer (positive or negative).

It is left as an exercise to show that the 2π-periodic extension of f is piecewise smooth on any open interval and that it is periodic with period 2π. It is also left as an exercise to show that

$$\int_c^d f(x)dx = \int_a^b f(x)dx \qquad \text{if } d - c = 2\pi = b - a$$

where f is *any* 2π-periodic function.

Let $f(x)$, $-\pi < x < \pi$, be a piecewise smooth function and let $\bar{f}(x)$, $-\infty < x < \infty$, be the 2π-periodic extension of f; $\bar{f}$ is a 2π-periodic function with $\bar{f}(x) = f(x)$ for $-\pi < x < \pi$.

The following theorem relates the convergence of a Fourier series to the normalized values of the function

THEOREM 1.1. *(Convergence theorem). Let $f(x), -\pi < x < \pi$, be piecewise smooth. Then the Fourier series of f converges for all x to the value $\frac{1}{2}[\bar{f}(x+0) + \bar{f}(x-0)]$, where $\bar{f}$ is the 2π-periodic extension of f.*

From the periodicity, we see that the left-hand limit $\bar{f}(\pi - 0)$ is equal to the left-hand limit $\bar{f}(-\pi - 0)$, with a corresponding statement for the right-hand limit. Therefore the average of the left- and right hand-limits at the endpoints agrees with the common average of the function at the endpoints; in symbols,

$$\begin{aligned} \frac{1}{2}[\bar{f}(-\pi - 0) + \bar{f}(-\pi + 0)] &= \frac{1}{2}[f(-\pi + 0) + f(\pi - 0)] \\ \frac{1}{2}[\bar{f}(\pi - 0) + \bar{f}(\pi + 0)] &= \frac{1}{2}[f(-\pi + 0) + f(\pi - 0)] \end{aligned}$$

The restriction to the interval $-\pi < x < \pi$ is of no significance. It has been made here so that, instead of writing $\cos(m\pi x/L)$ and $\sin(m\pi x/L)$, we may write $\cos mx$ and $\sin mx$.

Before proceeding with the proof, we need two lemmas.

Lemma 1 (Riemann). If f and f' are piecewise continuous on (a, b), then

$$\lim_{\lambda \to \infty} \int_a^b f(x) \sin \lambda x \, dx = 0$$

Proof. First we write

$$\int_a^b f(x)\sin\lambda x\,dx = \sum_{i=0}^{p}\int_{x_i}^{x_{i+1}} f(x)\sin\lambda x\,dx$$

It remains to show that

$$\lim_{\lambda\to\infty}\int_{x_i}^{x_{i+1}} f(x)\sin\lambda x\,dx = 0$$

For this we integrate by parts, with $u = f(x)$, $dv = \sin\lambda x\,dx$. Thus

$$\int_{x_i}^{x_{i+1}} f(x)\sin\lambda x\,dx = \frac{-f(x)\cos\lambda x}{\lambda}\bigg|_{x_i}^{x_{i+1}} + \frac{1}{\lambda}\int_{x_i}^{x_{i+1}} f'(x)\cos\lambda x\,dx$$

Each of these tends to zero when $\lambda\to\infty$, completing the proof of Lemma 1. •

We wish to examine the limit as $N\to\infty$ of

$$f_N(x) = A_0 + \sum_{m=1}^{N}(A_m\cos mx + B_m\sin mx)$$

Using the definitions of A_0, A_m, B_m given in Sec. 1.1, formulas (1.1.6), (1.1.7), (1.1.8), we have

$$\begin{aligned} f_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{m=1}^{N}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)(\cos mt\cos mx + \sin mt\sin mx)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{m=1}^{N}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos m(t-x)dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{m=1}^{N}\cos m(t-x)\right]dt \end{aligned}$$

Clearly, it would be useful to be able to write

$$\frac{1}{2} + \sum_{m=1}^{N}\cos m(t-x)$$

in a more compact form. Therefore we formulate a second lemma.

Lemma 2. For any α real, $\alpha\neq 0, \pm 2\pi, \ldots$, we have

$$\frac{1}{2} + \cos\alpha + \cdots + \cos N\alpha = \frac{\sin(N+\frac{1}{2})\alpha}{2\sin\frac{1}{2}\alpha}$$

Proof. Setting $S = \frac{1}{2} + \cos\alpha + \cdots + \cos N\alpha$, we have

$$S\sin\alpha = \frac{1}{2}\sin\alpha + \sin\alpha\cos\alpha + \cdots + \sin\alpha\cos N\alpha$$

From the addition formulas

$$\begin{aligned}\sin(a+b) &= \sin a\cos b + \cos a\sin b\\ \sin(a-b) &= \sin a\cos b - \cos a\sin b\end{aligned}$$

we have

$$\cos a\sin b = \frac{1}{2}[\sin(a+b) - \sin(a-b)]$$

so that

$$\begin{aligned}S\sin\alpha &= \frac{1}{2}[\sin\alpha + \sin 2\alpha - 0 + \sin 3\alpha - \sin\alpha\\ &\quad + \cdots + \sin(N+1)\alpha - \sin(N-1)\alpha]\\ &= \frac{1}{2}[\sin N\alpha + \sin(N+1)\alpha]\end{aligned}$$

To complete the proof, we average the addition formula as follows:

$$\frac{1}{2}[\sin(a+b) + \sin(a-b)] = \sin a\cos b$$

Setting $a+b=(N+1)\alpha$, $a-b=N\alpha$, we take $a=(N+\frac{1}{2})\alpha$, $b=\frac{1}{2}\alpha$, so that

$$\frac{1}{2}[\sin N\alpha + \sin(N+1)\alpha] = \sin\left(N+\frac{1}{2}\right)\alpha\,\cos\frac{1}{2}\alpha$$

and

$$S = \frac{\sin(N+\frac{1}{2})\alpha\cos\frac{1}{2}\alpha}{\sin\alpha}$$

Substituting the identity $\sin\alpha = 2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha$ completes the proof of Lemma 2. (For a shorter proof of Lemma 2, using complex numbers, see Exercise 13 at the end of this section.) •

In view of Lemma 2, we can write

$$f_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(N+\frac{1}{2})(t-x)}{2\sin\frac{1}{2}(t-x)}\,dt$$

This form is preferable because it makes no mention of the Fourier coefficients $\{A_n\}$, $\{B_n\}$.

1.2.2. Dirichlet kernel. To proceed further, we make the definition

$$D_N(u) = \frac{\sin(N+\frac{1}{2})u}{2\pi\sin u/2} \qquad u \neq 0, \pm 2\pi, \pm 4\pi, \ldots$$

and by continuity we define $D_N(u) = (2N+1)/2\pi, u = 0, \pm 2\pi, \pm 4\pi, \ldots$.

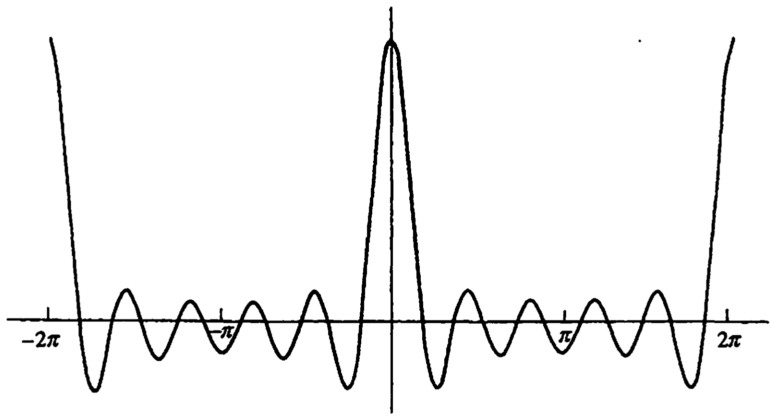


FIGURE 1.2.1 The Dirichlet kernel $D_N(u)$ for $N = 5$.

D_N is the *Dirichlet kernel*, an even, 2π-periodic function. From Lemma 2 we see that

$$\int_0^{\pi} D_N(u)du = \frac{1}{2} = \int_{-\pi}^{0} D_N(u)du$$

From Fig. 1.2.1 we see that $D_N(u)$ behaves roughly like a periodic function with period $2\pi/N$, except in the neighborhood of $u = 0, \pm 2\pi, \ldots$, where it is peaked. The most important property of the Dirichlet kernel is that it provides an explicit representation of the Fourier partial sum, through the formula

$$\boxed{f_N(x) = \int_{-\pi}^{\pi} f(t) D_N(t-x)\, dt} \tag{1.2.4}$$

1.2.3. Proof of convergence. To complete the proof of Theorem 1.1, we extend f to $\bar{f}$, a 2π-periodic function. Therefore the product $D_N(t-x)\bar{f}(t)$ is also a 2π-periodic function of t for each x. We now write

$$\begin{aligned}
f_N(x) &= \int_{-\pi}^{\pi} \bar{f}(t) D_N(t-x) dt \\
&= \int_{-\pi-x}^{\pi-x} \bar{f}(x+u) D_N(u) du && t - x = u \\
&= \int_{-\pi}^{\pi} \bar{f}(x+u) D_N(u) du && \text{periodicity} \\
&= \left\{ \int_{-\pi}^{0} + \int_{0}^{\pi} \right\} \bar{f}(x+u) D_N(u) du
\end{aligned}$$

We will analyze the two integrals separately and show that

$$\lim_{N\to\infty}\int_0^{\pi}\bar{f}(x+u)D_N(u)du = \frac{1}{2}\bar{f}(x+0) \tag{1.2.5}$$

$$\lim_{N\to\infty}\int_{-\pi}^{0}\bar{f}(x+u)D_N(u)du = \frac{1}{2}\bar{f}(x-0) \tag{1.2.6}$$

from which the result will follow. We carry out the analysis of only the first integral in detail; the second is identical in every respect. Define

$$g(u) \doteq [\bar{f}(x+u)-\bar{f}(x+0)]/u$$

Then

$$\int_0^{\pi}[\bar{f}(x+u)-\bar{f}(x+0)]D_N(u)du = \frac{1}{\pi}\lim_{N\to\infty}\int_0^{\pi} g(u)U(u)\sin\left(N+\frac{1}{2}\right)u\,du$$

where

$$\begin{aligned} U(u) &= \frac{u}{2\sin u/2} \qquad u\neq 0 \\ U(0) &= 1 \end{aligned}$$

Using L'Hospital's rule, we see that the function $U(u)$ is continuous and has a continuous derivative, $-\pi \le u \le \pi$. Similarly, we can use L'Hospital's rule to compute the limits

$$\lim_{u\downarrow 0} g(u) = \bar{f}'(x+0) \tag{1.2.7}$$

$$\lim_{u\downarrow 0} g'(u) = \frac{1}{2}\bar{f}''(x+0) \tag{1.2.8}$$

Therefore $g(u)$ is piecewise continuous with a piecewise continuous derivative. But $U(u)$ has a continuous derivative, and therefore the product $g(u)U(u)$ also has a piecewise continuous derivative. Applying Lemma 1, we have proved that

$$\lim_{N\to\infty}\int_0^{\pi} g(u)U(u)\sin\left(N+\frac{1}{2}\right)u\,du = 0$$

Writing this in terms of $\bar{f}$, we have

$$\lim_{N\to\infty}\int_0^{\pi}\bar{f}(x+u)D_N(u)du = \bar{f}(x+0)\lim_{N\to\infty}\int_0^{\pi}D_N(u)du = \frac{1}{2}\bar{f}(x+0)$$

which was to be proved. •

An examination of the graph of $D_N(u)$ (Fig. 1.2.1) helps to give an intuitive motivation of the proof. Since $\int_{-\pi}^{\pi}D_N(u)du = 1$, the graph suggests that, as N gets large, the area tends to concentrate around $u=0$, so that $\int_{-\pi}^{\pi}f(u)D_N(u)du$ tends to pick off the values of $f(u)$ near $u=0$. Thus

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}f(u)D_N(u)du = \frac{1}{2}[f(0+0)+f(0-0)]$$

for functions $f(x)$ that are piecewise smooth.

Having proved the convergence of the Fourier series, we can now obtain many useful conclusions. Referring to the first two examples in Sec. 1.1, we have the convergent Fourier series

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx = x \qquad -\pi < x < \pi$$

$$\frac{\pi}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos nx = |x| \qquad -\pi < x < \pi$$

Both of these examples are continuous functions, for which $f(x+0) = f(x-0) = f(x)$ for all x, $-\pi < x < \pi$. However, the periodic extension is not continuous in the first case, where $f(x) = x$, $-\pi < x < \pi$.

As an example of a discontinuous function, we have the convergent Fourier series

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx = \begin{cases} 1 & 0 < x < \pi \\ 0 & x = 0 \\ -1 & -\pi < x < 0 \end{cases}$$

These can also be used to obtain various numerical series. Taking $x = 0$ in the Fourier series for $|x|$, we have $0 = \pi/2 - (2/\pi)(2 + \frac{2}{9} + \frac{2}{25} + \cdots)$, $\pi^2/8 = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$. Similarly, taking $x = \pi/2$ in the third example, we obtain $\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$.

EXERCISES 1.2

1. Determine whether or not the indicated function is piecewise smooth.
   (a) $f(x) = |x|^{3/2}$, $-2 < x < 2$
   (b) $f(x) = [x] - x$, $0 < x < 3$ ($[x]$ = integer part of x)
   (c) $f(x) = x^4 \sin(1/x)$, $-1 < x < 1$
   (d) $f(x) = e^{-(1/x^2)}$, $-1 < x < 1$
2. Let $f(x) = x^2 \sin(1/x)$.
   (a) Show that $\lim_{x\to 0} f(x) = 0$.
   (b) Graph $f(x), -\pi < x < \pi$.
   (c) Show that $f'(0+0)$ does not exist by considering $f'(h)$ as $h \to 0$ through the values $1/2n\pi$ and $1/(2n+1)\pi$, $n = 1, 2, \ldots$.
3. Let f and g be piecewise smooth on (a, b).
   (a) Show that $f + g$ is piecewise smooth on (a, b).
   (b) Show that $f \cdot g$ is piecewise smooth on (a, b).
   (c) What restrictions must be made on g in order that f/g be piecewise smooth on (a, b)?

4. Let $\bar{f}$ be the 2π-periodic extension of the piecewise smooth function $f(x)$, $-\pi < x < \pi$.
   (a) Show that $\bar{f}(x), -\infty < x < \infty$, is piecewise smooth.
   (b) Show that $\bar{f}$ is 2π-periodic.
   (c) Show that
$$\int_c^d \bar{f}(x)dx = \int_a^b \bar{f}(x)dx \qquad \text{if } d - c = 2\pi = b - a$$
5. Define $U(0) = 1$ and
$$U(u) = \frac{u}{2\sin(u/2)} \qquad -\pi \le u \le \pi$$
Show that $U(u)$ is continuous and has a continuous derivative for $-\pi \le u \le \pi$. (*Hint:* Use L'Hospital's rule.)
6. Let $f(x)$, $a < x < b$, be a piecewise smooth function. Let $g(u) = [f(x+u) - f(x+0)]/u$ for $u \neq 0$. Show that $g(0+0) = f'(x+0)$, $g(0-0) = f'(x-0)$. (*Hint:* Use L'Hospital's rule.)
7. Let $g(u)$ be defined as in Exercise 6. Show that $g'(0+0) = \frac{1}{2}f''(x+0)$, $g'(0-0) = \frac{1}{2}f''(x-0)$.
8. Prove that $D_N(u)$ is even and 2π-periodic.
9. Use Lemma 1 and the properties of the Dirichlet kernel to compute the following limits:
   (a) $\lim_{N\to\infty} \int_{-\pi/2}^{\pi/2} D_N(u)du$ (b) $\lim_{N\to\infty} \int_0^{\pi/2} D_N(u)du$
   (c) $\lim_{N\to\infty} \int_{-\pi/6}^{\pi/6} D_N(u)du$ (d) $\lim_{N\to\infty} \int_{\pi/2}^{\pi} D_N(u)du$
10. What is the maximum value of $D_N(u)$, $-\pi \le u \le \pi$?
11. Find all solutions of the equation $D_N(u) = 0$.
12. Find all solutions of the equation $D_N'(u) = 0$.
13. There is another way of establishing Lemma 2. Recall that $e^{ix} = \cos x + i\sin x$.
   (a) Show that
$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$
   (b) Prove Lemma 2 using part (a) and the fact that for any complex number $r \neq 1$,
$$1 + r + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1} \qquad r \neq 1$$
14. This exercise establishes the formula
$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

(a) Let

$$f(u) = \left(\frac{1}{2\sin u/2} - \frac{1}{u}\right) \qquad u \neq 0 \qquad f(0) = 0$$

Show that f, f' are continuous on $(0, \pi)$. (The only trouble occurs at $u = 0$. Use L'Hospital's rule to show that the appropriate limits are finite.)

(b) Use Lemma 1 to conclude that

$$\lim_{N\to\infty} \int_0^\pi \left[\frac{1}{2\sin u/2} - \frac{1}{u}\right] \sin(N + \frac{1}{2})u\, du = 0$$

(c) Hence show that

$$\lim_{N\to\infty} \int_0^\pi D_N(u) du = \lim_{N\to\infty} \int_0^\pi \frac{\sin(N + \frac{1}{2})u}{u}\, du$$

(d) Make the appropriate substitution in the second definite integral and recall the appropriate facts about $D_N(u)$ to conclude that

$$\lim_{N\to\infty} \int_0^{(N+1/2)\pi} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

(e) If $(N - \frac{1}{2})\pi \leq X \leq (N + \frac{1}{2})\pi$, show that

$$\int_0^X \frac{\sin x}{x}\, dx = \int_0^{(N+1/2)\pi} \frac{\sin x}{x}\, dx + \epsilon_X$$

where $|\epsilon_X| \leq 1/(N - \frac{1}{2})$. Conclude that the improper integral converges to $\pi/2$ when $X \to \infty$.

15. (a) Set $x = \pi/2$ in the Fourier series for $f(x) = x$, $-\pi < x < \pi$, to obtain the formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(b) Set $x = \pi/4$ in the series part (a) to obtain

$$\frac{\pi}{4} = \sqrt{2}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots\right) - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

(c) Conclude from part (b) that

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \cdots$$

(d) If we set $x = \pi$ in the series in (a), we find that the series sums to zero. Why doesn't this contradict $f(x) = x$?

16. (a) Show that

$$x^2 = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \cdots + (-1)^m \frac{4}{m^2}\cos mx + \cdots$$

for $-\pi \leq x \leq \pi$.

(b) Setting $x = 0$ in (a), find the sum

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m^2}$$

(c) What is

$$\sum_{m=1}^{\infty} \frac{1}{m^2}$$

[*Hint:* Set $x = \pi$ in part (a).]

(d) What is

$$\sum_{m \text{odd}} \frac{1}{m^2}$$

[*Hint:* Add (b) and (c).]

17. Let $f(x) = x$, $-\pi < x < \pi$. What is the sum of the Fourier series for $x = -\pi$, $x = \pi$?
18. Let $f(x) = e^x$, $-\pi < x < \pi$. What is the sum of the Fourier series for $x = -\pi$, $x = \pi$?
19. Let $f(x)$, $g(x)$ be piecewise smooth functions for $a < x < b$. Show that

$$\begin{aligned}\int_a^b f(x)g'(x)dx + \int_a^b f'(x)g(x)dx = &-\sum_{i=1}^{p} [f(x_i + 0)g(x_i + 0) - f(x_i - 0)g(x_i - 0)] \\ &+ f(b-0)g(b-0) - f(a+0)g(a+0)\end{aligned}$$

20. Use Exercise 19 to prove the following integration-by-parts formula for piecewise smooth functions:

$$\begin{aligned}\int_a^b f(x)g'(x)dx = {} & f(b-0)g(b-0) - f(a+0)g(a+0) - \int_a^b f'(x)g(x)dx \\ &- \sum_{i=1}^{p} g(x_i - 0)[f(x_i + 0) - f(x_i - 0)] \\ &- \sum_{i=1}^{p} f(x_i - 0)[g(x_i + 0) - g(x_i - 0)] \\ &- \sum_{i=1}^{p} [f(x_i + 0) - f(x_i - 0)][g(x_i + 0) - g(x_i - 0)]\end{aligned}$$

21. By examining the proof of Theorem 1.1, show that the conclusion is valid if f, f', f'' are piecewise continuous.
22. On the basis of Exercise 21, for which $n \geq 1$, can we assert that the Fourier series of $x^n \sin 1/x$ is convergent for all x, $-\pi < x < \pi$?

23. Let $f(x)$, $-\pi < x < \pi$, be a piecewise smooth function with Fourier coefficients A_n, B_n. Apply Exercise 20 with $a = -\pi$, $b = \pi$, $g'(x) = \cos nx$ to find an asymptotic formula for A_n, B_n, $n \to \infty$.

1.3. Uniform Convergence and the Gibbs Phenomenon[2]

We have seen that the Fourier series of a piecewise smooth function converges to the function except at points of discontinuity, where it converges to the average of the function's left- and right-hand limits. Since we are interested in approximating functions by partial sums of their Fourier series, it is of interest how the Fourier series converge near a discontinuity, that is, how the partial sums of Fourier series behave near discontinuities of their functions. We turn first to an example.

