DYNAMICS OF QUADRATIC POLYNOMIALS, I-II.

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June 23, 1997

Abstract. In the first part of this paper we study combinatorics and geometry of the Yoccoz puzzle. We prove that the moduli of the principal nest of annuli grow at least linearly, and derive from there *a priori* bounds for a certain class of infinitely renormalizable quadratics. In the second part we prove for these quadratics local connectivity of the Julia and the Mandelbrot sets. Density of hyperbolic maps in the real quadratic family follows.

This is a unified version of the preprints IMS at Stony Brook # 1993/9, 1995/14 and the MSRI preprint # 026-95; to appear in Acta Math.

Contents

1. Introduction

Rigidity is a fundamental phenomenon in hyperbolic geometry and holomorphic dynamics. Its meaning is that the metric properties of certain manifolds or dynamical systems are determined by their combinatorics. Celebrated works of Mostow, Thurston, Sullivan, Yoccoz, among others, provide us with examples of rigid objects. Moreover, this phenomenon is intimately linked to the universality phenomenon, to basic measure-theoretical and topological properties of systems, to the problem of describing typical systems.

In the set up of holomorphic dynamics the general rigidity problem can be posed as follows:

Rigidity Problem. *Any two combinatorially equivalent rational maps are quasi-conformally equivalent. Except for the Lattés examples, the quasi-conformal deformations come from the dynamics on the Fatou set.*

Though the general problem is still far from being solved, there have been recently several breakthroughs in the quadratic case when the problem is equivalent to the famous MLC Conjecture ("the Mandelbrot set is locally connected"). In this case the problem has been directly linked to the Renormalization Theory. In 1990 Yoccoz proved MLC for all parameter values which are at most finitely renormalizable. In this paper we will prove MLC for a certain class of infinitely renormalizable maps. To this end we carry out a geometric analysis of Julia sets which has already found a number of other interesting applications.

Our analysis exploits a new powerful tool called "puzzle". It was introduced by Branner and Hubbard [BH] for cubic maps with one escaping critical point and by Yoccoz for quadratics (see [H], [M2]). The main geometric result of these works is the divergence property of moduli of a certain nest of annuli (provided the map is non-renormalizable). This implies that the corresponding domains ("puzzle pieces") shrink to points, which yields, for a non-renormalizable quadratic, local connectivity of the Julia set. Transferring this result to the parameter plane yields local connectivity of the Mandelbrot set at the corresponding parameter values.

The geometric results of Branner-Hubbard and Yoccoz don't contain information on the *rate* at which the pieces shrink to points. In this work we tackle this problem. We consider a smaller nest $V^0 \supset V^1 \supset \dots$ of puzzle pieces called *principal*, and prove that the moduli of the annuli $A^n = V^{n-1} \setminus V^n$ grow at linear rate over a certain combinatorially specified subsequence of levels:

Theorem III (moduli growth). *Let $n(k)$ counts the non-central levels in the principal nest $\{V^n\}$. Then*

$$\mathrm{mod}\, A^{n(k)+2} \geq Bk,$$

where the constant B depends only on the first modulus $\mu_1 = \mathrm{mod}\, A^1$.

To gain control of the first principal modulus, mod A^1, we consider a class $\mathcal{SL}$ of quadratics satisfying the *secondary limbs condition.* This class, in particular, contains

- Maps which are at most finitely renormalizable and don't have non-repelling periodic points (Yoccoz class);
- Infinitely renormalizable maps of bounded type;
- Real maps which don't have non-repelling periodic points.

In §4, Theorem I, we construct, for maps of class $\mathcal{SL}$, a dynamical annulus A^1 with a definite modulus.

A basic geometric quality of infinitely renormalizable maps are *a priori* bounds. They provide a key to the renormalization theory, problems of rigidity and local connectivity. In this paper we prove *a priori* bounds for maps of class $\mathcal{SL}$ with sufficiently big combinatorial type (§7, Theorems IV and IV′).