1.3.1. Example of Gibbs overshoot. Consider the function

$$f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

The cosine coefficients are all zero (f is odd), and the sine coefficients are given by

$$B_n = \frac{2}{\pi}\int_0^{\pi} \sin nx\, dx = \frac{2}{n\pi}[1-(-1)^n] \qquad n = 1, 2, \ldots$$

The partial sum of the Fourier series is therefore

$$f_{2n}(x) = f_{2n-1}(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \cdots + \frac{\sin(2n-1)x}{2n-1}\right]$$

From the graph of Fig. 1.3.1 we see that, just before the discontinuity, the partial sums overshoot the right- and left-hand limits and then slope rapidly toward their mean. On the interval $-\pi \le x \le \pi$, f_1 has one maximum and one minimum, f_3 has three maxima and three minima, f_5 has five maxima and five minima, etc. We can actually calculate the overshoot by computing the derivative

$$f'_{2n-1}(x) = \frac{4}{\pi}[\cos x + \cos 3x + \cos 5x + \cdots + \cos(2n-1)x] \tag{1.3.1}$$

and solving the equation $f'_{2n-1}(x) = 0$.

To solve this equation, we multiply (1.3.1) by $\sin x$ and use the identity

$$\sin x \cos kx = \frac{1}{2}[\sin(k+1)x - \sin(k-1)x]$$

and get

$$\begin{aligned} \pi \sin x\, f'_{2n-1}(x) &= 2\left\{\sin 2x + \sum_{k=1}^{n-1}[\sin 2(k+1)x - \sin 2kx]\right\} \\ &= 2\sin 2nx \end{aligned}$$

[2]This section treats theoretical material and can be omitted without loss of continuity.

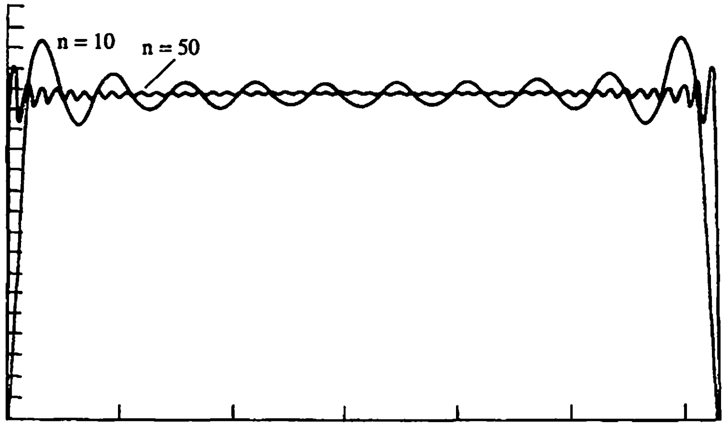


FIGURE 1.3.1 The Gibbs phenomenon for $n = 10$ and $n = 50$.

Therefore, the extrema occur at the points

$$2nx = \pm\pi, \pm 2\pi, \ldots, \pm 2n\pi$$

These points are equally spaced in $[-\pi, \pi]$. It is the maximum closest to the discontinuity (i.e., when $x = \pi/2n$) that is of interest, so we wish to compute

$$f_{2n-1}\left(\frac{\pi}{2n}\right) = \frac{4}{\pi}\left[\sin\frac{\pi}{2n} + \frac{1}{3}\sin\frac{3\pi}{2n} + \cdots + \frac{1}{2n-1}\sin\frac{(2n-1)\pi}{2n}\right]$$

for large n. The technique we will use for evaluating this sum consists of rewriting the sum so that it looks like the approximating sum of a Riemann integral and then evaluating the integral. Our answer will be exact when $n \uparrow \infty$ and so should give a good approximation for large n.

The function whose integral we will approximate is $g(x) = (\sin x)/x$. Consider the partition of $[0, \pi]$, given by the points $\{x_k\}$, where

$$\begin{aligned} x_k &= \frac{\pi k}{n} \qquad k = 1, \ldots, n \\ \Delta x_k &= \frac{\pi}{n} \end{aligned}$$

If we choose the midpoints x'_k of each of these intervals as our sampling points, then we have

$$\sum_{k=1}^{n} g(x'_k)\Delta x_k = \frac{\sin\pi/2n}{\pi/2n}\frac{\pi}{n} + \cdots + \frac{\sin(2n-1)\pi/2n}{(2n-1)\pi/2n}\frac{\pi}{n} \to \int_0^\pi \frac{\sin x}{x}\,dx$$

If we rearrange our sum, we see that it equals

$$\frac{2n}{\pi}\frac{\pi}{n}\left[\sin\frac{\pi}{2n}+\frac{\sin 3\pi/2n}{3}+\cdots+\frac{\sin(2n-1)\pi/2n}{2n-1}\right]=\frac{\pi}{2}f_{2n-1}\left(\frac{\pi}{2n}\right)$$

Therefore the *limit of the overshoot* is given by

$$\lim_{n\uparrow\infty} f_{2n-1}\left(\frac{\pi}{2n}\right)=\frac{2}{\pi}\int_0^\pi\frac{\sin x}{x}\,dx$$

We can approximate the integral numerically as follows:

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cdots$$

so

$$\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots$$

and

$$\begin{aligned}\frac{2}{\pi}\int_0^\pi\frac{\sin x}{x}\,dx &= \frac{2}{\pi}\int_0^\pi\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}\right)dx+\cdots\\ &= \frac{2}{\pi}\left(\pi-\frac{\pi^3}{18}+\frac{\pi^5}{600}-\frac{\pi^7}{35{,}280}\right)+\cdots\\ &= 2-\frac{\pi^2}{9}+\frac{\pi^4}{300}-\frac{\pi^6}{17{,}640}+\cdots\\ &= 2-1.11+0.33-0.04+\cdots\\ &= 1.18\ \text{ to two decimal places}\end{aligned}$$

This means that if we stand at any one point, we will land on the graph of $f(x)$ in the limit $n\uparrow\infty$. However, if we ride the crest of the worst point possible for each n, then we will never reach the graph of $f(x)$. When $n\uparrow\infty$, we will be left dangling approximately 1.18 units above the origin. This behavior can be described by saying that the partial sums *do not* converge *uniformly* to $f(x)$ (i.e., the *entire* curve is not arbitrarily close to the graph of f for sufficiently large n). Rather, they converge to the graph indicated in Fig. 1.3.2. This is known as the *Gibbs phenomenon.* Notice that the overshoot of 1.18 is 9 percent of the jump made at the discontinuity. This is characteristic of the overshoot due to any discontinuity in any piecewise smooth function f. In fact, we have the following general fact, whose proof is omitted.

Let f be piecewise smooth on $(-\pi,\pi)$. Then the amount of overshoot near a discontinuity, due to the Gibbs phenomenon, is approximately equal to

$$0.09|f(x_0+0)-f(x_0-0)|$$

for large n.

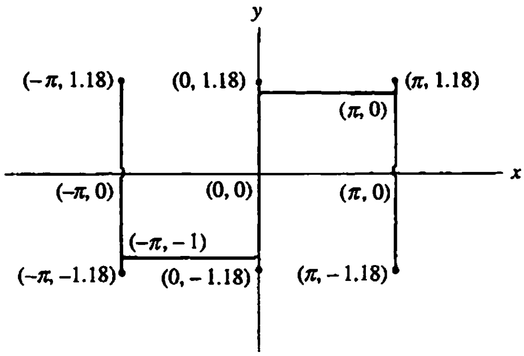


FIGURE 1.3.2 Limiting graph in Gibbs' phenomenon.

1.3.2. Implementation with Mathematica. The graphs of the Gibbs phenomenon can be easily produced using Mathematica. We will illustrate this with the function

$$f(x) = \begin{cases} -1 & -\pi < x \leq 0 \\ 1 & 0 < x < \pi \end{cases}$$

To implement this in Mathematica, we first define a step function by means of the "If" function:

```
u[a_,x_]:=If[a<x,1,0]
```

With this definition, the function f can be written

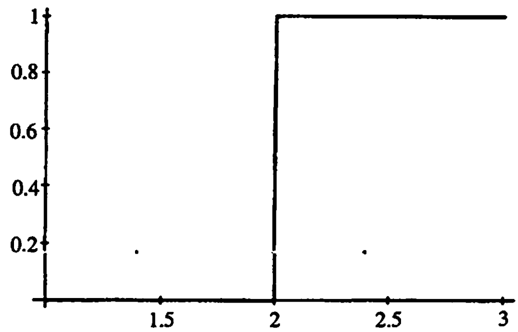
```
f[x_]:=1 - 2u[-Pi,x]+2u[0,x]
```

To see this in more detail, note that the function **If** takes three arguments; the first argument is a condition, the second argument is the value of the function when the condition is satisfied, and the third argument is the value of the function if the condition is not satisfied. In the case at hand, we see that if $-\pi < x < 0$, then the first condition is met but not the second, so that $f(x) = 1-2+0 = -1$. If $0 < x < \pi$, then both conditions are satisfied, so that $f(x) = 1 - 2 + 2 = 1$, as required. In case $x = 0$ only the first condition is satisfied, so that $f(0) = 1 - 2 + 0 = -1$, as required.

This function can be plotted in Mathematica by means of the commmand

```
U[a_]:=Plot[u[a,x],{x,a-1,a+1}]
```

For example, the graph of u[2,x] can be obtained by typing U[2]:



If we want to graph the Fourier series of f using Mathematica, we first recall the Fourier series representation for the partial sums:

$$f_{2n-1}(x) = \frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin((2k-1)x)}{2k-1}$$

To implement this in Mathematica, we define a function of two variables as follows:

```
f[n_,x_]:=(4/Pi) Sum[(1/(2 k -1)) Sin[(2k-1) x],{k,1,n}]
```

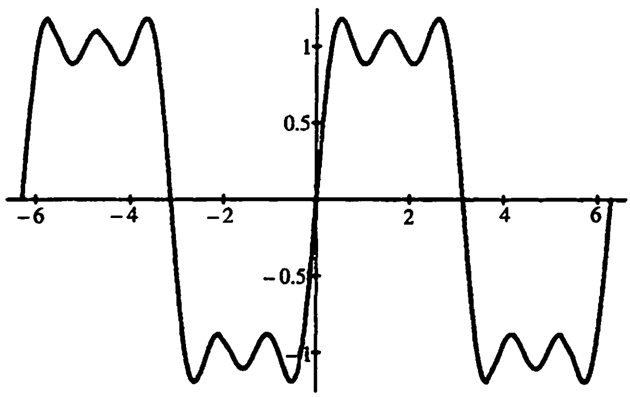
For example, if we now type f[3,x], we obtain the output

```
                 sin[3x]   Sin[5x]
     4(Sin[x] + -------+ --------)
                   3         5
Out[4]=---------------------------------
                 Pi
```

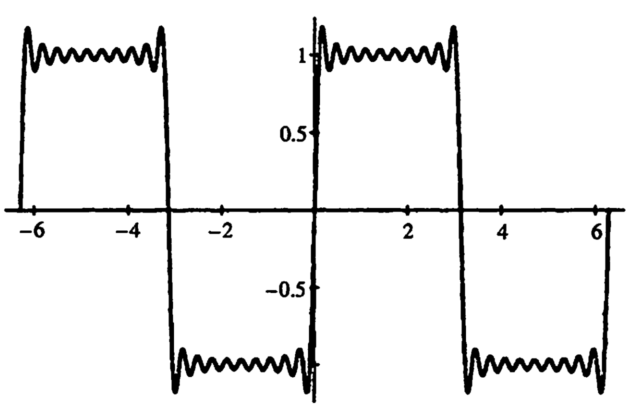
To graph the partial sum, we define a function as follows:

```
fgraph[n_]:=Plot[f[n,x],{x,-2Pi, 2Pi}]
```

If we type **fgraph[3]**, we obtain



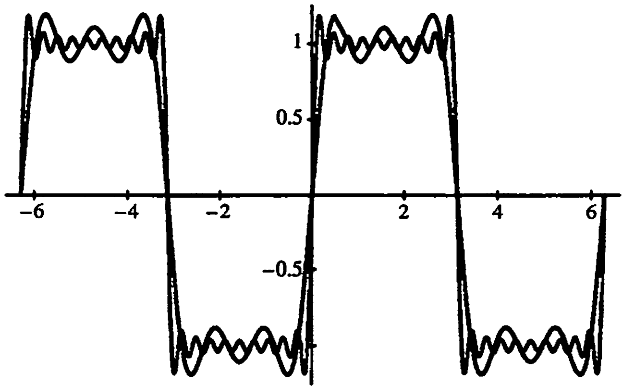
whereas if we type **fgraph[10]** we obtain



In order to display the two graphs simultaneously, we type

```
Plot[{f[3,x], f[10,x]},{x,-2Pi,2Pi}]
```

to obtain



1.3.3. Uniform and nonuniform convergence. In many problems it is important to avoid the Gibbs phenomenon—in other words, to be sure that the function $f(x)$ is well approximated by the partial sum $f_n(x)$ at all points of the interval $-L \le x \le L$. Recall that a sequence of functions $f_n(x)$, $a \le x \le b$, converges *uniformly* to a limit function $f(x)$, $a \le x \le b$, if

$$|f_n(x) - f(x)| \le \epsilon_n \qquad a \le x \le b,\ n = 1, 2, \ldots$$

where

$$\lim_{n\to\infty} \epsilon_n = 0$$

This is clearly violated in the Gibbs phenomenon, for in the previous example $\lim_{n\to\infty}[f_{2n-1}(\pi/2n) - f(\pi/2n)] = 0.18\ldots$.

1.3.4. Two criteria for uniform convergence. We shall give two general criteria for uniform convergence. The first of these can be tested on the series, while the second can be tested on the function.

PROPOSITION 1.3.1. **(First criterion for uniform convergence).** *Let $f(x)$, $-L < x < L$, be a piecewise smooth function. Suppose that the Fourier coefficients $\{A_n\}$, $\{B_n\}$ satisfy*

$$\sum_{n=1}^{\infty}(|A_n| + |B_n|) < \infty$$

Then the Fourier series converges uniformly.

For example, $\sum_{n=1}^{\infty}(\sin nx)/n^2$ is a uniformly convergent Fourier series.

PROPOSITION 1.3.2. **(Second criterion for uniform convergence).** *Let $f(x)$, $-L < x < L$, be a piecewise smooth function. Suppose in addition that*

$$f \text{ is continuous} \qquad -L < x < L \text{ and } f(-L+0) = f(L-0)$$

Then the Fourier series converges uniformly.

For example, $f(x) = |x|$ has a uniformly convergent Fourier series.

Within the class of piecewise smooth functions, these criteria are necessary and sufficient: If the Fourier series of a piecewise smooth function converges uniformly, then f is continuous, $f(-L+0) = f(L-0)$, and $\sum_{n=1}^{\infty}(|A_n| + |B_n|) < \infty$. Once we leave the domain of piecewise smooth functions, the theory becomes much more complicated; for example, the Fourier series $\sum_{n=2}^{\infty}(\sin nx)/(n \log n)$ is known to be uniformly convergent,[3] but it does not satisfy the first criterion. Of course the sum of this series must be a *continuous* function by the general properties of uniform convergence.

1.3.5. Differentiation of Fourier series. We now give a general criterion for differentiating a Fourier series.

PROPOSITION 1.3.3. *Let $f(x)$, $-L < x < L$, be a continuous piecewise smooth function with $f(L-0) = f(-L+0)$. Then*

$$\frac{1}{2}[f'(x+0) + f'(x-0)] = \sum_{n=1}^{\infty} \frac{n\pi}{L}\left(B_n \cos\frac{n\pi x}{L} - A_n \sin\frac{n\pi x}{L}\right)$$

Proof. It suffices to apply the convergence theorem to the piecewise smooth function $f'(x), -L < x < L$. Its Fourier coefficients are given by

$$\begin{aligned}
A_0' &= \frac{1}{2L}\int_{-L}^{L} f'(x)\,dx = \frac{1}{2L}\left(f(L-0) - f(-L+0)\right) \\
A_n' &= \frac{1}{2L}\int_{-L}^{L} f'(x)\cos(n\pi x/L)\,dx = \frac{n\pi}{L}\int_{-L}^{L} f\,(x)\sin(n\pi x/L)\,dx = \frac{n\pi}{L}B_n \\
B_n' &= \frac{1}{2L}\int_{-L}^{L} f'(x)\sin(n\pi x/L)\,dx = -\frac{n\pi}{L}\int_{-L}^{L} f\,(x)\cos(n\pi x/L)\,dx = -\frac{n\pi}{L}A_n
\end{aligned}$$

where we have integrated by parts and used the continuity of $f(x), -L < x < L$. The result now follows from Theorem 1.1. •

For example, suppose that we want to compute the Fourier series of $f(x) = x^2$, $-L < x < L$. The Fourier series of this even function is of the form $A_0 + \sum_{n=1}^{\infty} A_n \cos nx$, where $\{A_n\}$ are to be determined. From Proposition 1.3.3 we may write

$$2x = -\sum_{n=1}^{\infty} nA_n \sin nx$$

But from Example 1.1.1, Sec. 1.1, we know that

$$2x = 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

[3] A. Zygmund, *Trigonometrical Series*, Dover Publications, New York, 1955, p. 108.

Therefore $A_n = 4(-1)^n/n^2$ for $n = 1, 2, \ldots$. To compute A_0 we must return to the definition $A_0 = (1/2\pi)\int_{-\pi}^{\pi} x^2 dx = \pi^2/3$. Therefore we have the Fourier series

$$x^2 = \pi^2/3 + 4\sum_{n=1}^{\infty}[(-1)^n/n^2]\cos nx \qquad -\pi < x < \pi$$

1.3.6. Integration of Fourier series. The following proposition shows that a Fourier series may be integrated term by term under very general conditions.

PROPOSITION 1.3.4. *Let $f(x)$, $-\pi < x < \pi$, be a piecewise smooth function with Fourier series*

$$A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$$

If $-\pi \le x_0 < x \le \pi$, then

$$\begin{aligned}\int_{x_0}^{x} f(u)du &= A_0(x - x_0)\\ &\quad + \sum_{n=1}^{\infty}\left[\frac{A_n}{n}(\sin nx - \sin nx_0) + \frac{B_n}{n}(\cos nx_0 - \cos nx)\right]\end{aligned}$$

Proof. Let $F(x) = \int_{-\pi}^{x}[f(u) - A_0]du$. F is continuous and piecewise smooth with $F(-\pi) = F(\pi)$. Therefore by the basic convergence theorem (Theorem 1.1) we have

$$F(x) = \bar{A}_0 + \sum_{n=1}^{\infty}(\bar{A}_n \cos nx + \bar{B}_n \sin nx) \qquad -\pi \le x \le \pi$$

where $(\bar{A}_n, \bar{B}_n)$ are the Fourier coefficients of F. To compute these, we have, for $n \ne 0$,

$$\begin{aligned}\bar{A}_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\, dx\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx\left\{\int_{-\pi}^{x}[f(u) - A_0]du\right\}dx\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}[f(u) - A_0]\left(\int_{u}^{\pi}\cos nx\, dx\right)du\\ &= -\frac{1}{\pi}\int_{-\pi}^{\pi}[f(u) - A_0]\frac{\sin nu}{n}du\\ &= -\frac{B_n}{n}\end{aligned}$$

In the same fashion, we have

$$\begin{aligned}\bar{B}_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx \\ &= \frac{1}{n\pi}\int_{-\pi}^{\pi}[f(u)-A_0][\cos nu - \cos n\pi]du \\ &= \frac{A_n}{n}\end{aligned}$$

Recalling the definition of $F(x)$, we have proved that

$$\begin{aligned}\int_{-\pi}^{x} f(u)du &= A_0(x+\pi)+\bar{A}_0 \\ &\quad +\sum_{n=1}^{\infty}\frac{1}{n}(A_n\sin nx - B_n\cos nx) \qquad -\pi\le x\le\pi\end{aligned}$$

If we replace x by x_0 and subtract the result, then $\bar{A}_0$ cancels and we have proved the stated result. •

1.3.7. A continuous function with a divergent Fourier series. This example is constructed by a particular grouping of the terms in a special trigonometric series. Explicitly, we define the finite trigonometric sums

$$C_n(x) = \cos(N_n+1)x + \frac{\cos(N_n+2)x}{2} + \cdots + \frac{\cos(N_n+m_n)x}{m_n} \tag{1.3.2}$$

$$D_n(x) = \cos(N_n-1)x + \frac{\cos(N_n-2)x}{2} + \cdots + \frac{\cos(N_n-m_n)x}{m_n} \tag{1.3.3}$$

and the function

$$f(x) = \sum_{n=1}^{\infty}\frac{C_n(x)-D_n(x)}{n^2} \tag{1.3.4}$$

The integers N_n, m_n will be chosen so that

$$|C_n(x)-D_n(x)|\le 8 \qquad -\pi\le x\le\pi,\quad n=1,2,\ldots \tag{1.3.5}$$

$$\frac{C_n(0)}{n^2}\to\infty \qquad \frac{D_n(0)}{n^2}\to\infty \qquad n\to\infty \tag{1.3.6}$$

$$N_n+m_n < N_{n+1}-m_{n+1} \quad n=1,2,\ldots \tag{1.3.7}$$

To do this, we use the following two facts:

$$\sum_{k=1}^{n}\frac{1}{k} > \log n \qquad n=1,2,\ldots \tag{1.3.8}$$

$$\Big|\sum_{k=1}^{n}\frac{\sin kx}{k}\Big| \le 4 \qquad n=1,2,\ldots, \quad -\pi\le x\le\pi \tag{1.3.9}$$

To prove (1.3.5), we use the trigonometric identity

$$\cos a-\cos b=-2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$$

to write

$$C_n(x)-D_n(x)=-2\sin(N_nx)\left(\sin x+\frac{\sin 2x}{2}+\cdots+\frac{\sin m_nx}{m_n}\right)$$

From (1.3.9) the second factor is less than or equal to 4, and we have proved (1.3.5).

To prove (1.3.6), we write

$$C_n(0)=D_n(0)=1+\frac{1}{2}+\cdots+m_n>\log(m_n)$$

If we choose $m_n=2^{n^3}$, then $\log(m_n)=n^3\log 2$, and thus $C_n(0)/n^2$ and $D_n(0)/n^2$ tend to ∞, as required.

To prove (1.3.7) we define $N_1=3$ and for $n>1$, $N_{n+1}-N_n=2m_{n+1}$. With this choice, it immediately follows that $N_{n+1}-N_n>m_{n+1}+m_n$, as required.

Having defined N_n, m_n, it follows from (1.3.5) that the series (1.3.4) is uniformly convergent and therefore $f(x), -\pi<x<\pi$, is a continuous function. It remains to compute the Fourier series of f.

Since $f(x), -\pi<x<\pi$, is an even function, the Fourier sine coefficients $B_n\equiv 0$. To compute the Fourier cosine coefficients, we may multiply the uniformly convergent series (1.3.4) by $\cos nx$ and integrate on $-\pi<x<\pi$. From (1.3.7) there is exactly one nonzero term corresponding to each integer of the form $n=N_k\pm 1,\ldots,N_k\pm m_k$. These nonzero terms are of the form

$$A_n=\pm\frac{1}{j} \quad \text{if} \quad n=N_k\pm j \qquad 1\le j\le m_k$$

In particular, the partial sums at $x=0$ satisfy

$$\begin{aligned}
f_{N_k+m_k}(0) &= (C_1(0)-D_1(0))+\cdots+\frac{C_k(0)-D_k(0)}{k^2}\\
f_{N_{k+1}}(0) &= (C_1(0)-D_1(0))+\cdots+\frac{C_k(0)-D_k(0)}{k^2}-\frac{D_{k+1}(0)}{(k+1)^2}\\
f_{N_k+m_k}(0)-f_{N_{k+1}}(0) &= \frac{D_{k+1}(0)}{(k+1)^2}
\end{aligned}$$

If the sequence of partial sums $f_n(0)$ were convergent, it would follow that $\lim(f_{N_k+m_k}(0) - f_{N_{k+1}}(0)) = 0$, which contradicts (1.3.6). Therefore the Fourier series diverges at $x = 0$, which was to be proved.

EXERCISES 1.3

1. Let
$$f_{2n-1}(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \cdots + \frac{1}{2n-1}\sin(2n-1)x\right]$$
Show that
$$f_{2n-1}\left(\frac{k\pi}{2n}\right) \to \frac{2}{\pi}\int_0^{k\pi} \frac{\sin x}{x}\,dx \qquad k = 1, 2, \ldots$$
[*Hint:* Write the sum for $f_{2n-1}(k\pi/2n)$ as the approximating sum for an appropriate Riemannian integral.]
2. Estimate the integral $\int_0^{k\pi} (\sin x)/x\,dx$ for $k = 2, 3, 4$.
3. Let $f(x)$, $-L < x < L$, be a piecewise smooth function. Show that the first criterion for uniform convergence follows from the Weierstrass M-test (Appendix A.2.).
4. Let $f(x)$, $-L < x < L$, be a piecewise smooth function. Show that $A_n = O(1/n)$, $B_n = O(1/n)$ when $n \uparrow \infty$.
5. Let $f(x)$, $-L < x < L$, be a piecewise smooth function. Let A'_n, B'_n be the Fourier coefficients of f'.
$$\begin{aligned} A'_n &= \frac{1}{L}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}\,dx \\ B'_n &= \frac{1}{L}\int_{-L}^{L} f'(x)\sin\frac{n\pi x}{L}\,dx \end{aligned}$$
If f is continuous and $f(-L+0) = f(L-0)$, show that
$$A'_n = \frac{n\pi}{L}B_n \qquad B'_n = -\frac{n\pi}{L}A_n$$
6. Let $f(x)$, $-L < x < L$, be a continuous piecewise smooth function with $f(-L+0) = f(L-0)$. Use Exercises 4 and 5 to show that $A_n = O(1/n^2)$, $B_n = O(1/n^2)$ when $n \uparrow \infty$.
7. Let $f(x)$, $-L < x < L$, be a continuous piecewise smooth function with $f(-L+0) = f(L-0)$. Use Exercise 6 to prove the second criterion for uniform convergence (Proposition 1.3.2).
8. Use Exercise 5 and the main convergence theorem (Theorem 1.1) to prove the proposition on differentiating a Fourier series.

9. Let $f(x) = \sum_{n=1}^{\infty} e^{-(n^2\pi/L^2)} \sin n\pi x/L$ be the Fourier series of a piecewise smooth function. Show that

$$f'(x) = \sum_{n=1}^{\infty} \frac{n\pi}{L} e^{-(n^2\pi/L^2)} \cos \frac{n\pi x}{L}$$

$$f''(x) = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 e^{-(n^2\pi/L^2)} \sin \frac{n\pi x}{L}$$

10. Consider the Fourier series of $f(x) = x$ found in Example 1.1.1, Sec. 1.1. By formally differentiating the series at $x = 0$, show that it is not valid to differentiate a Fourier series term by term, even if the function is differentiable.
11. Consider the Fourier series of $f(x) = x$ found in Example 1.1.1, Sec. 1.1. By integrating this series, find a series for x^2.
12. Integrate the series of Exercise 11 and compare the result with Example 1.1.5.
13. Among the series for x, x^2, and $x^3 - L^2x$ found in Exercises 10 to 12, which are uniformly convergent?
14. Let $f(x) = x$, $-\pi < x < \pi$. Find the maximum of the partial sum $f_N(x)$ and verify the presence of Gibb's phenomenon.
15. This exercise provides the missing steps in the proof of (1.3.9).

(i) If $0 \le x \le \pi$, establish the identity

$$\sin x + \frac{\sin 2x}{2} + \cdots + \frac{\sin nx}{n} = \int_0^x (\cos t + \cdots + \cos nt)\, dt = \int_0^x \frac{\sin(n + (1/2))t}{2\sin(t/2)}\, dt - \frac{x}{2}$$

(ii) Rewrite the integral in (i) as

$$\int_0^x \sin(n + (1/2))t \left(\frac{1}{2\sin(t/2)} - \frac{1}{t}\right) dt + \int_0^x \frac{\sin(n + (1/2))t}{t}\, dt$$

(iii) Use the inequalities $|\sin\theta - \theta| \le \theta^3/6$, $\sin\theta \ge 2\theta/\pi$ for $0 \le \theta \le \pi$ to bound the first integral in the form

$$\left|\int_0^x \sin(n + (1/2))t \left(\frac{1}{2\sin(t/2)} - \frac{1}{t}\right) dt\right| \le \frac{\pi}{48}\int_0^x t\, dt = \frac{\pi x^2}{96}$$

(iv) Make the change of variable $u = (n + 1/2)t$ in the second integral to prove that

$$\left|\int_0^x \frac{\sin(n + (1/2))t}{t}\, dt\right| \le \int_0^\pi \frac{\sin u}{u} du = 1.852$$

(v) Conclude that $|\sin x + \cdots + (\sin nx)/n| \le 3.75$ for $0 \le x \le \pi$.

1.4. Parseval's Theorem and Mean Square Error

Having developed the convergence properties of Fourier series, we now turn to some concrete computations that show how Fourier series may be used in various problems.

1.4.1. Statement and proof of Parseval's theorem. The key to these applications is *Parseval's theorem*, a form of the pythagorean theorem that is valid in the setting of Fourier series.

THEOREM 1.2. *(Parseval's theorem). Let $f(x)$, $-L < x < L$, be a piecewise smooth function with Fourier series*

$$A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L}\right)$$

Then

$$\boxed{\frac{1}{2L} \int_{-L}^{L} f(x)^2 dx = A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (A_n^2 + B_n^2)} \tag{1.4.1}$$

The left side represents the mean square of the function $f(x)$, $-L < x < L$. The right side represents the sum of the squares of the Fourier components in the various coordinate directions $\cos n\pi x/L$, $\sin n\pi x/L$.

Proof. The proof of Parseval's theorem is especially simple if the piecewise smooth function is also continuous with $f(-L+0) = f(L-0)$. In that case we multiply the uniformly convergent Fourier series by $f(x)$ to obtain

$$f(x)^2 = A_0 f(x) + \sum_{n=1}^{\infty} \left[A_n f(x) \cos \frac{n\pi x}{L} + B_n f(x) \sin \frac{n\pi x}{L}\right]$$

This series is also uniformly convergent, and we may integrate term by term for $-L < x < L$, with the result

$$\begin{aligned} \int_{-L}^{L} f(x)^2 dx &= A_0 \int_{-L}^{L} f(x) dx \\ &\quad + \sum_{n=1}^{\infty} \left[A_n \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} + B_n \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\right] \end{aligned}$$

On the right we recognize the integrals that define the Fourier coefficients A_0, A_n, B_n. Dividing both sides by $2L$, we obtain equation (1.4.1), the desired form of Parseval's theorem in this case. The proof in the general case is outlined in the exercises. •

1.4.2. Application to mean square error. Our first application of Parseval's theorem is to the *mean square error* σ_N^2, defined by

$$\boxed{\sigma_N^2 = \frac{1}{2L}\int_{-L}^{L}[f(x) - f_N(x)]^2 dx} \tag{1.4.2}$$

This number measures the average amount by which $f_N(x)$ differs from $f(x)$. The Fourier series of $f(x) - f_N(x)$ is

$$\sum_{n=N+1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

and therefore, by Parseval's theorem, we have

$$\frac{1}{2L}\int_{-L}^{L}[f(x) - f_N(x)]^2 dx = \frac{1}{2}\sum_{n=N+1}^{\infty}(A_n^2 + B_n^2)$$

and the formula

$$\boxed{\sigma_N^2 = \frac{1}{2}\sum_{n=N+1}^{\infty}(A_n^2 + B_n^2)} \tag{1.4.3}$$

The mean square error is half the sum of the squares of the remaining Fourier coefficients. This formula shows, in particular, that the mean square error tends to zero when N tends to infinity.

EXAMPLE 1.4.1. *Let $f(x) = |x|$, $-\pi < x < \pi$. Find the mean square error and give an asymptotic estimate when $N \to \infty$.*

Solution. We have $B_n = 0$, $A_{2m} = 0$, $A_{2m-1} = -4/\pi(2m-1)^2$, so that

$$\begin{aligned}
\sigma_{2N-1}^2 = \sigma_{2N}^2 &= \frac{1}{2}\sum_{n=2N+1}^{\infty} A_n^2 \\
&= \frac{1}{2}\sum_{m=N+1}^{\infty}\left[\frac{4}{\pi(2m-1)^2}\right]^2 \\
&= \frac{8}{\pi^2}\sum_{m=N+1}^{\infty}\frac{1}{(2m-1)^4}
\end{aligned}$$

Although we cannot make a closed-form evaluation of this series, we can still make a useful *asymptotic* estimate. To do this, we compare the sum with the integral

$$\frac{8}{\pi^2}\int_N^{\infty}\frac{1}{(2x-1)^4}dx = \frac{4}{3\pi^2}\frac{1}{(2N-1)^3}$$

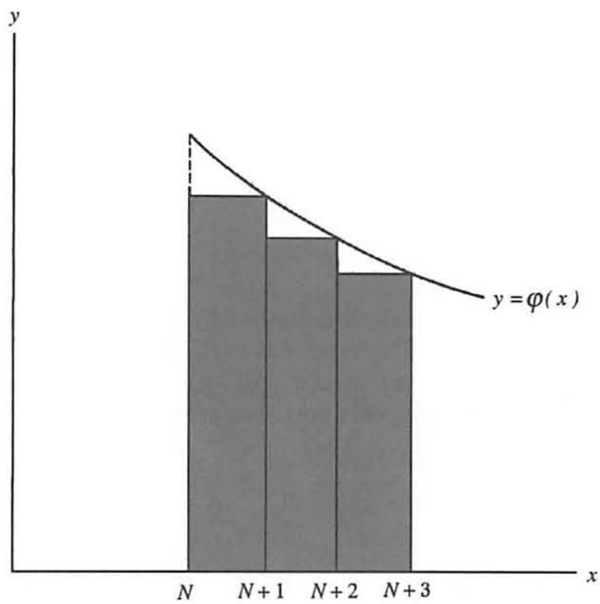


FIGURE 1.4.1 Illustrating the relation $\sum_{m=N+1}^{\infty} \varphi(m) \leq \int_N^{\infty} \varphi(x)\,dx$.

Figure 1.4.1 shows the comparison of a sum with an integral. This gives us the useful asymptotic statement

$$\sigma_N^2 = O(N^{-3}) \qquad N \to \infty$$

EXAMPLE 1.4.2. *Let $f(x) = x$, $-\pi < x < \pi$. Find the mean square error and give an asymptotic estimate when $N \to \infty$.*

Solution. We have $A_m = 0$, $B_m = (-1)^{m-1}(2/m)$, and therefore

$$\sigma_N^2 = \frac{1}{2} \sum_{m=N+1}^{\infty} \frac{4}{m^2} = 2 \sum_{m=N+1}^{\infty} \frac{1}{m^2}$$

To obtain a useful asymptotic estimate of this sum, we compare it with the integral

$$2 \int_N^{\infty} \frac{dx}{x^2} = \frac{2}{N}$$

so that

$$\sigma_N^2 = O(N^{-1}) \qquad N \to \infty \quad \bullet$$

1.4.3. Application to the isoperimetric theorem. We now give an application of Fourier series to geometry, the so-called isoperimetric theorem.

THEOREM 1.3. *Suppose that we have a smooth closed curve in the xy plane that encloses an area A and has perimeter P. Then*

$$P^2 \geq 4\pi A$$

with equality if and only if the curve is a circle.

Proof. Suppose that the curve is described by parametric equations $x = x(t)$, $y = y(t)$ where $-\pi \leq t \leq \pi$. The functions $x(t)$, $y(t)$ are supposed smooth and satisfy the normalization $x(-\pi) = x(\pi)$, $y(-\pi) = y(\pi)$ because the curve is closed. From calculus, the perimeter and area are given by the formulas

$$P = \int_{-\pi}^{\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt \qquad A = \int_{-\pi}^{\pi} x(t) y'(t)\, dt$$

where $x' = dx/dt$, $y' = dy/dt$. By reparametrizing the curve, we may suppose that $x'(t)^2 + y'(t)^2$ is constant (see Exercise 20); in fact, it must be

$$x'(t)^2 + y'(t)^2 = \frac{P^2}{4\pi^2}$$

Now we introduce the convergent Fourier series

$$\begin{aligned} x(t) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad -\pi \leq t \leq \pi \\ y(t) &= c_0 + \sum_{n=1}^{\infty} (c_n \cos nt + d_n \sin nt) \qquad -\pi \leq t \leq \pi \end{aligned}$$

Since the functions $x(t)$, $y(t)$ are supposed smooth, we also have the convergent Fourier series

$$\begin{aligned} x'(t) &= \sum_{n=1}^{\infty} n(-a_n \sin nt + b_n \cos nt) \qquad -\pi \leq t \leq \pi \\ y'(t) &= \sum_{n=1}^{\infty} n(-c_n \sin nt + d_n \cos nt) \qquad -\pi \leq t \leq \pi \end{aligned}$$

Applying Parseval's theorem, we have

$$\begin{aligned}
\frac{P^2}{2\pi} &= \int_{-\pi}^{\pi}[x'(t)^2 + y'(t)^2]dt = \pi\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2 + c_n^2 + d_n^2) \\
A &= \int_{-\pi}^{\pi} x(t)y'(t)dt \\
&= \frac{1}{4}\int_{-\pi}^{\pi}\{[x(t) + y'(t)]^2 - [x(t) - y'(t)]^2\}dt \\
&= \pi\sum_{n=1}^{\infty} n(a_n d_n - b_n c_n)
\end{aligned}$$

Performing the necessary algebraic steps, we have

$$\frac{P^2}{2\pi} - 2A = \pi\sum_{n=1}^{\infty}[n(a_n - d_n)^2 + n(b_n + c_n)^2 + n(n-1)(a_n^2 + b_n^2 + c_n^2 + d_n^2)]$$

The right side is a sum of squares with nonnegative coefficients; thus $P^2/2\pi - 2A \geq 0$. If the sum is zero, then all of the terms are zero; in particular, $a_n^2 + b_n^2 + c_n^2 + d_n^2 = 0$ for $n > 1$ and $a_1 - d_1 = 0$, $b_1 + c_1 = 0$. This means that

$$\begin{aligned}
x(t) &= a_0 + a_1\cos t - c_1\sin t \qquad -\pi \leq t \leq \pi \\
y(t) &= c_0 + c_1\cos t + a_1\sin t \qquad -\pi \leq t \leq \pi
\end{aligned}$$

which is the equation of a circle of radius $\sqrt{a_1^2 + c_1^2}$ with center at (a_0, c_0). The proof is complete. •

EXERCISES 1.4

Find the mean square errors for the Fourier series of the functions in Exercises 1 to 3.

1. $f(x) = 1$ for $0 < x < \pi, f(0) = 0,$ and $f(x) = -1$ for $-\pi < x < 0$.
2. $f(x) = x^2$, $-\pi \leq x \leq \pi$
3. $f(x) = \sin 10x$, $-\pi < x < \pi$
4. Write out Parseval's theorem for the Fourier series of Exercise 1.
5. Write out Parseval's theorem for the Fourier series of Exercise 2.
6. Show that, in Exercise 1, $\sigma_N^2 = O(N^{-1})$, $N \uparrow \infty$.
7. Show that, in Exercise 2, $\sigma_N^2 = O(N^{-3})$, $N \uparrow \infty$.
8. Let $f(x) = x(\pi - x)$, $0 \leq x \leq \pi$.
   (a) Compute the Fourier sine series of f.
   (b) Compute the Fourier cosine series of f.
   (c) Find the mean square error incurred by using N terms of each series and find asymptotic estimates when $N \to \infty$.
   (d) Which series gives a better mean square approximation of f?

9. Let $f(x)$, $g(x)$, $-L \leq x \leq L$, be piecewise smooth functions with Fourier series

$$\begin{aligned} f(x) &= A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \\ g(x) &= C_0 + \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi x}{L} + D_n \sin \frac{n\pi x}{L} \right) \end{aligned}$$

Show that

$$\frac{1}{2L} \int_{-L}^{L} f(x)g(x)dx = A_0C_0 + \frac{1}{2} \sum_{n=1}^{\infty} (A_nC_n + B_nD_n)$$

Note that this formula corresponds to the dot product formula

$$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \cdot (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) = a_1a_2 + b_1b_2 + c_1c_2$$

for vectors in the three-dimensional space $\mathbf{R}^3$.

10. Let $f(x) = (\cos ax / \sin a\pi)$, $-\pi \leq x \leq \pi$, where $0 < a < \frac{1}{2}$.

(a) Find the Fourier series of f.

(b) Give an asymptotic estimate for the mean square error incurred in approximating f by the first N terms of the Fourier series.

(c) Apply Parseval's theorem to obtain the following integral formula:

$$\sum_{n=-\infty}^{\infty} (a^2 - n^2)^{-2} = \frac{\pi}{2} (a \sin a\pi)^{-2} \int_{-\pi}^{\pi} \cos^2 ax \, dx$$

(d) Prove that $\sum_{n=1}^{\infty} n^{-4} = \pi^4/90$. [*Hint:* Make a three-term Taylor expansion of part (c) in powers of a and identify the coefficients.]

11. Let $\varphi(x)$ be defined for $x > 0$ with $\varphi(x) > 0$, $\varphi'(x) < 0$, and the integral $\int_1^{\infty} \varphi(x)dx$ convergent.

(a) Show that

$$\int_{N+1}^{\infty} \varphi(x)dx \leq \sum_{n=N+1}^{\infty} \varphi(n) \leq \int_N^{\infty} \varphi(x)dx$$

(b) Deduce from this that

$$-\varphi(N) \leq \sum_{n=N+1}^{\infty} \varphi(n) - \int_N^{\infty} \varphi(x)dx \leq 0$$

12. Let $\varphi(x) = 1/x^s$ where $s > 1$.

(a) Use Exercise 11 to show that

$$\frac{-1}{N^s} \leq \sum_{n=N+1}^{\infty} \frac{1}{n^s} - \frac{1}{s-1} \frac{1}{N^{s-1}} \leq 0$$

(b) Show that this may be written in the form

$$\sum_{n=N+1}^{\infty} \frac{1}{n^s} = \frac{1}{(s-1)N^{s-1}}\left[1+O\left(\frac{1}{N}\right)\right] \qquad N \to \infty$$

13. Let σ_N^2 be the mean square error in the Fourier series of $f(x) = x$, $-\pi < x < \pi$. Use Exercise 12 to show that $\sigma_N^2 = (1/N)[1 + O(1/N)]$, $N \to \infty$.
14. Let $\varphi(x) = 1/P(x)$ where $P(x)$ is a polynomial of degree s, $s > 1$. Modify Exercise 11(b) to show that

$$\sum_{n=N+1}^{\infty} \varphi(n) = \int_N^{\infty} \varphi(x)dx \left[1+O\left(\frac{1}{N}\right)\right] \qquad N \to \infty$$

15. Let σ_N^2 be the mean square error in the Fourier series of $f(x) = |x|$, $-\pi < x < \pi$. Use Exercise 14 to find an asymptotic estimate of the form $\sigma_N^2 = (C/N^s)[1 + O(1/N)]$, $N \to \infty$ for appropriate constants C, s.
16. Let $\varphi(x) = e^{-x}$, $x > 0$. Discuss the validity of the asymptotic estimate

$$\sum_{n=N+1}^{\infty} \varphi(n) = \int_N^{\infty} \varphi(x)dx[1 + O(1/N)] \qquad N \to \infty$$

17. Compute the ratio P^2/A for an equilateral triangle.
18. Compute the ratio P^2/A for a square.
19. Compute the ratio P^2/A for a regular polygon of n sides and compare it with the isoperimetric theorem in the limit when $n \to \infty$.
20. Let $x(t)$, $y(t)$ be smooth functions, $-\pi \le t \le \pi$, with $(x')^2 + (y')^2 \neq 0$. Let $s(t) = \int_{-\pi}^{t} \sqrt{(x')^2 + (y')^2}$, $P = s(\pi)$, $\tilde{t} = -\pi + (2\pi s/P)$, $\tilde{x}(\tilde{t}) = x(t)$, $\tilde{y}(\tilde{t}) = y(t)$. Show that $-\pi \le \tilde{t} \le \pi$ and $(d\tilde{x}/d\tilde{t})^2 + (d\tilde{y}/d\tilde{t})^2 = P^2/4\pi^2$.