Being specified for real quasi-quadratic maps of Epstein class, this result yields complex bounds on every renormalization level with sufficiently big "essential period" (§8, Theorem V). In a more recent work [LY] complex bounds were established for maps with essentially bounded combinatorics. Altogether this yields:

Complex Bounds Theorem (joint with Yampolsky). *Let f be an infinitely renormalizable quasi-quadratic map of Epstein class. Then for all sufficiently big m, the renormalization $R^m f$ is quadratic-like with a definite modulus: $\mathrm{mod}\, R^m f \geq \mu > 0$, with an absolute μ. If f is a quadratic polynomial, this occurs for all m.*

This result was independently proven by Levin & van Strien [LS].

In Part II we use the above geometric information to prove the following result:

Rigidity Theorem. *Any combinatorial class contains at most one quadratic polynomial satisfying the secondary limbs condition with a priori bounds.*

We also show that the quadratics satisfying the above assumptions have locally connected Julia sets (Theorem VI, §9). In particular, all real quadratics have locally connected Julia sets (see also [LS]).

Conjecture. *The secondary limbs condition implies a priori bounds.*

Let $\mathcal{QC}(c) \subset \mathcal{T}op(c) \subset \mathcal{C}om(c) \subset \mathbb{C}$ stand respectively for the quasi-conformal, topological and combinatorial classes of the quadratic map P_c. A map P_c is called combinatorially (respectively topologically or quasi-conformally) rigid if $\mathcal{C}om(c) = \{c\}$ (respectively $\mathcal{T}op(c) = \{c\}$ or $\mathcal{QC}(c) = \{c\}$).

The strongest, combinatorial, rigidity of a map P_c turns out to be equivalent to the local connectivity of the Mandelbrot set M at c (see [DH1, Sc1]). This property of M was conjectured by Douady and Hubbard under the name "MLC".

Corollary 1.1. *For a quadratic polynomial $P_c \in \mathcal{SL}$ of a sufficiently big type (that is, satisfying the assumptions of Theorem IV′) the Julia set $J(f)$ is locally connected, and the Mandelbrot set is locally connected at c.*

In particular, this gives first examples of infinitely renormalizable parameter values $c \in M$ of *bounded type* where MLC holds (though one needs a minor part of Corollary 1.1 to produce some examples of such kind).

One might wonder of how big is the set of infinitely renormalizable parameter values satisfying the assumptions of Corollary 1.1. It is obviously dense on the boundary of the Mandelbrot set. We can show that this set has Lebesgue measure zero and Hausdorff dimension at least 1 [L10]. Note that 1=(1/2)2 where 2 = HD(∂M) by Shishikura's Theorem [Sh1].

Let us now dwell on the case of real parameter values $c \in [-2, 1/4]$. Corollary 1.1 implies MLC (and thus complex rigidity) at real c with sufficiently big "essential period" on all renormalization levels (§12, Theorem VIII). For the rest of real parameters the Rigidity and Complex Bounds Theorems imply a weaker property, real rigidity. Let us say that a parameter value $c \in \mathbb{R}$ (or the corresponding quadratic polynomial P_c) is *rigid on the real line* if $\mathcal{C}om(c) \cap \mathbb{R} = \{c\}$. Thus we have:

Density Theorem. *Any real quadratic polynomial P_c without attracting cycles is rigid on the real line. Thus hyperbolic quadratics are dense on the real line.*

(The latter statement follows from the former by the Milnor-Thurston kneading theory [MT]).

Among other applications of the above results are the proof of the Feigenbaum-Coullet-Tresser Renormalization Conjecture [L9] and an advance in the problem of absolutely continuous invariant measures (joint with Martens & Nowicki [L8, MN]).

Let us now describe the structure of the paper.

In §2, we overview the necessary preliminaries in holomorphic dynamics, particularly Douady-Hubbard renormalization and the Yoccoz puzzle.

In §3 we present our approach to combinatorics of the puzzle. The main concepts involved are the *principal nest* of puzzle pieces, *generalized renormalization* and *central cascades.* As we indicated above, the principal nest $V^0 \supset V^1 \supset \dots$ contains the key combinatorial and geometric information about the puzzle. We describe the combinatorics of this nest by means of *generalized renormalizations*, that is, appropriately restricted first return maps considered up to rescaling.