The following exercises are designed to lead to a proof of Parseval's theorem for piecewise smooth functions.

21. Let $f(x), -L < x < L$, be a piecewise smooth function. Show that for each $\epsilon > 0$, there is a continuous piecewise smooth function $f^*(x), -L < x < L$, with $f^*(-L+0) = f^*(L-0)$ such that $(1/2L)\int_{-L}^{L}[f(x) - f^*(x)]^2\,dx < \epsilon$. [*Hint:* Across each subdivision point replace f by a linear function on the interval $x_i - h < x < x_i + h$, where h is chosen in terms of ϵ, p and the maximum of $|f(x)|, -L < x < L$.]
22. Let $f(x), -L < x < L$, be a piecewise smooth function and let $f^*(x), -L < x < L$, be the continuous function constructed in the previous exercise. Use Proposition 0.3.2 to show that $||f - f_N|| \le ||f - f_N^*||$, where f_N is the Nth partial sum of the Fourier series for the function $f(x), -L < x < L$, and f_N^* is the Nth partial sum of the Fourier series for the function $f^*(x), -L < x < L$.

23. Use the triangle inequality from Sec. 0.3 to prove the inequality $||f - f_N^*|| \le ||f - f^*|| + ||f^* - f_N^*||$.
24. Show that there is an integer N_0 so that for $N \ge N_0$ we have $||f - f_N^*|| < \epsilon$. [*Hint:* Combine Proposition 0.3.4 with the Parseval theorem already proved for the function $f^*(x), -L < x < L$.]
25. Conclude the validity of Parseval's theorem for the piecewise smooth function $f(x), -L < x < L$.

1.5. Complex Form of Fourier Series

1.5.1. Fourier series and Fourier coefficients. It is often useful to rewrite the formulas of Fourier series using complex numbers. To do this, we begin with Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.5.1}$$

and the immediate consequences

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

We apply these to a Fourier series:

$$\begin{aligned} f(x) &= A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right) \\ &= A_0 + \frac{1}{2}\sum_{n=1}^{\infty}[(A_n - iB_n)e^{in\pi x/L} + (A_n + iB_n)e^{-(in\pi x/L)}] \end{aligned}$$

Therefore we let $\alpha_n = \frac{1}{2}(A_n - iB_n)$, $n = 1, 2, \ldots$; $\alpha_n = \frac{1}{2}(A_{-n} + iB_{-n})$, $n = -1, -2, \ldots$; and $\alpha_0 = A_0$. With this convention the Fourier series assumes the form

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}} \tag{1.5.2}$$

To obtain integral formulas for the coefficients $\{\alpha_n\}$, we use (1.1.7) and (1.1.8),

$$\begin{aligned} 2\alpha_n = (A_n - iB_n) &= \frac{1}{L}\int_{-L}^{L} f(x) \times \left(\cos\frac{n\pi x}{L} - i\sin\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L}\int_{-L}^{L} f(x)e^{-(in\pi x/L)}dx \end{aligned}$$

with a corresponding formula for the plus sign. When $n = 0$, (1.1.6) shows that α_0 is given appropriately. Thus we have

$$\boxed{\alpha_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-(in\pi x/L)}\,dx \qquad n = 0, \pm 1, \pm 2, \ldots} \tag{1.5.3}$$

1.5.2. Parseval's theorem in complex form. Finally, we retrieve the appropriate form of Parseval's theorem. To do this, multiply (1.5.2) by $f(x)$ and integrate on $(-L, L)$. The result is

$$\boxed{\frac{1}{2L}\int_{-L}^{L} f(x)^2 dx = \sum_{n=-\infty}^{\infty} |\alpha_n|^2} \tag{1.5.4}$$

1.5.3. Applications and examples. The functions $e^{(in\pi x/L)}$ satisfy an orthogonality relation, which may be written in the form

$$\int_{-L}^{L} e^{(in\pi x/L)} e^{-(im\pi x/L)} dx = \begin{cases} 0 & n \neq m \\ 2L & n = m \end{cases}$$

These may be proved by using Euler's formula and the orthogonality of the trigonometric functions $\cos(n\pi x/L)$, $\sin(n\pi x/L)$. Knowing these orthogonality relations, we can develop the complex form of Fourier series in its own right, without reference to the original formulas of Sec. 1.1.

The theory of Fourier series may also be extended to *complex-valued functions* $f(x)$, $-L < x < L$. These are of the form $f(x) = f_1(x) + if_2(x)$, where f_1, f_2 are real-valued functions. The Fourier coefficients are defined by the same formulas $\alpha_n = (1/2L)\int_{-L}^{L} f(x)e^{-(in\pi x/L)}dx$. If both f_1 and f_2 are piecewise smooth functions, then the complex Fourier series converges for all x to $\frac{1}{2}[\bar{f}(x+0) + \bar{f}(x-0)]$, where $\bar{f}$ is the periodic extension of the piecewise smooth function $f(x)$, $-L < x < L$. This convergence is understood as the limit of the sum $\sum_{-N}^{N}$ when N tends to infinity.

The Fourier coefficients of a real-valued function are characterized by the relation

$$\alpha_{-n} = \bar{\alpha}_n$$

where the bar indicates the *complex conjugate* of a complex number: if $c = a+ib$, then $\bar{c} = a - ib$.

To simplify the computation of complex Fourier series, we indicate some formulas that are of frequent use. If $c = a+ib$ is a complex number, the exponential function $e^{cx} = e^{ax}e^{ibx} = e^{ax}(\cos bx + i\sin bx)$. From this we have $(d/dx)e^{cx} = ae^{ax}\cos bx - be^{ax}\sin bx + i(ae^{ax}\sin bx + be^{ax}\cos bx) = (a+ib)e^{ax}(\cos bx + i\sin bx) = ce^{cx}$. Hence the differentiation formula

$$\frac{d}{dx}e^{cx} = ce^{cx}$$

is valid for any complex number c.

EXAMPLE 1.5.1. *Compute the complex Fourier series of* $f(x) = e^{ax}$, $-\pi < x < \pi$, *where* a *is a real number.*

Solution. The Fourier coefficients are given by the formula

$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}dx$$

Noting that $(d/dx)e^{(a-in)x} = (a-in)e^{(a-in)x}$, we have

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi}\frac{1}{a-in}\left(e^{(a-in)\pi} - e^{(a-in)(-\pi)}\right)\\ &= \frac{1}{2\pi}\frac{1}{a-in}(-1)^n(e^{a\pi}-e^{-a\pi})\\ &= \frac{1}{\pi}\sinh a\pi\frac{(-1)^n(a+in)}{a^2+n^2}\end{aligned}$$

The complex Fourier series of $f(x) = e^{ax}$, $-\pi < x < \pi$, is

$$\frac{1}{\pi}\sinh a\pi\sum_{n=-\infty}^{\infty}\frac{(-1)^n(a+in)}{a^2+n^2}e^{inx} \quad \bullet$$

As our next application of complex Fourier series, we compute the Fourier series of

$$f(x) = \cos^m x \qquad -\pi < x < \pi$$

If we were to use the real form of Fourier series, we would encounter many cumbersome trigonometric identities. With the complex approach, we avoid these. We begin with the identity

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

We expand the mth power, using the binomial theorem:

$$(e^{ix} + e^{-ix})^m = \sum_{j=0}^{m}\binom{m}{j}e^{ijx}e^{-i(m-j)x}$$

Therefore

$$\cos^m x = \frac{1}{2^m}\sum_{j=0}^{m}\binom{m}{j}e^{i(2j-m)x}$$

This is the complex form of the Fourier series for $\cos^m x$. As a by-product, we can obtain some useful integrals. To do this, we multiply the previous equation by e^{-inx} and integrate for $-\pi < x < \pi$. By orthogonality all the integrals are zero except when $2j - m - n = 0$, in which case the integral is 2π. In particular, $m + n$ must be even. Therefore we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos x)^m e^{-inx}dx = \begin{cases} 0 & m+n \text{ odd} \\ \frac{1}{2^m}\binom{m}{j} & 0 \le m+n = 2j \le 2m \end{cases}$$

The Fourier series for $\cos^m x$ can also be written in a real form, to obtain familiar trigonometric identities. It is simpler to consider separately the cases m even and m odd. Thus, if $m = 2k+1$, we can group the terms of the Fourier series in pairs: $j = 0$ with $j = m$ and $j = 1$ with $j = m - 1$, etc. To each pair, we apply Euler's formula, with the result

$$\cos^{2k+1} x = \left(\frac{1}{2}\right)^{2k} \left[\cos(2k+1)x + \cdots + \binom{2k+1}{k} \cos x\right]$$

In particular, this gives the identities

$$\begin{aligned} \cos^3 x &= \frac{1}{4}(\cos 3x + 3\cos x) \\ \cos^5 x &= \frac{1}{16}(\cos 5x + 5\cos 3x + 10\cos x) \end{aligned}$$

If m is even, we group the term $j = 0$ with $j = m$, etc., as before and finish with one ungrouped term in the middle. Applying Euler's theorem again, we have, with $m = 2k$,

$$\cos^{2k} x = \left(\frac{1}{2}\right)^{2k} \left[2\cos 2kx + \cdots + 2\binom{2k}{k-1} \cos 2x + \binom{2k}{k}\right]$$

In particular, we retrieve the identities

$$\begin{aligned} \cos^2 x &= \frac{1}{2}(\cos 2x + 1) \\ \cos^4 x &= \frac{1}{8}(\cos 4x + 4\cos 2x + 3) \end{aligned}$$

1.5.4. Fourier series of mass distributions. The theory of Fourier series is especially natural in the case of a *mass distribution*. This is defined by a *mass distribution function* $F(x), -L < x < L$, which can be any increasing function. The left and right limits are denoted by $F(x-0)$ and $F(x+0)$, respectively. The *mass of the interval* $a < x < b$ is defined by $m(a,b) = F(b-0) - F(a+0)$. The mass of a point is defined by $m(\{a\}) = F(a+0) - F(a-0)$.

For example, the Dirac δ distribution of mass m at the point x_0 is defined by setting $F(x) = 0$ for $x < x_0$ and $F(x) = m$ for $x > x_0$. At the other extreme, a mass distribution with density $f(x), -L < x < L$, is defined by the mass distribution function $F(x) = \int_{-L}^{x} f(y)\, dy$.

The Fourier coefficients of a mass distribution function are defined by the integrals

$$\alpha_n = \frac{1}{2L} \int_{-L}^{L} e^{-\imath n\pi x/L} dF(x) \qquad n = 0, \pm 1, \pm 2, \ldots$$

For $n = 0$ this is the total mass per unit length: $\alpha_0 = [F(L-0) - F(-L+0)]/(2L)$. The precise meaning for $n \neq 0$ can be defined by partial integration. If the mass distribution consists of several point masses plus a density, then each of the terms can be done separately.

EXAMPLE 1.5.2. *Find the Fourier coefficients of the mass distribution that consists of a uniform distribution of mass M on the interval $-L < x < L$, together with a Dirac δ distribution of mass m situated at the point $x = 0$.*

Solution. The mass distribution function is linear with a jump at the point $x = 0$. In detail, we have

$$F(x) = \begin{cases} (M/2L)(x+L) & \text{if } -L < x < 0 \\ m + (M/2L)(x+L) & \text{if } 0 < x < L \end{cases}$$

The Fourier coefficients are obtained as

$$\alpha_0 = (1/2L)(m+M), \quad \alpha_n = (m/2L) + (M/2L)\int_{-L}^{L} e^{-in\pi x/L}dx = (m/2L)$$

since the last integral is zero for $n \neq 0$. •

The following theorem shows that the theory of Fourier inversion of mass distributions is especially simple.

THEOREM 1.4. *(Convergence theorem). Suppose that $F(x), -L < x < L$, defines a mass distribution m with Fourier coefficients α_n. Define the Fourier partial sum by*

$$f_N(x) = \sum_{n=-N}^{N} \alpha_n e^{in\pi x/L} \qquad -L < x < L, \quad N = 1, 2, \ldots$$

Then if $a < b$, we have

$$\lim_{N\to\infty} \int_a^b f_N(x)\,dx = m(a,b) + \frac{1}{2}m(\{a\}) + \frac{1}{2}m(\{b\})$$

Proof. We can repeat the steps of the proof of Fourier convergence, noting that

$$\begin{aligned} f_N(x) &= \int_{-L}^{L} D_N(x-y)dF(y) \\ \int_a^b f_N(x)\,dx &= \int_{-L}^{L}\left(\int_a^b D_N(x-y)\,dx\right)dF(y) \end{aligned}$$

where D_N is the Dirichlet kernel introduced in Sec. 1.2. From the properties of the Dirichlet kernel proved there, it follows that

$$\begin{aligned} \lim_{N\to\infty}\int_a^b D_N(x-y)\,dx &= 1 \qquad a < y < b \\ \lim_{N\to\infty}\int_a^b D_N(x-y)\,dx &= 1/2 \quad y = a, b \\ \lim_{N\to\infty}\int_a^b D_N(x-y)\,dx &= 0 \qquad \text{otherwise} \end{aligned}$$

and that the integral is uniformly bounded by a constant. Therefore one may take the limit inside the sign of integration to obtain the result. •

EXERCISES 1.5

1. Verify that the orthogonality relations hold, in the form

$$\int_{-L}^{L} e^{in\pi x/L} e^{-im\pi x/L} dx = \begin{cases} 0 & \text{if } n \neq m \\ 2L & \text{if } n = m \end{cases}$$

2. Use the formulas in Exercise 1 to prove (1.5.3) from (1.5.2). You may assume that the series (1.5.2) converges uniformly for $-L < x < L$.
3. Use the complex form to find the Fourier series of $f(x) = e^x$, $-L < x < L$.
4. Let $0 < r < 1$, $f(x) = 1/(1 - re^{ix})$, $-\pi < x < \pi$. Find the Fourier series of f. (*Hint:* First expand f as a power series in r.)
5. Use Exercise 4 to derive the real formulas

$$\frac{1 - r\cos x}{1 + r^2 - 2r\cos x} = 1 + \sum_{n=1}^{\infty} r^n \cos nx, \qquad 0 \leq r < 1$$

$$\frac{r\sin x}{1 + r^2 - 2r\cos x} = \sum_{n=1}^{\infty} r^n \sin nx, \qquad 0 \leq r < 1$$

6. Show that the convergence theorem from Sec. 1.2 can be written in complex form as

$$\frac{1}{2}[\bar{f}(x + 0) + \bar{f}(x - 0)] = \lim_{N\to\infty} \sum_{n=-N}^{N} \alpha_n e^{in\pi x/L}$$

7. Show that the unrestricted double limit

$$\lim_{M,N\to\infty} \sum_{n=-M}^{N} \alpha_n e^{in\pi x/L}$$

does not exist in general. (*Hint:* Try Example 1.1.4 at $x = 0$.)

In the following exercises, find the Fourier coefficients of the indicated mass distributions.

8. A mass m at the point x_0.
9. A row of three equally spaced masses of mass $m/3$ at the points $x = -L/2, x = 0, x = L/2$.
10. A uniform distribution of mass M on the interval $-L/2 < x < L/2$.
11. A triangular mass distribution described by the density function $f(x) = M(L - |x|)/L^2, -L < x < L$.

12. Theorem 1.4 in the text gives no information in case $a = b$. Show that in this case
$$m(\{a\}) = \lim_{N\to\infty} \frac{f_N(a)}{2N+1}$$
[*Hint:* Examine the behavior of $D_N(x)/(2N+1)$ when N is large.]
13. Show that the following analogue of Parseval's identity is valid:
$$\lim_{N\to\infty} \frac{\sum_{n=-N}^{N} |\alpha_n|^2}{2N+1} = \sum_a m(\{a\})^2$$
where the sum is over all of the point masses of the mass distribution.
14. A sequence of functions $f_n(x), -\pi < x < \pi$, is said to *converge weakly* to the function $f(x), -\pi < x < \pi$, if for every piecewise smooth function $g(x), -\pi < x < \pi$, we have
$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f_n(x)\, g(x)\, dx = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$
Suppose that $f(x), -\pi < x < \pi$, is an arbitrary continuous function with Fourier partial sum $f_n(x), -\pi < x < \pi$. Prove that $f_n(x), -\pi < x < \pi$, converges weakly to $f(x), -\pi < x < \pi$. [*Hint:* First establish the identity $\int_{-\pi}^{\pi} f_n(x)g(x)\, dx = \int_{-\pi}^{\pi} g_n(x)f(x)$ where $g_n(x), -\pi < x < \pi$, is the Fourier partial sum of $g(x), -\pi < x < \pi$.]

1.6. Sturm-Liouville Eigenvalue Problems

Fourier series may be formulated as the orthogonal expansion in terms of functions $\phi(x)$ that are solutions of the differential equation

$$\phi''(x) + \lambda\,\phi(x) = 0 \tag{1.6.1}$$

on the interval $-L < x < L$ and that satisfy the periodic boundary conditions
$$\phi(-L) = \phi(L) \qquad \phi'(-L) = \phi'(L)$$
Indeed, the functions $\phi(x) = \sin(n\pi x/L)$ and $\phi(x) = \cos(n\pi x/L)$ satisfy these conditions with the value $\lambda = (n\pi/L)^2$.

More generally, we can study the solutions of the differential equation (1.6.1) that satisfy other sets of boundary conditions arising in problems of heat conduction and wave propagation. The general *two-point boundary condition* on the interval $a \le x \le b$ is written

$$\cos\alpha\,\phi(a) - L\sin\alpha\,\phi'(a) = 0 \tag{1.6.2}$$

$$\cos\beta\,\phi(b) + L\sin\beta\,\phi'(b) = 0 \tag{1.6.3}$$

where $L = b - a$ and α, β are dimensionless parameters that may be assumed to satisfy $0 \le \alpha < \pi, 0 \le \beta < \pi$. The number λ is called an *eigenvalue* and $\phi(x)$ is called an *eigenfunction* of the Sturm-Liouville (S-L) eigenvalue problem defined by (1.6.1), (1.6.2), and (1.6.3). Clearly, $\phi(x) \equiv 0$ is always a solution of the Sturm-Liouville eigenvalue problem, the so-called trivial solution. A solution $\phi(x)$ of (1.6.1), (1.6.2), and (1.6.3) that is not identically zero is called a *nontrivial solution.*

1.6.1. Examples of Sturm-Liouville eigenvalue problems. Fourier sine series and Fourier cosine series both arise from Sturm-Liouville problems with a two-point boundary condition on the interval $0 < x < L$. In the first case we use $\alpha = 0$, $\beta = 0$, corresponding to the boundary conditions $\phi(0) = 0, \phi(L) = 0$; in the second case we use $\alpha = \pi/2$, $\beta = \pi/2$ corresponding to the boundary conditions $\phi'(0) = 0, \phi'(L) = 0$.

The following worked examples demonstrate that no other solutions exist. In order to simplify the writing, we ignore arbitrary constants that may occur in the nontrivial solutions.

EXAMPLE 1.6.1. ($\alpha = 0$, $\beta = 0$) *Find all nontrivial solutions of (1.6.1) on the interval* $0 < x < L$ *satisfying the boundary conditions* $\phi(0) = 0, \phi(L) = 0$.

Solution. We consider separately the cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

In case $\lambda = 0$, the general solution of (1.6.1) is $\phi(x) = Ax + B$. The boundary conditions further require that $0 = \phi(0) = B$, $0 = \phi(L) = AL + B$, which is satisfied if and only if $(A, B) = (0, 0)$.

In case $\lambda = -\mu^2 < 0$, the general solution of (1.6.1) is $\phi(x) = Ae^{\mu x} + Be^{-\mu x}$. The boundary conditions further require that $0 = A + B$, $0 = Ae^{\mu L} + Be^{-\mu L}$, which is satisfied if and only if $(A, B) = (0, 0)$.

In case $\lambda > 0$, the general solution is $\phi(x) = A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda})$. The boundary conditions further require that $0 = A$, $0 = A\cos(L\sqrt{\lambda}) + B\sin(L\sqrt{\lambda})$. A nontrivial solution is obtained by taking $B \neq 0$, $L\sqrt{\lambda} = n\pi$, where $n = 1, 2, \ldots$. Therefore we have found all of the eigenvalues and eigenfunctions, in the form

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, \ldots \quad \bullet$$

EXAMPLE 1.6.2. ($\alpha = \pi/2$, $\beta = \pi/2$) *Find all nontrivial solutions of (1.6.1) on the interval* $0 < x < L$ *satisfying the boundary conditions* $\phi'(0) = 0, \phi'(L) = 0$.

Solution. We consider separately the cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

In case $\lambda = 0$, the general solution of (1.6.1) is $\phi(x) = Ax + B$. The boundary conditions further require that $0 = \phi'(0) = A$, $0 = \phi'(L) = A$, which gives a nontrivial solution if and only if $A = 0$ and B is nonzero.

In case $\lambda = -\mu^2 < 0$, the general solution of (1.6.1) is $\phi(x) = Ae^{\mu x} + Be^{-\mu x}$. The boundary conditions further require that $0 = \mu A - \mu B$, $0 = A\mu e^{\mu L} - B\mu e^{-\mu L}$, which is satisfied if and only if $(A, B) = (0, 0)$.

In case $\lambda > 0$, the general solution is $\phi(x) = A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda})$. The boundary conditions further require that $0 = \phi'(0) = B\sqrt{\lambda}$, $0 = \phi'(L) =$

$-A\sqrt{\lambda}\sin L\sqrt{\lambda} + B\sqrt{\lambda}\cos(L\sqrt{\lambda})$. A nontrivial solution is obtained by taking $B = 0$, $L\sqrt{\lambda} = n\pi$, where $n = 1, 2, \ldots$. Therefore we have found all of the eigenvalues and eigenfunctions, in the form

$$\lambda_0 = 0,\ \phi_0(x) = 1 \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2,\ \phi_n(x) = \cos\left(\frac{n\pi\, x}{L}\right) \qquad n = 1, 2, \ldots \quad \bullet$$

1.6.2. Some general properties of S-L eigenvalue problems. The solutions of Sturm-Liouville eigenvalue problems with two-point boundary conditions have some general properties, which are summarized in the following theorem.

THEOREM 1.5. *Consider the Sturm-Liouville eigenvalue problem represented by (1.6.1), (1.6.2), and (1.6.3).*

1. *Suppose that $\phi(x), \psi(x)$ are nontrivial solutions of (1.6.1)-(1.6.3) with the same eigenvalue λ. Then there is a constant $C \neq 0$ such that*

$$\phi(x) = C\psi(x)$$

2. *Suppose that $\phi_1(x), \phi_2(x)$ are nontrivial solutions of (1.6.1)-(1.6.3) with different eigenvalues $\lambda_1 \neq \lambda_2$. Then the eigenfunctions are orthogonal:*

$$\int_a^b \phi_1(x)\,\phi_2(x)\,dx = 0$$

Proof.

1. First consider the case $\alpha = 0$, where the boundary condition at the left end requires $\phi(a) = 0$, $\psi(a) = 0$. Both $\phi(x)$ and $\psi(x)$ satisfy the same second-order linear homogeneous differential equation, and so does any linear combination. We set

$$f(x) = \psi'(a)\phi(x) - \phi'(a)\psi(x)$$

The function $f(x), a < x < b$, also satisfies (1.6.1) and the initial conditions $f(a) = 0, f'(a) = 0$. This requires that $f(x) \equiv 0$. But if $\psi'(a) = 0$ (resp. $\phi'(a) = 0$), then $\psi(x) \equiv 0$ (resp. $\phi(x) \equiv 0$), a contradiction, so that we have proved (1) with the value $C = \phi'(a)/\psi'(a)$.

In the general case $\alpha \neq 0$, we set

$$f(x) = \psi(a)\phi(x) - \phi(a)\psi(x)$$

The function $f(x), a < x < b$, also satisfies (1.6.1) and the initial conditions $f(a) = 0, f'(a) = 0$. This requires that $f(x) \equiv 0$. But if $\psi(a) = 0$ (resp. $\phi(a) = 0$), then from (1.6.2) it follows that $\psi'(a) = 0$ (resp. $\phi'(a) = 0$), so that $\phi(x) \equiv 0$ (resp. $\psi(x) \equiv 0$), a contradiction. We have proved the theorem with the value $C = \phi(a)/\psi(a)$.

2. To prove the orthogonality, we write (1.6.1) for $\phi_1(x)$:

$$\phi_1''(x) + \lambda_1\phi_1(x) = 0 \tag{1.6.4}$$

Multiply (1.6.4) by $\phi_2(x)$ and integrate on the interval $a < x < b$:

$$\int_a^b \phi_2(x)\phi_1''(x)\,dx + \lambda_1 \int_a^b \phi_1(x)\phi_2(x)\,dx = 0$$

The first integral can be integrated by parts, to obtain

$$\phi_2(x)\phi_1'(x)|_{x=a}^{x=b} - \int_a^b \phi_1'(x)\phi_2'(x)\,dx + \lambda_1 \int_a^b \phi_1(x)\phi_2(x)\,dx = 0$$

Now we interchange the roles of (ϕ_1, λ_1) and (ϕ_2, λ_2) to obtain

$$\phi_1(x)\phi_2'(x)|_{x=a}^{x=b} - \int_a^b \phi_2'(x)\phi_1'(x)\,dx + \lambda_2 \int_a^b \phi_2(x)\phi_1(x)\,dx = 0$$

When we subtract these two equations, the first integrals cancel, and we are left with

$$\left(\phi_2(x)\phi_1'(x) - \phi_1(x)\phi_2'(x)\right)|_{x=a}^{x=b} + (\lambda_1 - \lambda_2) \int_a^b \phi_1(x)\phi_2(x)\,dx = 0$$

From the boundary conditions, we conclude that the endpoint terms contribute zero, so we are left with the statement

$$(\lambda_1 - \lambda_2) \int_a^b \phi_1(x)\phi_2(x)\,dx = 0$$

But we have assumed that $\lambda_1 - \lambda_2 \neq 0$; hence we conclude the required orthogonality. •

1.6.3. Example of transcendental eigenvalues. The next example illustrates the possibility of numerical/graphical determination of the eigenvalues.

EXAMPLE 1.6.3. ($\alpha = 0$, $0 < \beta < \pi/2$) *Find all nontrivial solutions of (1.6.1) on the interval* $0 < x < L$ *satisfying the boundary conditions* $\phi(0) = 0, h\,\phi(L) + \phi'(L) = 0$, *where* $h > 0$.

Solution. In case $\lambda = 0$, the general solution of (1.6.1) is $\phi(x) = Ax + B$. The boundary conditions further require that $0 = \phi(0) = B$, $0 = h\,\phi(L) + \phi'(L) = h(AL + B) + A = A(1 + hL)$, which requires that $A = 0, B = 0$—hence a trivial solution.

In case $\lambda = -\mu^2 < 0$, the general solution of (1.6.1) is $\phi(x) = Ae^{\mu x} + Be^{-\mu x}$. The boundary conditions further require that $0 = A + B$, $0 = h(Ae^{\mu L} + Be^{-\mu L}) + (A\mu e^{\mu L} - B\mu e^{-\mu L})$, which is satisfied if and only if $(A, B) = (0, 0)$.

In case $\lambda > 0$, the general solution is $\phi(x) = A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda})$. The boundary conditions further require that $0 = \phi(0) = A$, $0 = h\,\phi(L) + \phi'(L) = hB\sin(L\sqrt{\lambda}) + B\sqrt{\lambda}\cos(L\sqrt{\lambda})$. Clearly, neither term can be zero, so we can divide and obtain a nontrivial solution if and only if λ satisfies the equation

$$\cot(L\sqrt{\lambda}) = -\frac{h}{\sqrt{\lambda}} = -\frac{hL}{L\sqrt{\lambda}} \tag{1.6.5}$$

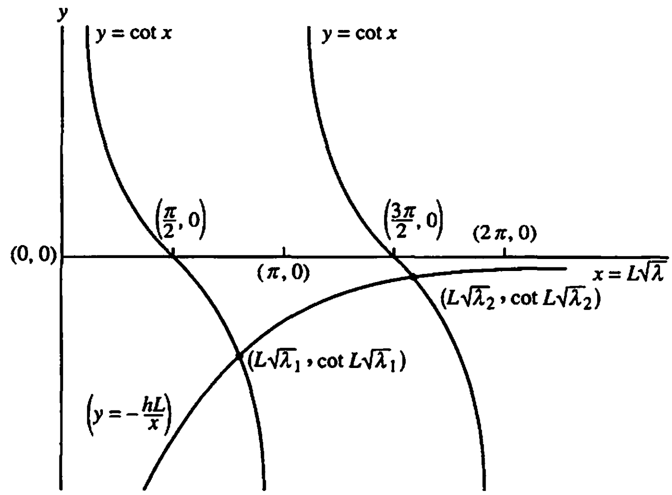


FIGURE 1.6.1 Graphical solution of $\cot(L\sqrt{\lambda}) = -h/\sqrt{\lambda}$.

Therefore we have found all eigenfunctions in the form

$$\phi_n(x) = \sin(x\sqrt{\lambda_n}) \qquad n = 1, 2, \ldots$$

where the eigenvalues λ_n are determined by solving (1.6.5). •

From the graph of the cotangent function (Fig. 1.6.1), it is seen that the eigenvalues satisfy the inequalities

$$\frac{\pi}{2} < L\sqrt{\lambda_1} < \pi, \; \frac{3\pi}{2} < L\sqrt{\lambda_2} < 2\pi, \quad L\sqrt{\lambda_n} - \left(n - \frac{1}{2}\right)\pi \to 0, \; n \to \infty$$

It is possible to make a more refined asymptotic analysis of the eigenvalues as follows. Writing $L\sqrt{\lambda} = (n - (1/2))\pi + \epsilon_n$, we invoke the Taylor expansion of the cotangent function about the point $(n - (1/2))\pi$:

$$\cot\left((n - (1/2))\pi + \epsilon\right) = -\epsilon + O(\epsilon^3) \qquad \epsilon \to 0$$

Substituting in (1.6.5), we find that

$$-\epsilon_n + O(\epsilon^3) = -\frac{hL}{(n - (1/2))\pi + \epsilon_n}$$

from which we conclude that $\epsilon_n = -hL/n\pi + O(1/n^2)$ and we get the asymptotic formula

$$L\sqrt{\lambda_n} = (n - (1/2))\pi - \frac{hL}{n\pi} + O(1/n^2) \qquad n \to \infty$$

1.6.4. Further properties: completeness and positivity. By analogy with Fourier series, we may expect to be able to expand a piecewise smooth function in a series of Sturm-Liouville eigenfunctions in the form

$$f(x) \sim \sum_{n=1}^{\infty} A_n \phi_n(x) \tag{1.6.6}$$

where the Fourier coefficients are defined by

$$A_n = \frac{\int_a^b f(x)\,\phi_n(x)\,dx}{\int_a^b \phi_n(x)^2\,dx} \qquad n = 1, 2, \ldots \tag{1.6.7}$$

The following theorem shows that we may always expect a complete set of eigenfunctions for the Sturm-Liouville eigenvalue problem.

THEOREM 1.6. *There exist an infinite sequence of solutions $\lambda_n, \phi_n(x)$ of the Sturm-Liouville eigenvalue problem defined by (1.6.1)-(1.6.3) that possess the following properties.*

- $\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \to \pi/L, \quad n \to \infty$
- *If $f(x), a < x < b$, is a piecewise smooth function, the series (1.6.6) converges to $f(x+0)/2 + f(x-0)/2$, $a < x < b$.*
- *Parseval's relation holds, in the form*

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b \phi_n(x)^2\,dx = \int_a^b f(x)^2\,dx$$

The proof will not be given here, but can be found in more advanced texts of analysis.[4]

A final point of detail regarding Sturm-Liouville eigenvalue problems is the question of *positivity* of the eigenvalues. From the previous theorem, we see that we must have $\lambda_n > 0$ for all large n, but it may happen that in some cases $\lambda_1 \le 0$—for example, $\lambda_1 = 0$ in case $\alpha = \pi/2, \beta = \pi/2$. The following *sufficient* condition is easily proved.

THEOREM 1.7. *Suppose that the parameters α, β satisfy the inequalities $0 \le \alpha < \pi/2$, $0 \le \beta < \pi/2$. Then all eigenvalues of the Sturm-Liouville eigenvalue problem (1.6.1) with the boundary conditions (1.6.2),(1.6.3) satisfy $\lambda_n > 0$.*

Proof. Suppose that $\phi(x)$ is a nontrivial solution of the Sturm-Liouville problem (1.6.1)-(1.6.3). We multiply (1.6.1) by $\phi(x)$ and integrate on the interval $a \le x \le b$:

$$\int_a^b \phi(x)\phi''(x)\,dx + \lambda \int_a^b \phi(x)^2\,dx = 0$$

[4]See, e.g., G. Birkhoff and G. C. Rota, *Ordinary Differential Equations*, Ginn, Lexington, MA, 1962.

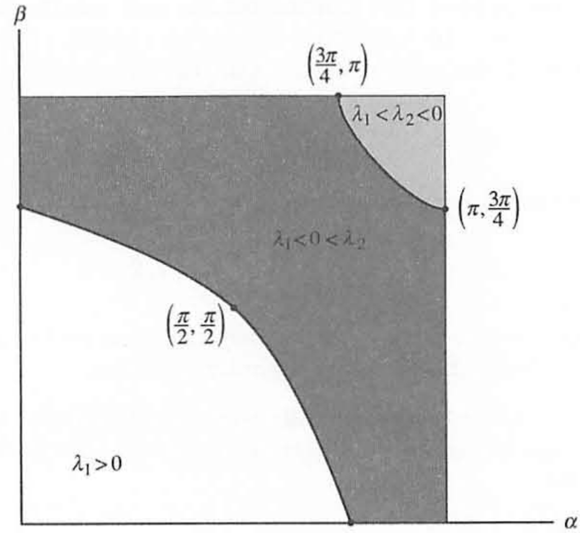


FIGURE 1.6.2 Regions of positive and negative eigenvalues.

The first integral can be integrated by parts, which leads to the identity

$$\lambda \int_a^b \phi(x)^2 \, dx = \int_a^b \phi'(x)^2 \, dx + \phi(a)\phi'(a) - \phi(b)\phi'(b)$$

The new integral on the right-hand side is strictly positive, since otherwise $\phi(x)$ would be a constant function, which is possible if and only if $\alpha = \pi/2$, $\beta = \pi/2$, which is excluded. On the other hand, we can rewrite the boundary conditions in the form $\phi(a) = L \tan\alpha \, \phi'(a)$, $\phi(b) = -L \tan\beta \, \phi'(b)$, which leads to

$$\lambda \int_a^b \phi(x)^2 \, dx > L\phi'(a)^2 \tan\alpha + L\phi'(b)^2 \tan\beta \geq 0$$

since α and β both lie in the first quadrant $0 \leq \alpha, \beta < \pi/2$. •

We emphasize that the previous theorem only provides a sufficient condition for the positivity of the eigenvalues. In order to obtain more precise results, we can plot the set of points (α, β) for which $\lambda_1 > 0$. Figure 1.6.2 shows that this region contains the square $0 \leq \alpha, \beta < \pi/2$ and is bounded by a curve whose equation is $\sin(\alpha+\beta) + \cos\alpha\cos\beta = 0$. This curve passes through the three points $(\alpha, \beta) = (3\pi/4, 0)$, $(\pi/2, \pi/2)$, and $(0, 3\pi/4)$. The complete analysis of negative eigenvalues is described next. Further details are described in the exercises.

We now present the complete analysis of the existence of negative eigenvalues for the Sturm-Liouville eigenvalue problem (1.6.1)–(1.6.3). If $\lambda = -\mu^2 < 0$ is a negative eigenvalue, then the corresponding eigenfunction must be of the form

$$\phi(x) = A \sinh \mu x + B \cosh \mu x$$

We may assume, without loss of generality, that $\mu > 0$. Applying the boundary conditions (1.6.2), (1.6.3) yields the two simultaneous linear equations

$$\begin{aligned} \cos\alpha\,(A\sinh\mu a + B\cosh\mu a) - L\sin\alpha\,(A\mu\cosh\mu a + B\mu\sinh\mu a) &= 0 \\ \cos\beta\,(A\sinh\mu b + B\cosh\mu b) + L\sin\beta\,(A\mu\cosh\mu b + B\mu\sinh\mu b) &= 0 \end{aligned}$$

For a nontrivial solution we must have $(A, B) \neq (0,0)$, which can happen if and only if the determinant of the coefficients is zero. After some algebra, this is written

$$\frac{\tanh \mu L}{\mu L} = -\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta + (L\mu)^2 \sin\alpha\sin\beta} \tag{1.6.8}$$

We consider four separate cases:

(i) $0 < \alpha < \pi/2,\ 0 < \beta < \pi/2$
(ii) $0 < \alpha < \pi/2 < \beta < \pi$
(iii) $0 < \beta < \pi/2 < \alpha < \pi$
(iv) $\pi/2 < \alpha < \pi,\ \pi/2 < \beta < \pi$

In case (i), the left side of (1.6.8) is positive, while the right side is negative for $\mu > 0$; hence there are no solutions—in accord with Theorem 1.7.

In case (ii), the denominator of the right side of (1.6.8) is zero when $\mu L = \sqrt{|\cot\alpha\cot\beta|}$, yielding a vertical asymptote, to the right of which the right side of (1.6.8) is negative. The number of solutions to (1.6.8) depends on the initial value of the right side at $\mu = 0$, which is seen to be

$$-\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} \tag{1.6.9}$$

We consider two subcases:

(iia) $\sin(\alpha+\beta) + \cos\alpha\cos\beta > 0$
(iib) $\sin(\alpha+\beta) + \cos\alpha\cos\beta < 0$

In subcase (iia) the initial value (1.6.9) is greater than 1 and the right side of (1.6.8) increases to infinity, whereas the left side remains less than 1 and tends to zero. Hence the graphs do not intersect, and we have no solution. In subcase (iib) the initial value (1.6.9) is less than 1 and the right side of (1.6.8) increases to infinity, so that the graphs must intersect at some point to the left of the vertical asymptote. Hence there exists exactly one solution μ_1 that satisfies $0 < \mu_1 L < \sqrt{|\cot\alpha\cot\beta|}$.

Case (iii) is identical to (ii) with the roles of α and β interchanged; hence the analysis is identical.

For case (iv) we rewrite (1.6.8) in the form

$$\tanh \nu = \frac{A\nu}{B + C\nu^2} \qquad \nu = L\mu \tag{1.6.10}$$

Note that the function $\nu \to A\nu/(B+C\nu^2)$ begins from the origin; it rises steadily to a maximum value, strictly larger than 1, at $\nu = \sqrt{B/C} = \sqrt{|\cot\alpha\cot\beta|}$, and then steadily decreases to zero. The number of solutions depends on the slope at $\nu = 0$, leading again to the consideration of subcases:

(iva) $\sin(\alpha+\beta) + \cos\alpha\cos\beta < 0$
(ivb) $\sin(\alpha+\beta) + \cos\alpha\cos\beta > 0$

In subcase (iva) the slope of the right side of (1.6.10) at $\nu = 0$ is greater than 1, the slope of the hyperbolic tangent; hence we have no intersection to the left of the maximum. To the right of the maximum the right side of (1.6.10) tends to zero; hence there is exactly one intersection with the graph of the hyperbolic tangent.

In subcase (ivb) the slope of the right side of (1.6.10) at $\nu = 0$ is less than the slope of the hyperbolic tangent; therefore initially it lies below the hyperbolic tangent. But at the maximum the order is reversed; hence there is precisely one solution to the left of the maximum. To the right of the maximum the right side of (1.6.10) tends steadily to zero, whereas the hyperbolic tangent tends to 1; hence there is another solution to the right.

Summarizing the preceding analysis, we have the following breakdown:

- There are no negative eigenvalues if either $0 < \alpha < \pi/2$ and $\sin(\alpha+\beta)+\cos\alpha\cos\beta > 0$ *or* $0 < \beta < \pi/2$ and $\sin(\alpha+\beta)+\cos\alpha\cos\beta > 0$.
- There is precisely one negative eigenvalue if $\sin(\alpha+\beta) + \cos\alpha\cos\beta < 0$. This is in the interval $0 < L\sqrt{-\lambda} < \sqrt{|\cot\alpha\cot\beta|}$.
- There are precisely two negative eigenvalues if $\pi/2 < \alpha < \pi$, $\pi/2 < \beta < \pi$, and $\sin(\alpha+\beta) + \cos\alpha\cos\beta > 0$. The first one satisfies $0 < L\sqrt{-\lambda_1} < \sqrt{|\cot\alpha\cot\beta|}$ while the second one satisfies $L\sqrt{-\lambda_2} > \sqrt{|\cot\alpha\cot\beta|}$.

In other words, the equation $\sin(\alpha+\beta) + \cos\alpha\cos\beta = 0$ defines two curves that divide the square $0 < \alpha < \pi$, $0 < \beta < \pi$ into three regions, corresponding to two, one, or zero negative eigenvalues. This is depicted in Fig. 1.6.2, where the unshaded region corresponds to no negative eigenvalues, the darker shaded region corresponds to one negative eigenvalue, and the lighter shaded region corresponds to two negative eigenvalues.

1.6.5. General Sturm-Liouville problems. Many of the properties of the eigenfunctions of the simple differential equation $\phi''(x) + \lambda\phi(x) = 0$ are shared by the eigenfunctions of the more general equation

$$[s(x)\phi'(x)]' + [\lambda\rho(x) - q(x)]\phi(x) = 0 \qquad a < x < b \tag{1.6.11}$$

where $s(x)$, $\rho(x)$, $q(x)$ are given functions on the interval $a < x < b$ with $\rho(x) > 0$. We have already studied the special case $s(x) \equiv 1$, $\rho(x) \equiv 1$, $q(x) = 0$. The new feature here is that the eigenfunctions will satisfy a property of *weighted orthogonality* with respect to the weight function $\rho(x), a < x < b$.