It may happen that a quadratic-like map $g_n : V^n \to V^{n-1}$ has "almost connected" Julia set. This phenomenon often requires a special treatment. Such a map generates a subnest of the principal nest called a *central cascade.* The number of central cascades in the principal nest is called the *height* $\chi(f)$ of a map f. In other words, $\chi(f)$ is the number of different quadratic-like *germs* among the g_n's. It will play a crucial role for our discussion.

In §4 we study the initial geometry of the puzzle. The main result of this section is the construction of an initial annulus $A^1 = V^0 \setminus V^1$ with definite modulus, provided the hybrid class of a map is selected from a truncated secondary limb (Theorem I).

In §5 we define a new geometric parameter (worked out jointly with J. Kahn), *the asymmetric modulus*, and prove that it is *monotonically non-decreasing* when we go through the principal nest (Theorem II). This already provides us with lower bounds for the principal moduli $\mu_n = \operatorname{mod} A^n$ (which, by the way, implies the Branner-Hubbard-Yoccoz divergence property), and upper bounds on the distortion. We reach these results by means of a purely combinatorial analysis plus the standard Grötzsch inequality.

Our main geometric result, Theorem III, is proven in §6. The above analysis does not always yield the linear growth of moduli. In particular, it is not good enough for the basic example called the *Fibonacci map*. Proof of the moduli growth for the Fibonacci combinatorics is the heart of the whole paper (§6.4). This crucial step is based on the *Definite Grötzsch inequality*, estimates of hyperbolic distances between puzzle pieces and analysis of their shapes. The key observation is that sufficiently pinched pieces make a definite extra contribution to the moduli growth.

In the next section, §7 we prove a priori bounds for infinitely renormalizable quadratics of sufficiently big type (Theorems IV and IV′). The meaning of this condition is that certain combinatorial parameters of the renormalized maps $R^n f$ are sufficiently big. The main such a parameter is the above mentioned height, but there are also a few others. These conditions together mean roughly that the *periods* of $R^n f$ are sufficiently big, except for a possibility of long "parabolic or Siegel cascades".

In the last section of Part I, §8, the above discussion is specified and refined for real maps of Epstein class. We define a notion of "essential period" and prove that $\operatorname{mod}(Rf)$ is big if and only if the essential period $\operatorname{per}_e(f)$ is big. This discussion exploits essentially Martens' real bounds [Ma] and complex bounds of [L4].

Let us now pass to Part II. In §9 we show that the secondary limbs condition and *a priori* bounds yield a definite space between the bouquets of little Julia sets. This provides us with special disjoint neighborhoods of little Julia bouquets with bounded geometry (called "standard"). Together with the work of Hu & Jiang [HJ, J] and McMullen [McM3] this yields local connectivity of the big Julia set (Theorem VI).

In the next two sections we prove the Rigidity Theorem. We start §10 with a discussion of reductions which boil the Rigidity Theorem down to the following problem: Two topologically equivalent maps (satisfying the assumptions of the theorem) are Thurston equivalent. Then we set up an inductive construction of approximations to the Thurston conjugacy. In particular, we adjust an approximate conjugacy in such a way that it respects the standard neighborhoods of little Julia bouquets.

The next section, §11, presents the proof of the Main Lemma. This lemma gives a uniform bound on the pseudo-Teichmüller distance between the generalized renormalizations of two combinatorially equivalent quadratic-like maps (the bound depends only on the selected secondary limbs and *a priori* bounds). The main geometric ingredient which makes this work is the linear growth of the principal moduli (Theorem III).

In the last section, §12, we discuss rigidity and deformations of real quasi-quadratic maps.

In Appendix A we collect necessary background material in conformal and quasi-conformal geometry.

In Appendix B we make further reference comments.

Acknowledgement. I would like to thank Jeremy Kahn for a fruitful suggestion (see §14) and for telling me his proof of the Yoccoz Rigidity Theorem. I thank Marco Martens, Curt McMullen, Yair Minsky, Mary Rees and Mitsuhiro Shishikura for useful comments on the results. I also had useful discussions with John Milnor on combinatorics of external rays. Feedback from the Stony Brook dynamics group during my course in the fall 1994 was helpful for cleaning up the exposition. I also thank Scott Sutherland and Brian Yarrington for help with the computer pictures.