As before, we also need to consider boundary conditions at the endpoints $x = a$, $x = b$. These are written in the form (1.6.2)-(1.6.3), exactly as in the previous cases. We state and prove the corresponding orthogonality properties of the Sturm-Liouville eigenfunctions.

THEOREM 1.8. *Consider the Sturm-Liouville problem (1.6.11),(1.6.2)-(1.6.3). Suppose that $\phi_1(x), \phi_2(x)$ are nontrivial solutions with different eigenvalues $\lambda_1 \neq \lambda_2$. Then the eigenfunctions are orthogonal with respect to the weight function $\rho(x), a < x < b$:*

$$\int_a^b \phi_1(x)\, \phi_2(x) \rho(x)\, dx = 0$$

If the two eigenfunctions belong to the same eigenvalue $\lambda_1 = \lambda_2$, then the eigenfunctions must be proportional: $\phi_2(x) = C\phi_1(x)$ for some constant C.

Proof. Write the Sturm-Liouville equation satisfied by ϕ_1:

$$[s\phi_1']' + (\lambda_1 \rho - q)\phi_1 = 0$$

Multiply this equation by ϕ_2 and integrate the resulting equation on the interval $a < x < b$:

$$\int_a^b \phi_2(x)(s\phi_1'(x))'\, dx + \int_a^b \phi_2(x)(\lambda_1 \rho(x) - q(x))\phi_1(x)\, dx = 0$$

The first integral can be integrated by parts to yield

(1.6.12)

$$\phi_2(x)s(x)\phi_1'(x)|_a^b - \int_a^b \phi_2'(x)s(x)\phi_1'(x)\, dx + \int_a^b \phi_2(x)(\lambda_1\rho(x) - q(x))\phi_1(x)\, dx = 0$$

Now we interchange the roles of $\phi_1(x)$ and $\phi_2(x)$ to yield

(1.6.13)

$$\phi_1(x)s(x)\phi_2'(x)|_a^b - \int_a^b \phi_1'(x)s(x)\phi_2'(x)\, dx + \int_a^b \phi_1(x)(\lambda_2\rho(x) - q(x))\phi_2(x)\, dx = 0$$

When we subtract (1.6.12) and (1.6.13) and apply the boundary conditions, all of the terms cancel except for the final integrals. This yields the statement that $(\lambda_1 - \lambda_2) \int_a^b \phi_1(x)\phi_2(x)\rho(x)\, dx = 0$; if $\lambda_1 - \lambda_2 \neq 0$, it follows that ϕ_1 and ϕ_2 must be orthogonal with respect to the weight function ρ, which was to be proved. •

EXAMPLE 1.6.4. *Find the orthogonality relation for eigenfunctions of the Bessel equation of order zero: $(x\phi')' + \lambda x\phi = 0$.*

Solution. In this case we have $s(x) = x$, $\rho(x) = x$, $q(x) = 0$. If $\phi_1(x)$ and $\phi_2(x)$ both satisfy the same two-point boundary conditions with different eigenvalues $\lambda_1 \neq \lambda_2$, then we must have the orthogonality in the form $\int_a^b \phi_1(x)\phi_2(x)\, x\, dx = 0$. •

EXAMPLE 1.6.5. *Find the orthogonality relation for eigenfunctions of the Bessel equation of order m:* $(x\phi')' + (\lambda x - m^2/x)\phi = 0$.

Solution. In this case we have $s(x) = x$, $\rho(x) = x$, $q(x) = m^2/x$. If $\phi_1(x)$ and $\phi_2(x)$ both satisfy the same two-point boundary conditions with different eigenvalues $\lambda_1 \neq \lambda_2$, then we must have the orthogonality in the form $\int_a^b \phi_1(x)\phi_2(x)\, x\, dx = 0$. •

The orthogonality asserted in Theorem 1.8 also applies in the case of other types of boundary conditions, specifically

Periodic boundary conditions: $s(a) = s(b)$, $\phi(a) = \phi(b)$, $\phi'(a) = \phi'(b)$

Singular Sturm-Liouville problems: $s(a) = 0$, $s(b) = 0$

In each of these cases we simply need to verify that the boundary term is zero. In detail,

$$s(x)\left(\phi_1(x)\phi_2'(x) - \phi_1'(x)\phi_2(x)\right)\big|_a^b = 0$$

EXAMPLE 1.6.6. *Verify the orthogonality of eigenfunctions for the Legendre equation* $[(1-x^2)\phi']' + \lambda\phi = 0$, *where* $-1 < x < 1$.

Solution. This is a singular Sturm-Liouville problem with $s(x) = (1 - x^2)$, $\rho(x) = 1$, $q(x) = 0$, since $s(1) = 0$, $s(-1) = 0$. The weight function is $\rho(x) = 1$, so that the orthogonality relation is $\int_{-1}^1 \phi_1(x)\phi_2(x)\, dx = 0$. •

In some cases we may have a singular Sturm-Liouville problem with respect to one end. In that case we require only that the boundary condition be satisfied at the nonsingular end, where $s(x) \neq 0$. The Bessel equation on the interval $0 < x < b$ provides an example of this type.

EXAMPLE 1.6.7. *Find the orthogonality relation for eigenfunctions of the Bessel equation of order m:* $(x\phi')' + (\lambda x - m^2/x)\phi = 0$ *on the interval* $0 < x < b$.

Solution. In this case we have $s(x) = x$, $\rho(x) = x$, $q(x) = m^2/x$. If $\phi_1(x)$ and $\phi_2(x)$ both satisfy the same separable boundary conditions at $x = b$ with different eigenvalues $\lambda_1 \neq \lambda_2$, then we must have the orthogonality in the form $\int_0^b \phi_1(x)\phi_2(x)\, x\, dx = 0$. •

The case of periodic boundary conditions can be applied to give a new proof of the orthogonality of $\sin(n\pi x/L)$, $\cos(n\pi x/L)$ as follows.

EXAMPLE 1.6.8. *Consider the Sturm-Liouville eigenvalue problem for the equation $\phi'' + \lambda\phi = 0$ on the interval $-L < x < L$ with the periodic boundary conditions $\phi(-L) = \phi(L)$, $\phi'(-L) = \phi'(L)$. Find the eigenfunctions and the associated orthogonality relation for $\lambda > 0$.*

Solution. The general solution of the equation $\phi'' + \lambda\phi = 0$ with $\lambda > 0$ is $\phi(x) = A\cos x\sqrt{\lambda} + B\sin x\sqrt{\lambda}$. The periodic boundary conditions translate into the following system of two simultaneous linear equations:

$$\begin{aligned} A\cos L\sqrt{\lambda} - B\sin L\sqrt{\lambda} &= A\cos L\sqrt{\lambda} + B\sin L\sqrt{\lambda} \\ -\sqrt{\lambda}A\cos L\sqrt{\lambda} - \sqrt{\lambda}B\sin L\sqrt{\lambda} &= -A\cos L\sqrt{\lambda} + B\sin L\sqrt{\lambda} \end{aligned}$$

This system has a nontrivial solution if and only if $\sin L\sqrt{\lambda} = 0$, namely, $L\sqrt{\lambda} = n\pi$. The eigenfunctions are of the form $\phi_n(x) = A\cos(n\pi x/L) + B\sin(n\pi x/L)$, and the orthogonality relation is $\int_{-L}^{L} \phi_m(x)\phi_n(x)\,dx = 0$ if $m \neq n$. •

1.6.6. Complex-valued eigenfunctions and eigenvalues. In the above discussion of Sturm-Liouville eigenvalue problems, it has been tacitly assumed that both the eigenvalue and eigenfunction are real-valued. We now demonstrate that this leads to no loss of generality.

PROPOSITION 1.6.1. *Suppose that $\phi(x)$ is a complex-valued function and λ is a (possibly) complex number that satisfies the Sturm-Liouville equation (1.6.11) where $s(x), \rho(x), q(x)$ are real-valued functions. Suppose further that $\phi(x)$ satisfies one of the above boundary conditions. Then λ is a real number, and both the real and imaginary parts of $\phi(x)$ are eigenfunctions of the Sturm-Liouville eigenvalue problem.*

Proof. We multiply the Sturm-Liouville equation (1.6.11) by the complex conjugate of $\phi(x)$ and integrate over the basic interval:

$$\int_a^b \bar{\phi}(x)[s(x)\phi'(x)]'\,dx + \int_a^b [\lambda\rho(x) - q(x)]\bar{\phi}(x)\phi(x)\,dx = 0$$

Similarly,

$$\int_a^b \phi(x)[s(x)\bar{\phi}'(x)]'\,dx + \int_a^b [\bar{\lambda}\rho(x) - q(x)]\phi(x)\bar{\phi}(x)\,dx = 0$$

We subtract these and apply integration by parts on each of the first terms as follows:

$$\int_a^b \left(\bar{\phi}(x)[s(x)\phi'(x)]' - \phi(x)[s(x)\bar{\phi}'(x)]'\right)dx = s(x)[\bar{\phi}(x)\phi'(x) - \bar{\phi}'(x)\phi(x)|_a^b$$

But the boundary conditions imply that this term is zero. When we subtract the second terms, the result is

$$(\lambda - \bar{\lambda})\int_a^b \rho(x)|\phi(x)|^2\,dx = 0$$

which proves that the imaginary part of λ is zero; in other words, λ must be a real number. Writing $\phi(x) = u(x) + iv(x)$, we see that both $u(x)$ and $v(x)$ satisfy the same Sturm-Liouville equation that was satisfied by the complex function $\phi(x)$, which was to be proved. •

EXAMPLE 1.6.9. *Consider the Sturm-Liouville eigenvalue problem for the equation $\phi''(x) + \lambda\phi(x) = 0$. Find the complex-valued eigenfunctions satisfying the periodic boundary conditions $\phi(-L) = \phi(L)$, $\phi'(-L) = \phi'(L)$.*

Solution. From the previous work, all of the real-valued solutions are written $\sin(n\pi x/L)$, $\cos(n\pi x/L)$ with the eigenvalue $\lambda = (n\pi/L)^2$, where $n = 0, 1, 2 \dots$. The corresponding complex-valued functions may be written

$$\phi(x) = e^{in\pi x/L} \qquad \phi(x) = e^{-in\pi x/L} \quad \bullet$$

By contrast, it should be noted that in the case of two-point boundary conditions, Theorem 1.5 implies that the real and imaginary parts of a complex eigenfunction must be proportional to one another; put differently, any complex eigenfunction is a complex multiple of a real-valued eigenfunction.

EXERCISES 1.6

In Exercises 1–6, find the eigenvalues and eigenfunctions of the Sturm-Liouville eigenvalue problem (1.6.1).

1. $\phi(0) = 0, \phi'(L) = 0$
2. $\phi'(0) - h\phi(0) = 0, \phi'(L) + h\phi(L) = 0,\ h > 0$
3. $\phi'(0) = 0, \phi(L) = 0$
4. $\phi(0) = \phi(L), \phi'(0) = \phi'(L)$
5. $\phi(0) = 0, \phi'(L) - \phi(L) = 0$
6. $\phi'(0) - \phi(0) = 0, \phi'(L) = 0$
7. Show that $\lambda = 0$ is an eigenvalue of the Sturm-Liouville problem defined by (1.6.1)–(1.6.3) if and only if the parameters α, β satisfy the relation $\sin(\alpha+\beta)+\cos\alpha\,\cos\beta = 0$, which can be written in the form $\tan\alpha+\tan\beta = -1$ when $\alpha \neq \pi/2, \beta \neq \pi/2$.
8. Suppose the boundary conditions (1.6.2), (1.6.3) are written in the form $h_1\phi(0) - \phi'(0) = 0$, $h_2\phi(L) + \phi'(L) = 0$. Show that $\lambda = 0$ is an eigenvalue of the Sturm-Liouville problem if and only if the parameters h_1, h_2 satisfy the equation of the two-sheeted hyperbola: $h_1 + h_2 + Lh_1h_2 = 0$.
9. On the basis of the results in this section, how many negative eigenvalues exist for the Sturm-Liouville problem (1.6.1)–(1.6.3) in the following cases?
   (a) $\alpha = \pi/4, \beta = \pi/2$
   (b) $\alpha = \pi/4, \beta = 3\pi/4$
   (c) $\alpha = 7\pi/8, \beta = 7\pi/8$

10. Suppose $\alpha = 0$ and $0 \leq \beta < 3\pi/4$. Show directly that all eigenvalues of the Sturm-Liouville eigenvalue problem (1.6.1)–(1.6.3) satisfy $\lambda_n > 0, n = 1, 2, \ldots$. [*Hint:* If $\phi(x) = A\sinh(\mu(x-a))$ is an eigenfunction satisfying the boundary condition at $x = a$, find a transcendental equation for μ and show that it has no solution. Also check $\lambda = 0$ separately.]
11. Suppose that $\beta = 0$ and $0 \leq \alpha < 3\pi/4$. Show directly that all eigenvalues of the Sturm-Liouville eigenvalue problem (1.6.1)–(1.6.3) satisfy $\lambda_n > 0, n = 1, 2, \ldots$. [*Hint:* Use instead $\phi(x) = A\sinh(\mu(x-b))$ to find the appropriate transcendental equation.]
12. Show that the Sturm-Liouville eigenvalue problem (1.6.1)–(1.6.3) has a negative eigenvalue if and only if the the parameters α, β satisfy the inequality $\sin(\alpha+\beta) + \cos\alpha\cos\beta < 0$. [*Hint:* If $\phi(x) = A\sinh(\mu x) + B\cosh(\mu x)$ is an eigenfunction, show that μ must be a solution of the transcendental equation

$$\tanh(\mu L) = -L\mu\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta + (L\mu)^2\sin\alpha\sin\beta}$$

and that this equation will have a nonzero solution if and only if the slope at $\mu = 0$ is larger than 1.]
13. With reference to the generalized Sturm-Liouville problem, let L be the linear differential operator defined by $L\varphi = (s\varphi')' - q\varphi$. Prove the *Lagrange identity* $\varphi_2 L\varphi_1 - \varphi_1 L\varphi_2 = (s(\varphi_1'\varphi_2 - \varphi_1\varphi_2'))$, where φ_1, φ_2 are twice-differentiable functions.
14. Use the Lagrange identity to give an alternative proof of Theorem 1.8.
15. Show that if $s(x) \geq 0$, $q(x) \geq 0$, then all eigenvalues of the generalized Sturm-Liouville problem with the two-point boundary conditions $\varphi(a) = 0$, $\varphi(b) = 0$ satisfy $\lambda_n \geq 0$. [*Hint:* Apply the Lagrange identity with $\varphi_2 = 1$.]

CHAPTER 2

BOUNDARY-VALUE PROBLEMS IN RECTANGULAR COORDINATES

INTRODUCTION

In this chapter we will derive the general form of the heat equation and the wave equation for the vibrating string. These PDEs will eventually be solved in regions with rectangular, cylindrical, and spherical boundaries. In this chapter we focus attention on the case of rectangular boundaries, where we can use the usual cartesian coordinates (x, y, z), coupled with trigonometric Fourier series, which were introduced in Chapter 1. Regions with cylindrical or spherical boundaries will be treated in Chapter 3 and Chapter 4, respectively.

2.1. The Heat Equation

In this and the next two sections we will apply Fourier series to some typical problems of heat conduction. These are concerned with the flow of heat—specifically, with representing changes in temperature as a function of space and time. We denote by $u(x, y, z; t)$ the temperature measured at the point (x, y, z) at the time instant t. We suppose that u is a smooth function of $(x, y, z; t)$ and will proceed to determine a partial differential equation for u.

2.1.1. Fourier's law of heat conduction. We consider a solid material that occupies a portion of three-dimensional space. A basic quantity of importance is the *heat current density* $\mathbf{q}(\mathbf{x}; t)$. This vector quantity represents the rate of heat flow at the point $\mathbf{x} = (x, y, z)$. If $\mathbf{n}$ is any unit vector, the scalar quantity $\mathbf{q} \cdot \mathbf{n}$ is called the *heat flux* in the direction $\mathbf{n}$. It measures the rate of heat flow per unit time per unit area across a plane with normal vector $\mathbf{n}$. *Fourier's law* states that

$$\mathbf{q} = -k \operatorname{grad} u$$

where k is the thermal conductivity of the material. From calculus we know that grad u points in the direction of the maximum increase of u. Since heat is expected to flow from warmer to cooler regions, we insert the minus sign in Fourier's law. Thus $\mathbf{q}$ points in the direction of maximum *decrease* of u and $|\mathbf{q}|$ is the rate of heat flow in that direction.

2.1.2. Derivation of the heat equation. During a small time interval $(t, t+\Delta t)$ heat flows through the material and may also be generated by internal sources, at a rate $s(\mathbf{x}, t)$. Therefore the amount of heat that enters any region R of the material within the time interval $(t, t+\Delta t)$ is, to first order in Δt, given by

$$Q = \left(-\iint_{\partial R} \mathbf{q}\cdot\mathbf{n}\, dS + \iiint_R s\, dV\right)\Delta t + O(|\Delta t|^2)$$

where $\mathbf{n}$ is the outward-pointing normal vector, ∂R denotes the boundary of R, and the minus sign is in front of the surface integral because $\mathbf{q}\cdot\mathbf{n}\, dS$ is the density of heat flowing *out* of the surface element dS per unit time.

On the other hand, this heat Q has the effect of raising the temperature by the amount $u_t\,\Delta t$, to first order in Δt. Therefore we can write

$$Q = \iiint_R c\,\rho\, u_t\, dV\,\Delta t + O(|\Delta t|^2)$$

where c is the heat capacity per unit mass and ρ is the mass density of the material. Equating these, dividing by Δt, and letting $\Delta t \to 0$, we have the *continuity equation*

$$\iiint_R c\rho u_t\, dV = -\iint_{\partial R} \mathbf{q}\cdot\mathbf{n}\; dS + \iiint_R s\, dV$$

This equation is valid for any region, no matter how large or small. In particular, we take a small spherical region R about the point (x, y, z), divide by the volume, and take the limit when the diameter of the sphere tends to zero. The surface integral can be handled using the divergence theorem,

$$\iint_{\partial R} \mathbf{q}\cdot\mathbf{n}\, dS = \iiint_R (\operatorname{div}\mathbf{q})\, dV$$

and we obtain the differential form of the continuity equation:

$$c\rho u_t = \operatorname{div}(k\,\operatorname{grad} u) + s$$

This is the general form of the heat equation.

In most problems k is independent of $\mathbf{x}$, and we can bring it outside and thus obtain the heat equation in the form

$$\boxed{u_t = K\,\operatorname{div}(\operatorname{grad} u) + r = K\nabla^2 u + r} \tag{2.1.1}$$

where $K = k/c\rho$ and $r = s/c\rho$ are the renormalized conductivity and source terms, respectively. K is called the *thermal diffusivity* of the material. The *Laplacian* of a function u is defined by

$$\nabla^2 u = \operatorname{div}(\operatorname{grad} u) = u_{xx} + u_{yy} + u_{zz}$$

Remark. We can derive the heat equation without using the divergence theorem, by the following direct argument. Let R be the rectangular box defined by the inequalities $x_1 \le x \le x_2$, $y_1 \le y \le y_2$, $z_1 \le z \le z_2$, and let q^x, q^y, q^z be the components of the heat current density vector. Then

$$\begin{aligned}\int\int_{\partial R} \mathbf{q}\cdot\mathbf{n}\,dS &= \int_{z_1}^{z_2}\int_{y_1}^{y_2} [q^x(x_2,y,z) - q^x(x_1,y,z)]dy\,dz \\ &+ \int_{z_1}^{z_2}\int_{x_1}^{x_2} [q^y(x,y_2,z) - q^y(x,y_1,z)]dx\,dz \\ &+ \int_{y_1}^{y_2}\int_{x_1}^{x_2} [q^z(x,y,z_2) - q^z(x,y,z_1)]dx\,dy\end{aligned}$$

We must show that

$$\frac{1}{(x_2-x_1)(y_2-y_1)(z_2-z_1)}\int\int_{\partial R} \mathbf{q}\cdot\mathbf{n}\,dS$$

tends to div $\mathbf{q} = (q_x^x + q_y^y + q_z^z)(x_1, y_1, z_1)$ when $x_2 \to x_1$, $y_2 \to y_1$, $z_2 \to z_1$. To do this, we consider each of the three integrals separately. For the first integral we have to examine

$$\frac{1}{(y_2-y_1)(z_2-z_1)}\int_{z_1}^{z_2}\int_{y_1}^{y_2} \frac{q^x(x_2,y,z) - q^x(x_1,y,z)}{x_2-x_1}\,dy\,dz$$

When $x_2 \to x_1$, the integrand tends to $q_x^x(x_1, y, z)$, a continuous function. When $y_2 \to y_1$, $z_2 \to z_1$, the resulting integral tends to $q_x^x(x_1, y_1, z_1)$. The same result is obtained if we first let $y_2 \to y_1$, $z_2 \to z_1$. The second integral, where q^x is replaced by q^y, tends to $q_y^y(x_1, y_1, z_1)$ when $x_2 \to x_1$, $y_2 \to y_1$, $z_2 \to z_1$ in any order, and similarly for the third integral. This proves that $\int\int_R \mathbf{q}\cdot\mathbf{n}\,dS$, divided by the volume of the box R, tends to div $\mathbf{q}$ when the sides tend to zero, in any order. Referring to the continuity equation and letting $x_2 \to x_1$, $y_2 \to y_1$, $z_2 \to z_1$, we have proved that $c\rho u_t(x_1, y_1, z_1) = -\operatorname{div}\mathbf{q}(x_1, y_1, z_1) + s(x_1, y_1, z_1)$, which was to be shown.

2.1.3. Boundary conditions. The heat equation describes the flow of heat within the solid material. To completely determine the time evolution of temperature, we must also consider boundary conditions of various forms. For example, if the material is in contact with an ice-water bath, it is natural to suppose that $u = 32°\mathrm{F}$ on the boundary. Alternatively, we can imagine that the heat flux across the boundary is given; therefore by Fourier's law the appropriate boundary condition is of the type $\nabla u \cdot \mathbf{n} = a$, a given function on the boundary. For example, an insulated surface would necessitate $\nabla u \cdot \mathbf{n} = 0$ on the boundary. A third type of boundary condition results from Newton's law of cooling, written in the form

$$\mathbf{q}\cdot\mathbf{n} = h(u - T)$$

The heat flux across the boundary is proportional to the difference between the temperature u of the body and the temperature T of the surrounding medium.

2.1.4. Steady-state solutions in a slab. An important class of solutions of the heat equation are the *steady-state solutions.* This means that $\partial u/\partial t = 0$ or that u is a function of (x, y, z), independent of t. Thus we must have $K\nabla^2 u + r = 0$, a form of Poisson's equation. If in addition there are no internal sources of heat, then we have $r = 0$ and u satisfies Laplace's equation $\nabla^2 u = 0$. We restate this as follows.

PROPOSITION 2.1.1. *Steady-state solutions of the heat equation, with no internal heat sources, are solutions of Laplace's equation.*

Thus, Laplace's equation is a special case of the heat equation.

In the next three sections we will make a detailed study of the heat equation in a slab, defined by the inequalities $0 < z < L$, $-\infty < x < \infty$, $-\infty < y < \infty$. This mathematical model is appropriate for a wall of thickness L, where we ignore the variations of temperature in the x, y directions. The boundary conditions at the surfaces $z = 0$ and $z = L$ reflect the thermal properties of the inside (resp. outside) of the wall.

EXAMPLE 2.1.1. *Find the steady-state solution of the heat equation* $u_t = K\nabla^2 u$ *in the slab* $0 < z < L$ *satisfying the boundary conditions* $u(x, y, 0) = T_1$, $(\partial u/\partial z + hu)(x, y, L) = 0$, *where* T_1 *and* h *are positive constants.*

Solution. Steady-state solutions of the heat equation are solutions of Laplace's equation, $u_{xx} + u_{yy} + u_{zz} = 0$. Since the boundary conditions are independent of (x, y), we look for the solution in the form $u(x, y, z) = U(z)$, independent of (x, y). Thus U must satisfy $U''(z) = 0$, whose general solution is $U(z) = A + Bz$. The boundary condition at $z = 0$ requires $T_1 = A$, while the boundary condition at $z = L$ requires $B + h(A + BL) = 0$. Thus $B(1 + hL) = -hA = -hT_1$, and the solution is $U(z) = T_1 - hT_1 z/(1 + hL)$. •

In many problems it is important to compute the flux through the faces of the slab. From our earlier discussion, the flux is given by $-k\nabla u \cdot \mathbf{n}$; here $\mathbf{n} = (0, 0, 1)$ for the upper face and $\mathbf{n} = (0, 0, -1)$ for the lower face. Thus in Example 2.1.1, the flux from the upper face is $-k\partial U/\partial z = khT_1/(1 + hL)$, while the flux from the lower face is $k\partial U/\partial z = -khT_1/(1 + hL)$. •

We now consider an example with internal heat sources.

EXAMPLE 2.1.2. *Find the steady-state solution of the heat equation* $u_t = K\nabla^2 u + r$ *in the slab* $0 < z < L$ *satisfying the boundary conditions* $u(x, y, 0) = T_1$, $(\partial u/\partial z + hu)(x, y, L) = 0$, *where* r, K, h, *and* T_1 *are positive constants. Find the flux through the upper and lower faces.*

Solution. The boundary conditions are independent of (x, y); hence we look for the solution in the form $u(x, y, z) = U(z)$, independent of (x, y). Thus U must satisfy $KU''(z) + r = 0$, whose general solution is $U(z) = -rz^2/2K + A + Bz$.

The boundary condition at $z = 0$ requires $T_1 = A$, while the boundary condition at $z = L$ requires $-rL/K + B + h(-rL^2/2K + A + BL) = 0$. Thus $B(hL+1) = rL/K + hrL^2/2K - hT_1$. The solution is $U(z) = -rz^2/2K + T_1 + Bz$, where $B(1+b) = (rL/K)(1+\frac{1}{2}b) - hT_1$ and the *Biot modulus* b is defined as $b = hL$. The flux through the upper face is $-kU'(L) = krL/K - kB$. The flux through the lower face is $kU'(0) = kB$. •

In some cases the steady-state solution is not uniquely determined by the boundary conditions. For example, the heat equation $u_t = K\nabla^2 u$ with the boundary conditions $u_z(x,y,0) = 0$, $u_z(x,y,L) = 0$ has the solution $U(z) = A$ for *any* constant A. This phenomenon of nonuniqueness is equivalent to the statement that $\lambda = 0$ is an eigenvalue of the Sturm-Liouville problem with the associated homogeneous boundary conditions. Indeed, if we have two different steady-state solutions $U_1(z)$, $U_2(z)$ with the same nonhomogeneous boundary conditions, then the difference $U(z) = U_1(z) - U_2(z)$ is a nonzero solution of the homogeneous equation $U''(z) = 0$, satisfying the homogeneous boundary conditions. This is exactly the statement that $\lambda = 0$ is an eigenvalue of the Sturm-Liouville problem with these homogeneous boundary conditions. We will come back to this point in Sec. 2.3.

2.1.5. Time-periodic solutions. Another important class of solutions of the heat equation are the *periodic solutions*. These correspond to a stationary regime, where the solution exists for all time, $-\infty < t < \infty$. Typically the solution is specified by a boundary condition of boundedness. We illustrate with the following problem from geophysics.

The temperature at the surface of the earth is a given periodic function of time, and we seek the temperature z units below the surface. We assume that there are no internal heat sources and the thermal diffusivity is constant throughout the earth.

To formulate this problem, we suppose that the earth is flat and that the surface is given by the equation $z = 0$. (In Chapter 4 we show that the flat earth is a valid approximation for shallow depths.) The temperature on the surface is independent of location and depends only on time. Therefore we must solve the problem

$$\begin{aligned} u_t &= Ku_{zz} && z > 0,\ -\infty < t < \infty \\ u(0;t) &= u_0(t) && -\infty < t < \infty \end{aligned}$$

where $u_0(t)$ is periodic with period τ. In addition we require that the temperature be bounded,

$$|u(z;t)| \le M$$

since we do not expect that the temperature variations within the earth will exceed the variations on the surface.

To solve this problem, we first look for complex separated solutions, of the form

$$u(z;t) = Z(z)T(t)$$

Since the heat equation has real coefficients, the real and imaginary parts of a complex-valued solution are again solutions. Thus we may allow $Z(z)$, $T(t)$ to be complex-valued. Substituting into the heat equation, we have

$$\frac{KZ''(z)}{Z(z)} = \frac{T'(t)}{T(t)}$$

Both sides must be a constant, which we call $-\lambda$. Thus we have the ordinary differential equations

$$\begin{aligned} T'(t) + \lambda T(t) &= 0 \\ Z''(z) + \frac{\lambda}{K} Z(z) &= 0 \end{aligned}$$

The first equation has the solution $T(t) = e^{-\lambda t}$. Since we require bounded solutions for $-\infty < t < \infty$, λ must be pure imaginary, $\lambda = i\beta$ with β real. To solve the second equation, we try $Z(z) = e^{\gamma z}$. Thus we must have $\gamma^2 e^{\gamma z} + (\lambda/K)e^{\gamma z} = 0$, yielding the quadratic equation

$$\gamma^2 + \frac{i\beta}{K} = 0$$

In the case where $\beta > 0$, this has two solutions:

$$\gamma = \pm(-1+i)\sqrt{\frac{\beta}{2K}}$$

Since we require bounded solutions for $z > 0$, we must take the solution with $\operatorname{Re}\gamma < 0$, that is, the plus sign. Therefore we have the *complex separated solutions*

$$e^{-i\beta t} e^{(-1+i)z\sqrt{\beta/2K}}$$

Taking the real and imaginary parts, we have the real solutions

$$e^{-cz}\cos(\beta t - cz), \qquad e^{-cz}\sin(\beta t - cz), \qquad c = \sqrt{\beta/2K}$$

(If $\beta < 0$, it can be shown that no new solutions are obtained.) We refer to these as the *quasi-separated solutions.*

To solve the original problem, we suppose that the boundary temperature has been expanded as a Fourier series.

$$u_0(t) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{2n\pi t}{\tau} + B_n \sin\frac{2n\pi t}{\tau}\right)$$

We take $\beta_n = 2n\pi/\tau$, $c_n = \sqrt{n\pi/K\tau}$ in the quasi-separated solutions just developed to obtain the solution in the form

$$u(z;t) = A_0 + \sum_{n=1}^{\infty} e^{-c_n z}[A_n \cos(\beta_n t - c_n z) + B_n \sin(\beta_n t - c_n z)]$$

To verify that this is indeed a rigorous solution to the original problem, we may suppose that A_n, B_n are bounded by some constant. Then it may be shown that the formal series for u_z, u_{zz}, u_t converge uniformly, and hence u indeed satisfies the heat equation.

EXAMPLE 2.1.3. *Solve the heat equation* $u_t = Ku_{zz}$ *for* $z > 0$, $-\infty < t < \infty$, *with the boundary condition*

$$u(0;t) = A_0 + A_1 \cos \frac{2\pi t}{\tau}$$

where A_0, A_1, *and* τ *are positive constants. Graph the solution as a function of* t *for* $z\sqrt{\pi/K\tau} = 0$, $\pi/2$, π, $3\pi/2$, 2π *and* $0 \le t \le \tau$.

Solution. Referring to the general solution just obtained, we let $B_n = 0$ for $n \ge 1$ and $A_n = 0$ for $n \ge 2$. The solution is

$$u(z;t) = A_0 + A_1 e^{-c_1 z} \cos\left(\frac{2\pi t}{\tau} - z\sqrt{\frac{\pi}{K\tau}}\right)$$

In Fig. 2.1.1 we plot the temperature as a function of time for the depths indicated.

2.1.6. Applications to geophysics. This theory can be used to study the seasonal variations of temperature within the earth. For $z = 0$, the maximum of $u(z;t)$ is attained at $t = 0, \pm\tau, \pm 2\tau, \ldots$. For $z = \sqrt{\pi K\tau}$, $u(z;t)$ attains its *minimum* value for the same times, $t = 0, \pm\tau, \pm 2\tau, \ldots$. Stated differently, when it is summer on the earth's surface, it is winter at a depth of $z = \sqrt{\pi K\tau}$.

EXAMPLE 2.1.4. *Suppose that* $K = 2 \times 10^{-3}\text{cm}^2/\text{s}$, $\tau = 3.15 \times 10^7\text{s}$. *Find the depth necessary for a change from summer to winter.*

Solution. We have $\sqrt{\pi K\tau} = 4.45 \times 10^2$ cm. Therefore when it is summer on the earth's surface, it is winter at a depth of 4.4 meters. •

This theory can also be used to estimate the thermal diffusivity of the earth. To do this, we define the *amplitude variation* of the solution $u(z;t)$ as

$$A(z) = \max_{-\infty<t<\infty} u(z;t) - \min_{-\infty<t<\infty} u(z;t)$$

By measuring $A(z)$ at different depths, we may determine the diffusivity K. Indeed, using the solution obtained in Example 2.1.3, we have $\max u(z;t) = A_0 + A_1 e^{-c_1 z}$, $\min u(z;t) = A_0 - A_1 e^{-c_1 z}$, and thus $A(z) = 2A_1 e^{-c_1 z}$, $A(z)/A(0) = e^{-c_1 z}$. Let z_1 be the depth for which $e^{-c_1 z} = \frac{1}{2}$. Since $c_1 = \sqrt{\pi/K\tau}$, we have $\sqrt{\pi/K\tau}\, z_1 = \ln 2$, $K = \pi z_1^2/\tau(\ln 2)^2$.

EXAMPLE 2.1.5. *Estimate the thermal diffusivity of the earth if the summer-winter amplitude variation decreases by a factor of 2 at a depth of 1.3 meters.*

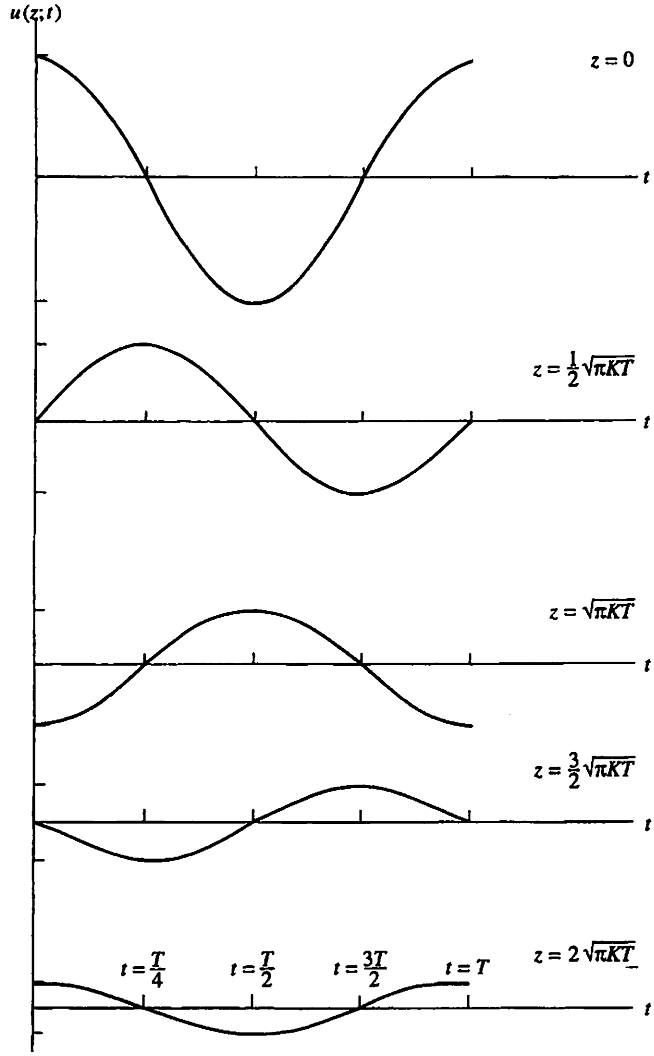


FIGURE 2.1.1 Temperature as a function of time at different depths.

Solution. We take $\tau = (365)(24)(3600) = 3.15 \times 10^7\,\text{s}$, $z_1 = 1.3$ m. Thus

$$K = \frac{\pi(1.3)^2}{(3.15 \times 10^7)(0.69)^2} = 3.5 \times 10^{-7}\,\text{m}^2/\text{s} \quad \bullet$$

2.1.7. Implementation with Mathematica. We can use Mathematica to do a three-dimensional plot of the bounded function $u(z;t)$ that satisfies the heat equation

$$u_t = K\, u_{zz}, \qquad z > 0, -\infty < t < \infty$$

with the boundary condition

$$u(0;t) = \cos\frac{2\pi t}{T}$$

From Example 2.1.3, the solution is

$$u(z;t) = e^{-c_1 z}\cos\left(\frac{2\pi t}{T} - c_1\, z\right), \qquad c_1 = \sqrt{\frac{\pi}{KT}}$$

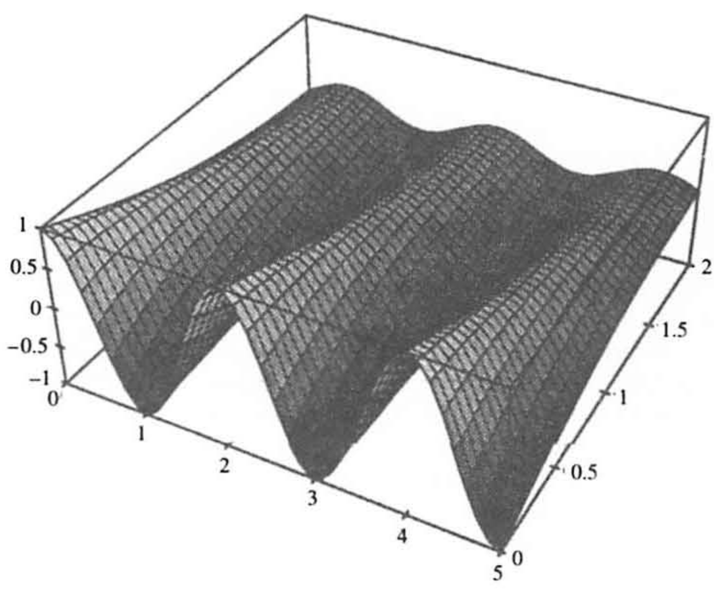
This function can be defined in Mathematica using the command

```
u[z_,t_,K_,T_]:=E^(-z Sqrt[Pi/(KT)])*Cos[2 Pi t/T - z Sqrt[Pi/(KT)]]
```

In the following graph we have chosen the parameter values $T = K = 2$; the independent variables range over the intervals $0 \le z \le 2$, $0 \le t \le 5$. The plot is realized by typing

```
Plot3D[u[z,t,2,2],{t,0,5},{z,0,2},PlotPoints->40,PlotRange->{-1,1}]
```

to yield



At the front of this graph, moving from left to right, we see the change of seasons at the surface of the earth, while at the back of the graph, moving from left to right, we see the change of seasons at a depth of 2 feet.

EXERCISES 2.1

1. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u$ in the slab $0 < z < L$, satisfying the boundary conditions $u(x, y, 0) = T_1$, $u(x, y, L) = T_2$, where T_1 and T_2 are positive constants.
2. For the solution found in Exercise 1, find the flux through the upper face $z = L$.
3. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u$ in the slab $0 < z < L$, satisfying the boundary conditions $(\partial u/\partial z)(x, y, 0) = \Phi$, $u(x, y, L) = T_0$, where Φ and T_0 are positive constants.
4. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u$ in the slab $0 < z < L$, satisfying the following boundary conditions: $[k(\partial u/\partial z) - h(u - T_0)](x, y, 0) = 0$, $[k(\partial u/\partial z) + h(u - T_1)](x, y, L) = 0$.
5. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u - \beta(u - T_3)$ in the slab $0 < z < L$, satisfying the boundary conditions $u(x, y, 0) = T_1$, $u(x, y, L) = T_2$ where T_1, T_2, T_3, and β are positive constants.
6. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u + r$ in the slab $0 < z < L$, satisfying the boundary conditions $(\partial u/\partial z)(x, y, 0) = 0$, $u(x, y, L) = T_1$ where K, r, and T_1 are positive constants. Find the flux through the face $z = L$.
7. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u + r$ in the slab $0 < z < L$, satisfying the boundary conditions $u(x, y, 0) = T_1$, $u(x, y, L) = T_2$, where K, r, T_1, and T_2 are positive constants. If $T_1 = T_2$, show that the flux across the plane $z = \frac{1}{2}L$ is zero.
8. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u + r(z)$ in the slab $0 < z < L$, satisfying the boundary condition $u(x, y, 0) = 0$, $u(x, y, L) = 0$, where $r(z) = r_0$ for $L/3 < z < 2L/3$, $r(z) = 0$ for $0 < z < L/3$ and $2L/3 < z < L$, and r_0 and K are positive constants. (*Hint:* Although u is not smooth, it may be supposed that u and u_z are both continuous.)
9. A wall of thickness 25 cm has outside temperature −10°C and inside temperature 18°C. The conductivity is $k = 0.0016$ cal/s-cm-°C and there are no internal heat sources. Find the steady-state heat flux through the outer wall, per unit area.
10. Find the solution of the heat equation $u_t = K\nabla^2 u$ in the half-space $z > 0$ for $-\infty < t < \infty$ satisfying the conditions $|u(z; t)| \leq M$, $u(0; t) = A_0 + A_1 \cos 2\pi t/\tau_1 + A_2 \cos 2\pi t/\tau_2$, where A_0, A_1, A_2, τ_1, and τ_2 are positive constants.

11. Let $u(z;t) = e^{-cz}\cos(\beta t - cz)$, where β and c are constants. Show that u satisfies the heat equation $u_t = Ku_{zz}$ if and only if $c^2 = \beta/2K$.