The main results of this work were obtained in June 1993 during the Warwick Workshop on hyperbolic geometry. I am grateful to the organizers, particularly David Epstein and Caroline Series, for that wonderful time. I also thank MSRI for their hospitality: this paper was partly prepared during the Complex Dynamics and Hyperbolic Geometry spring program 1995. This work has been partially supported by NSF grants DMS-8920768 and DMS-9022140, and the Sloan Research Fellowship.

Part I. Combinatorics and geometry of the Yoccoz puzzle

2. Douady-Hubbard renormalization and Yoccoz puzzle

2.1. General terminology and notations. We will use the following notations:
$\mathbb{D}_r = \{z : |z| < r\}$ is the standard disk of radius r, $\mathbb{D} \equiv \mathbb{D}_1$ is the unit disk;
$\mathbb{T}_r = \partial\mathbb{D}_r$ is the standard circle of radius r, $\mathbb{T} \equiv \mathbb{T}_1$ is the unit circle;
$\mathbb{A}(r, R) = \{z : r < |z| < R\}$ is a standard annulus; similar notation is used for a closed annulus $\mathbb{A}[r, R]$ (or a semi-closed one).

Given two sets A and B, let $\mathrm{dist}(A, B) = \inf\{\mathrm{dist}(z, \zeta) : z \in A, \zeta \in B\}$.

Given two subsets V and W of the complex plane, we say that V is *strictly contained* in W, $V \Subset W$, if $\mathrm{cl}\, V \subset \mathrm{int}\, W$.

By a *topological disk* we will mean a simply connected region in C. By an *annulus* we mean a doubly connected region. A *horizontal curve* in an annulus A is a preimage of a circle centered at 0 by the Riemann mapping $A \to \{z : r < |z| < R\}$ (here $0 \le r < R \le \infty$).

Let us consider a family of two topological nested disks $D_1 \subset D_2$ with $\Gamma_i = \partial D_i$, and $A = D_2 \setminus D_1$. The statement that $\mathrm{mod}(A) > \epsilon$ with an $\epsilon > 0$ uniform over the family will be freely expressed in the following ways: "The annulus A has a definite modulus", or "D_1 is well inside D_2", or "There is a definite space in between Γ_1 and Γ_2."

Quasi-conformal and quasi-symmetric maps will be abbreviated as qc and qs correspondingly.

By $\mathrm{orb}\, z$ we denote the forward orbit $\{f^n z\}_{n=0}^{\infty}$ of z, and by $\omega(z)$ its ω-limit set. Let also $\mathrm{orb}_n z = \{f^m z\}_{m=0}^{n}$. Let $P_c : z \mapsto z^2 + c$.

2.2. Polynomials. By now there are many surveys and books on holomorphic dynamics. The reader can consult, e.g., [Be], [CG], [L1, Mi1] for general reference, and [Br], [DH1] for the quadratic case. Below we will remind the main definitions and facts required for discussion. However we assume that the reader is familiar with classification of periodic points as attracting, neutral, parabolic and repelling.

Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree $d \geq 2$, $f(z) = z^d + a_1 z^{d-1} + \cdots + a_d$. The *basin of* ∞ is the set of points escaping to ∞:

$$D_f(\infty) \equiv D(\infty) = \{z \in \mathbb{C} : f^n z \to \infty\}.$$

Its complement is called *the filled Julia set*: $K(f) = \mathbb{C} \setminus D(\infty)$. The *Julia set* is the common boundary of $K(f)$ and $D(\infty)$: $J(f) = \partial K(f) = \partial D(\infty)$. The Fatou set $F(f)$ is defined as $\mathbb{C} \setminus J(f)$. The Julia set (and the filled Julia set) is connected if and only if non of the critical points escape to ∞, that is, all of them belong to $K(f)$.

Given a polynomial f, there is a conformal map (the *Böttcher function*)

$$B_f : U_f \to \{z : |z| > r_f \geq 1\}$$

of a neighborhood U_f of infinity onto the exterior of a disk such that $B_f(fz) = (B_f z)^d$ and $B_f(z) \sim z$ as $z \to \infty$. There is an explicit dynamical formula for this map:

$$B_f(z) = \lim_{n\to\infty} (f^n z)^{1/d^n} \tag{2.1}$$

with an appropriate choice of the branch of the d^nth root.

If the Julia set $J(f)$ is disconnected then ∂U_f contains a critical point b of f. Otherwise B_f coincides with the Riemann mapping of the whole basin of infinity $D(\infty)$ onto $\{z : |z| > 1\}$ (in this case $r_f = 1$).

The *external rays* $R^\theta \equiv R^\theta_f$ of angle θ and *equipotentials* $E^r \equiv E^r_f$ of level r are defined as the B_f-preimages of the straight rays $\{e^r e^{i\theta} : r_f < r < \infty\}$ and the round circles $\{re^{i\theta} : 0 \leq \theta \leq 2\pi\}$. They form two orthogonal invariant foliations of U_f. Moreover, even in the disconnected case, a ray R^θ can be infinitely extended towards the Julia set, unless it "bounces off" a critical point, and the Böttcher function can be analytically continued along this ray (see [GM], Appendix B, for a detailed discussion).

Let $\mathcal{R}^{\theta,(\rho,r)} = \mathcal{R}^{\theta,(\rho,r)}_f$ stand for the arc of the external ray of angle θ in between the equipotential levels $0 \leq \rho < r \leq \infty$ (with the usual meaning of notations $[\rho, r]$, $[\rho, r)$ etc.). Note that if the ray lands at some point $a \in J(f)$ then $\mathcal{R}^{\theta,[0,r)}$ also makes sense.

Each ray comes together with the natural parametrization by the equipotential levels.

Theorem 2.1 (see [M1], §18, or [H]). *Assume that $J(f)$ is connected. Then for any repelling periodic point a, there is at least one but at most finitely many external rays landing at a.*

Thus the external rays landing at a are organized in several cycles. The rotation number of these cycles is the same, and is called the *combinatorial rotation number* $\rho(a)$ of a. Let $\mathcal{R}(a) \equiv \mathcal{R}_f(a)$ denote the union of the closed external rays landing at a, and

$$\mathcal{R}(\bar{a}) \equiv \mathcal{R}_f(\bar{a}) = \cup_{k=0}^{p-1} \mathcal{R}(f^k a)$$

(where p is the period of a and $\bar{a} = \operatorname{orb} a$ is the corresponding periodic cycle). This configuration, with the external angles marked at the rays, is called the *rays portrait* of the cycle $\bar{a}$. The class of isotopic portraits is called the *abstract rays portrait.*

2.3. Quadratic family. Let now $f \equiv P_c : z \mapsto z^2 + c$ be a quadratic polynomial. In this case the rays portraits of periodic cycles have quite special combinatorial properties. The reader can consult [DH1], [At], [GM], [Sc2], [M4] for the proofs of the results quoted below.

Proposition 2.2 (see [M4]). *Let $\bar{a} = \{a_k\}_{k=0}^{p-1}$ be a repelling periodic cycle such that there are at least two rays landing at each point a_k.*

(i) Let S_1 be the components of $\mathbb{C} \setminus \mathcal{R}(\bar{a})$ containing the critical value c. Then S_1 is a sector bounded by two external rays.

(ii) Let S_0 be the component of $\mathbb{C} \setminus f^{-1}\mathcal{R}(\bar{a})$ containing the critical point 0. Then S_0 is bounded by four external rays: two of them land at a periodic point a_k, and two others land at the symmetric point $-a_k$.

(iii) The rays of $\mathcal{R}(\bar{a})$ form either one or two cycles under iterates of f.

A particular situation of such kind is the following. Let $\bar{b} = \{b_k\}_{k=0}^{p-1}$ be an attracting cycle, $p > 1$. Let D_k be the components of its basin of attraction containing b_k. Then the boundaries of D_k are Jordan curves, and the restrictions $f^p|\partial D_k$ are topologically conjugate to the doubling map $z \mapsto z^2$ of the unit circle. Hence there is a unique f^p-fixed point $a_k \in \partial D_k$. Altogether these points form a repelling periodic cycle $\bar{a}$ (whose period may be smaller than p), with at least two rays landing at each a_k. The portrait $\mathcal{R}(\bar{a})$ will be also called the rays portrait associated to the attracting cycle $\bar{b}$.

A case of special interest for what follows is the fixed points portraits. There is always a fixed point called β which is the landing point of the invariant ray $\mathcal{R}_0$. Moreover, this is the only ray landing β, so that this point is non-dividing: the set $K(f) \setminus \{\beta\}$ is connected.