Exercises 12 to 14 require the solution of the heat equation in the slab $0 < z < L$, where one face is maintained at temperature zero. Thus we have the boundary-value problem

$$\begin{aligned} u_t &= Ku_{zz} && 0 < z < L, -\infty < t < \infty \\ u(0;t) &= A_0 + A_1\cos(2\pi t/\tau) && -\infty < t < \infty \\ u(L;t) &= 0 && -\infty < t < \infty \end{aligned}$$

12. Find all complex separated solutions satisfying the heat equation that are of the form $u(z;t) = e^{\gamma z}e^{i\beta t}$, where β is positive.
13. By taking the real and imaginary parts of the complex-valued solutions found in Exercise 12, show that we have the quasi-separated solutions

$$\begin{aligned} u(z;t) &= e^{cz}\cos(\beta t + cz) \qquad & u(z;t) &= e^{-cz}\cos(\beta t - cz) \\ u(z;t) &= e^{cz}\sin(\beta t + cz) \qquad & u(z;t) &= e^{-cz}\sin(\beta t - cz) \end{aligned}$$

    where $c = \sqrt{\beta/2K}$.
14. By taking suitable linear combinations of the quasi-separated solutions found in Exercise 12 and steady-state solutions, solve the boundary-value problem in the slab $0 < z < L$.
15. Suppose that the daily temperature variation at the earth's surface is a periodic function $\varphi(t) = A_0 + A_1\cos(2\pi t/\tau)$. Find the depth necessary for a change from maximum to minimum daily temperature if $K = 2 \times 10^{-3}$ cm^2/s and $\tau = 24 \times 3600$ s.
16. Find the bounded solution of the heat equation $u_t = Ku_{zz}$ for $z > 0$, $-\infty < t < \infty$, satisfying the boundary conditions $u(0;t) = 1$ for $0 < t < \frac{1}{2}\tau$, $u(0;t) = -1$ for $\frac{1}{2}\tau < t < \tau$, where $u(0;t)$ is periodic with period τ.
17. Find the bounded solution of the heat equation $u_t = Ku_{zz}$ for $z > 0$, $-\infty < t < \infty$, satisfying the boundary condition $u_z(0;t) = A_1\cos\beta t$, where β and A_1 are positive constants.
18. Find the bounded solution of the heat equation $u_t = Ku_{zz}$ for $z > 0$, $-\infty < t < \infty$, satisfying the boundary condition $u_z(0;t) - hu(0;t) = A_1\cos\beta t$, where h, β, and A_1 are positive constants.
19. For the solution found in Exercise 14, find the limit of $u(z;t)$ when $L \to \infty$ and compare it with the solution for Example 2.1.3.
20. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u + r$ in the slab $0 < z < L$ satisfying the boundary conditions $u_z(0;t) = h[u(0;t) - T_1]$, $u_z(L;t) = -h[u(L;t) - T_2]$, where r, h, T_1, and T_2 are positive constants.
21. For which values of the constants K, r, Φ_1, and Φ_2 does there exist a steady-state solution of the equation $u_t = K\nabla^2 u + r$ satisfying the boundary conditions $u_z(x,y,0;t) = \Phi_1$, $u_z(x,y,L;t) = \Phi_2$?

2.2. Homogeneous Boundary Conditions on a Slab

Many problems in mathematical physics and engineering involve a partial differential equation with initial conditions and boundary conditions. In this section we consider the case of homogeneous boundary conditions for the heat equation in the slab $0 < z < L$. In Sec. 2.3 we will consider the general nonhomogeneous boundary condition.

A homogeneous boundary condition at $z = 0$ has one of the following forms:

$$u(0;t) = 0 \quad \text{or} \quad u_z(0;t) = 0 \quad \text{or} \quad u_z(0;t) = hu(0;t)$$

where h is a nonzero constant that has the dimension of length^{-1}. All three of these may be included in the following succinct form:

$$\cos\alpha\, u(0;t) - L \sin\alpha\, u_z(0;t) = 0 \tag{2.2.1}$$

where the dimensionless parameter α satisfies $0 \le \alpha < \pi$. When $\alpha = 0$ we have the first boundary condition, $u(0;t) = 0$; when $\alpha = \pi/2$ we have the second boundary condition, $u_z(0;t) = 0$; and when $\cot\alpha = hL$ we have the third boundary condition, $u_z(0;t) = hu(0;t)$. Similarly, the general homogeneous boundary condition at $z = L$ is written in the form

$$\cos\beta\, u(L;t) + L \sin\beta\, u_z(L;t) = 0 \tag{2.2.2}$$

where $0 \le \beta < \pi$. The constant β is not related to α, in general.

2.2.1. Separated solutions with boundary conditions. We now discuss separated solutions of the heat equation $u_t = Ku_{zz}$ with the homogeneous boundary conditions (2.2.1) and (2.2.2). A separated solution of the heat equation is written

$$u(z;t) = \phi(z)T(t)$$

Substituting in the heat equation $u_t = Ku_{zz}$, we obtain

$$\phi(z)T'(t) = K\phi''(z)T(t)$$

Dividing by $K\phi(z)T(t)$, we obtain $T'(t)/KT(t) = \phi''(z)/\phi(z)$. The left side depends on t alone, and the right side depends on z alone; therefore each is a constant, which we call $-\lambda$. Thus we have the ordinary differential equations

$$T'(t) + \lambda K T(t) = 0 \tag{2.2.3}$$

$$\phi''(z) + \lambda\phi(z) = 0 \tag{2.2.4}$$

Equation (2.2.3) has the solution $T(t) = e^{-\lambda K t}$, which is never zero. To the second equation, (2.2.4), we must add the boundary conditions (2.2.1) and (2.2.2). The product $u(z;t) = \phi(z)T(t)$ satisfies (2.2.1) if and only if $\phi(z)$ satisfies the boundary condition $\cos\alpha\phi(0) - L\sin\alpha\phi'(0) = 0$. Similarly, $u(z;t)$ satisfies (2.2.2) if and only if $\phi(z)$ satisfies the boundary condition $\cos\beta\phi(L) + L\sin\beta\phi'(L) = 0$. This leads us to the following proposition.

PROPOSITION 2.2.1. *The separated solutions of the heat equation* $u_t = Ku_{zz}$ *with the boundary conditions (2.2.1) and (2.2.2) are of the form* $u_n(z;t) = e^{-\lambda_n Kt}\phi_n(z)$ *where* λ_n *is an eigenvalue and* $\phi_n(z)$ *is an eigenfunction of the Sturm-Liouville eigenvalue problem* $\phi''(z) + \lambda\phi(z) = 0$ *with the boundary conditions* $\cos\alpha\,\phi(0) - L\sin\alpha\,\phi'(0) = 0$, $\cos\beta\,\phi(L) + L\sin\beta\,\phi'(L) = 0$. *These eigenfunctions satisfy the orthogonality relation* $\int_0^L \phi_n(z)\phi_m(z)dz = 0$ *for* $m \neq n$.

Our first example corresponds to a slab with both faces maintained at temperature zero.

EXAMPLE 2.2.1. *Find all the separated solutions of the heat equation* $u_t = Ku_{zz}$ *for* $0 < z < L$ *satisfying the boundary conditions* $u(0;t) = 0$, $u(L;t) = 0$.

Solution. The associated Sturm-Liouville problem is $\phi''(z) + \lambda\phi(z) = 0$ with the boundary conditions $\phi(0) = 0$, $\phi(L) = 0$. In Sec. 1.6, we found that the solutions are $\phi_n(z) = \sin(n\pi z/L)$, $\lambda_n = (n\pi/L)^2$. Thus we have the separated solutions

$$u_n(z;t) = \sin(n\pi z/L)\, e^{-(n\pi/L)^2 Kt}, \qquad n = 1, 2, \ldots \quad \bullet$$

The next example corresponds to a slab with one face insulated and the other face maintained at temperature zero.

EXAMPLE 2.2.2. *Find all the separated solutions of the heat equation* $u_t = Ku_{zz}$ *for* $0 < z < L$ *satisfying the boundary conditions* $u(0;t) = 0$, $u_z(L;t) = 0$.

Solution. The associated Sturm-Liouville problem is $\phi''(z) + \lambda\phi(z) = 0$ with the boundary conditions $\phi(0) = 0$, $\phi'(L) = 0$. For $\lambda = 0$ the general solution of the differential equation is $\phi(z) = Az + B$. The first boundary condition requires $B = 0$, while the second boundary condition requires $A = 0$. Hence $\lambda = 0$ is not an eigenvalue. For $\lambda = -\mu^2 < 0$ the general solution satisfying the first boundary condition is $\phi(z) = A\sinh(\mu z)$, but this satisfies the second boundary condition if and only if $A = 0$; hence $\lambda < 0$ is not a possible eigenvalue. For $\lambda > 0$ the general solution of the differential equation is $\phi(z) = A\sin z\sqrt{\lambda} + B\cos z\sqrt{\lambda}$. The first boundary condition requires that $B = 0$, while the second boundary condition requires that $A\cos L\sqrt{\lambda} = 0$. For a nonzero solution we must take $L\sqrt{\lambda} = (n - \frac{1}{2})\pi$, $n = 1, 2, \ldots$. Therefore the solutions are $\phi_n(z) = \sin(n - \frac{1}{2})\pi z/L$, $\lambda_n = (n - \frac{1}{2})^2\pi^2/L^2$. The separated solutions of the heat equation are

$$u_n(z;t) = \sin\left(\left(n - \frac{1}{2}\right)\frac{\pi z}{L}\right)\exp\left[-\left(n - \frac{1}{2}\right)^2 \pi^2 \frac{Kt}{L^2}\right], \quad n = 1, 2, \ldots \quad \bullet$$

2.2.2. Solution of the initial-value problem in a slab. Having obtained the separated solutions of the heat equation with homogeneous boundary conditions, we can solve the following *initial-value problem:*

$$\begin{aligned} u_t &= K u_{zz} && t > 0, 0 < z < L \\ \cos\alpha\, u(0;t) - L \sin\alpha\, u_z(0;t) &= 0 && t > 0 \\ \cos\beta\, u(L;t) + L \sin\beta\, u_z(L;t) &= 0 && t > 0 \\ u(z;0) &= f(z) && 0 < z < L \end{aligned}$$

where $f(z), 0 < z < L$, is a piecewise smooth function.

To solve this initial-value problem, we first expand $f(z)$ in a series of eigenfunctions of the Sturm-Liouville problem, in the form

$$f(z) = \sum_{n=1}^{\infty} A_n \phi_n(z) \qquad 0 < z < L$$

[If f is discontinuous at z, the series converges to $\frac{1}{2}f(z+0) + \frac{1}{2}f(z-0)$.] The formal solution of the initial-value problem is given by the series

$$\boxed{u(z;t) = \sum_{n=1}^{\infty} A_n \phi_n(z) e^{-\lambda_n K t}} \tag{2.2.5}$$

The solution has been written as a superposition of separated solutions of the heat equation satisfying the indicated homogeneous boundary conditions. The Fourier coefficients A_n are obtained from the orthogonality relations by the formulas

$$\int_0^L f(z)\phi_n(z)dz = A_n \int_0^L \phi_n(z)^2 dz \qquad n = 1, 2, \ldots$$

To prove that the formal solution (2.2.5) is a rigorous solution of the heat equation, we must check that, for each $t > 0$, the series for u, u_z, u_{zz}, and u_t are uniformly convergent for $0 \le z \le L$. This can be shown for each type of boundary condition we consider.

EXAMPLE 2.2.3. *Solve the initial-value problem $u_t = K u_{zz}$ for $t > 0$, $0 < z < L$, with the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$ and the initial condition $u(z;0) = 1$.*

Solution. The separated solutions of the heat equation satisfying the boundary conditions are $\sin(n\pi z/L)\, e^{-(n\pi/L)^2 K t}$, $n = 1, 2, \ldots$. To satisfy the initial condition, we must expand the function $f(z) = 1$ in a Fourier sine series. The Fourier coefficients are given by

$$A_n \int_0^L \sin^2 \frac{n\pi z}{L}\, dz = \int_0^L \sin \frac{n\pi z}{L}\, dz = \frac{L}{n\pi}[1 - (-1)^n]$$

Thus $A_n = (2/n\pi)[1-(-1)^n]$ and the solution is

$$u(z;t) = \frac{2}{\pi}\sum_1^\infty \frac{1-(-1)^n}{n} \sin\frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt}$$

For $t > 0$ and $0 \le z \le L$, this series converges uniformly, owing to the exponential factor. Likewise, the series for u_z, u_{zz}, and u_t converge uniformly for $0 \le z \le L$ and each $t > 0$. Thus u is a rigorous solution of the heat equation. •

2.2.3. Asymptotic behavior and relaxation time. In Example 2.2.3 we obtained a *transient solution* of the heat equation, meaning that $u(z;t)$ tends to zero when t tends to infinity. To analyze this more generally, we assume that the boundary conditions are $u(0;t) = 0$, $u(L;t) = 0$ and the initial condition is $u(z;0) = f(z)$, a piecewise smooth function. The solution is

$$u(z;t) = \sum_{n=1}^\infty A_n \sin\frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt}$$

where A_n are the Fourier sine coefficients of the piecewise smooth function $f(z)$, $0 < z < L$. Thus

$$A_n = \frac{2}{L}\int_0^L f(z)\sin(n\pi z/L)\,dz \quad \text{and} \quad |A_n| \le 2M$$

where M is the maximum of $|f(z)|$, $0 < z < L$. Writing $a = \pi^2 K/L^2$ and noting that $|\sin n\pi z/L| \le 1$, we have

$$|u(z;t)| \le 2M\sum_{n=1}^\infty e^{-n^2 at}$$

But $n^2 \ge n$ for $n \ge 1$, and thus $e^{-n^2at} \le e^{-nat} = (e^{-at})^n$. Hence

$$\begin{aligned} |u(z;t)| &\le 2M\sum_{n=1}^\infty (e^{-at})^n \\ &= 2M\frac{e^{-at}}{1-e^{-at}} \end{aligned}$$

where we have used the formula for the sum of a geometric series $\sum_{n=1}^\infty \gamma^n = \gamma/(1-\gamma)$, $0 \le \gamma < 1$. When $t \to \infty$, $e^{-at} \to 0$, and we have shown that

$$u(z;t) = O(e^{-at}) \qquad t \to \infty$$

In particular $u(z;t) \to 0$ when $t \to \infty$, which means that $u(z;t)$ is a transient solution.

We define the *relaxation time* τ by the formula

$$\frac{1}{\tau} = -\lim_{t\to\infty}\frac{1}{t}\ln|u(z;t)|$$

provided that the limit exists and is independent of z, $0 < z < L$. For transient solutions of the heat equation, the relaxation time can be computed explicitly from the first nonzero term of the series solution. The following theorem extends the previous example to the general set of homogeneous boundary conditions.

THEOREM 2.1. *For the heat equation $u_t = Ku_{zz}$ with the boundary conditions (2.2.1) and (2.2.2), suppose that all eigenvalues λ_n are positive. Then $u(z;t) = \sum_{n=1}^{\infty} A_n\phi_n(z)e^{-\lambda_n Kt}$ is a transient solution of the heat equation, and the relaxation time is given by $\tau = 1/\lambda_1 K$ if $A_1 \neq 0$.*

EXAMPLE 2.2.4. *Compute the relaxation time for the solution*

$$u(z;t) = \sum_{n=1}^{\infty} A_n \sin(n\pi z/L)\, e^{-(n\pi/L)^2 Kt}$$

Solution. We write

$$u(z;t) = A_1 \sin\frac{\pi z}{L}\, e^{-(\pi/L)^2 Kt} + \sum_{n=2}^{\infty} A_n \sin\frac{n\pi z}{L}\, e^{-(n\pi/L)^2 Kt}$$

From the preceding analysis the last series is $O(e^{-(4\pi^2 Kt/L^2)})$ when $t \to \infty$. If $A_1 \neq 0$, we may write

$$\begin{aligned} u(z;t) &= A_1 \sin\frac{\pi z}{L}\, e^{-(\pi/L)^2 Kt}[1 + O(e^{-(3\pi^2 Kt/L^2)})] \\ \ln|u(z;t)| &= \ln|A_1| + \ln\sin\frac{\pi z}{L} - (\pi/L)^2 Kt + O(e^{-(3\pi^2 Kt/L^2)}) \end{aligned}$$

Thus $\lim_{t\to\infty} t^{-1}\ln|u(z;t)| = -\pi^2 K/L^2$. We have proved that $\tau = L^2/\pi^2 K$ provided that $A_1 \neq 0$. •

This analysis of relaxation time shows that, for large t, the solution $u(z;t)$ is well approximated by the first term of the series. This can also be seen graphically, by plotting the function $z \to u(z;t)$ for various values of t. When t is small, the solution is close to the initial function $f(z)$. As t increases, the solution tends to zero and assumes the shape of a sine curve, corresponding to the first term of the series solution. The graphs in Fig. 2.2.1 plot the solution of the initial-value problem $u_t = 2u_{zz}$ for $0 < z < \pi$, with the boundary conditions $u(z;0) = 0$, $u(\pi;0) = 0$ and the initial conditions $u(z;0) = 2z$ for the times $t = 0$, 0.005, 0.01, 0.05, 0.1, 0.2, 0.3, 0.5, 0.7, and 0.8.

2.2.4. Uniqueness of solutions. We now discuss the *uniqueness* of the solution of the initial-value problem. We have found a solution as a series of separated solutions, but it is conceivable that by another method we might produce a distinct solution of the heat equation with the same initial conditions and boundary conditions. We shall prove that this is impossible. To be specific, we take the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$.

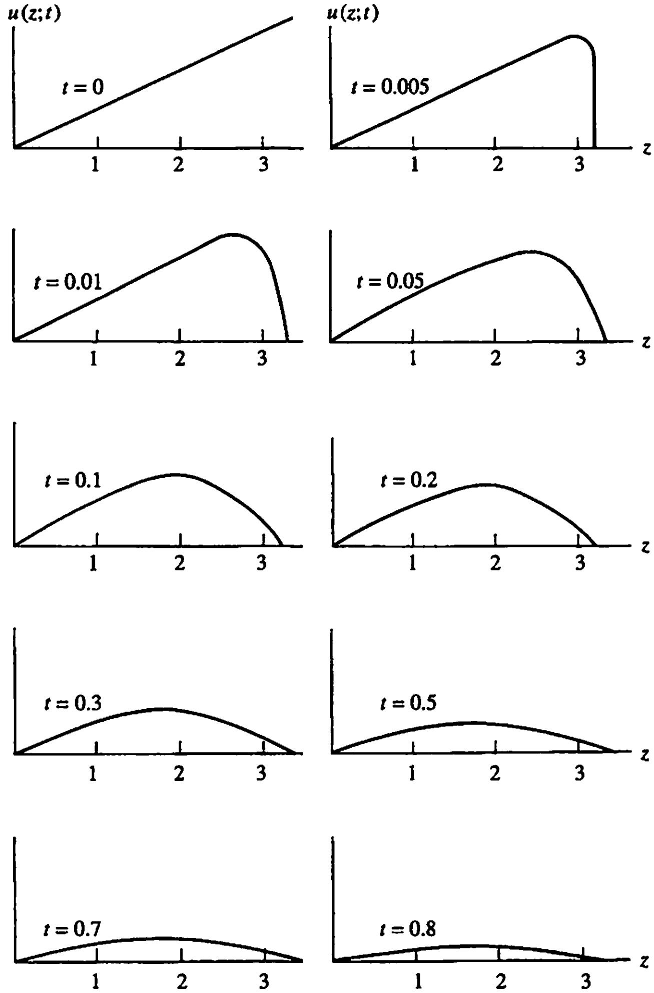


FIGURE 2.2.1 Solution of the heat equation at 10 different times.

For this purpose, suppose that u_1 and u_2 are two solutions with the same initial and boundary conditions, and set $u = u_1 - u_2$. Then u satisfies the heat equation with zero boundary conditions and zero initial conditions. Let

$$w(t) = \frac{1}{2}\int_0^L u(z;t)^2 dz$$

Then

$$w'(t) = \int_0^L u(z;t)u_t(z;t)dz \tag{2.2.6}$$

$$= K\int_0^L u(z;t)u_{zz}(z;t)dz \tag{2.2.7}$$

$$= Ku(z;t)u_z(z;t)|_0^L - K\int_0^L u_z(z;t)^2 dz \tag{2.2.8}$$

where we have used the heat equation to obtain (2.2.7) and integration by parts to obtain (2.2.8). Using the boundary conditions, we see that the first term in (2.2.8) is zero. Therefore we must have

$$w'(t) = -K\int_0^L u_z(z;t)^2 dz$$

But K is a positive constant and $u_z(z;t)^2 \geq 0$, since squares of real numbers are greater than or equal to zero. Thus we have both

$$w'(t) \leq 0 \quad \text{and} \quad w(t) \geq 0$$

But $u(z;0) = 0$, which means that $w(0) = 0$. To complete the proof, we use the fundamental theorem of calculus:

$$w(t) = w(0) + \int_0^t w'(s)ds \leq 0$$

Since $w(t) \geq 0$, we are forced to conclude that $w(t) \equiv 0$, which means that $u(z;t) = 0$ for each t, that is, $u_1(z;t) = u_2(z;t)$. Hence we have proved uniqueness of the solution.

The careful reader will notice that we have used the boundary conditions only to show that $uu_z\,|_0^L = 0$. Hence our proof applies also to other boundary conditions, for example, $u_z(0) = 0$, $u_z(L) = 0$.

2.2.5. Examples of transcendental eigenvalues. In certain cases we must solve the heat equation with the homogeneous boundary conditions

$$u(0;t) = 0, \qquad u_z(L,t) + hu(L;t) = 0 \tag{2.2.9}$$

where h is a positive constant. We will see that the eigenvalues are obtained by solving a transcendental equation. The separated solutions of the problem are of the form $u(z;t) = \phi(z)T(t)$, where $T(t) = e^{-\lambda Kt}$, λ is an eigenvalue, and $\phi(z)$ is an eigenfunction of the Sturm-Liouville problem $\phi''(z) + \lambda\phi(z) = 0$ with the boundary conditions $\phi(0) = 0$, $\phi'(L) + h\,\phi(L) = 0$. This was solved as Example 1.6.3 in Sec. 1.6, where we found the solutions $\phi(z) = B\sin(z\sqrt{\lambda})$, where λ is determined as a solution of the transcendental equation

$$\sqrt{\lambda}\cos(L\sqrt{\lambda}) + h\sin(L\sqrt{\lambda}) = 0 \tag{2.2.10}$$