If the second fixed point called α is also repelling, it turns out to be dividing: there are at least two external rays landing at it, so that $K(f) \setminus \{\alpha\}$ is disconnected. These rays are cyclically permuted by dynamics with a certain combinatorial rotation number q/p.

The *Mandelbrot set* M is defined as the set of $c \in \mathbb{C}$ for which $J(P_c)$ is connected, that is, 0 does not escape to ∞ under iterates of P_c. If $c \in \mathbb{C} \setminus M$, then $J(P_c)$ is a Cantor set.

The Mandelbrot set itself is connected (see [DH1], [CG]). This is proven by constructing explicitly the Riemann mapping $B_M : \mathbb{C} \setminus M \to \{z : |z| > 1\}$. Namely, let

$D_c(\infty)$ be the basin of ∞ of P_c, and B_c be the Böttcher function (2.1) of P_c. Then

$$B_M(c) = B_c(c). \tag{2.2}$$

The meaning of this formula is that the "conformal position" of a parameter $c \in \mathbb{C} \setminus M$ coincides with the "conformal position" of the critical value c in the basin $D_c(\infty)$. This relation is a key to the similarity between dynamical and parameter planes.

Using the Riemann mapping B_M we can define the *parameter external rays* and *equipotentials* as the preimages of the straight rays going to ∞ and round circles centered at 0. This gives us two orthogonal foliations in the complement of the Mandelbrot set.

A quadratic polynomial P_c with $c \in M$ is called *hyperbolic* if it has an attracting cycle. The set of hyperbolic parameter values is the union of some components of $\operatorname{int} M$ called *hyperbolic components.* Conjecturally all components of $\operatorname{int} M$ are hyperbolic. This Conjecture would follow from the MLC Conjecture asserting that the Mandelbrot set is locally connected (Douady & Hubbard [DH1]).

The *main cardioid* of M is defined as the set of points c for which P_c has a neutral fixed point α_c, that is, $|P_c'(\alpha_c)| = 1$. It encloses the *main hyperbolic component* where P_c has an attracting fixed point. In the exterior of the main cardioid both fixed points are repelling.

Let $H \subset \operatorname{int} M$ be a hyperbolic component of the Mandelbrot set, and let $\bar{b}(c) = \{b_k(c)\}_{k=0}^{p-1}$ be the corresponding attracting cycle. On the boundary of H the cycle $\bar{b}$ becomes neutral, and there is a single point $d \in \partial H$ where $(P_d^p)'(b_0) = 1$ [DH1]. This point is called the *root* of H.

If H is not the main component then for any $c \in H$ there is the rays portrait $\mathcal{R}_c$ associated to the corresponding attracting basin. Let θ_1 and θ_2 be the external angles of the two rays bounding the sector S_1 of Proposition 2.2.

Theorem 2.3 (see [DH1], [M4], [Sc2]). *The parameter rays with angles θ_1 and θ_2 land at the root d of H. There are no other rays landing at d.*

The region W_d in the parameter plane bounded by the above two rays and containing H is called the *wake* of W_d. The part of the Mandelbrot set contained in the wake together with the root d is called the *limb* L_d of the Mandelbrot set originated at H. The root d is also referred to as the root of the wake W_d or the limb L_d.

Recall that for $c \in H$, $\bar{a}_c$ denotes the repelling cycle associated to the basin of the attracting cycle $\bar{b}_c$. The dynamical meaning of the wakes is reflected in the following statement.

Proposition 2.4 (see [GM]). *Under the circumstances just described, the repelling cycle $\bar{a}_c$ stays repelling throughout the wake W_d originated at H. The corresponding rays portrait $\mathcal{R}(\bar{a}_c)$ preserves its isotopic type throughout this wake.*

The limbs attached to the main cardioid are called *primary.* Let H be a hyperbolic component attached to the main cardioid. The limbs attached to such a component are called *secondary.* More generally, if H is a hyperbolic component obtained from the main cardioid by means of n consecutive bifurcations, then the limbs originated at such a component will be called *limbs of order* n.

A *truncated limb* is obtained from a limb by removing a neighborhood of its root.

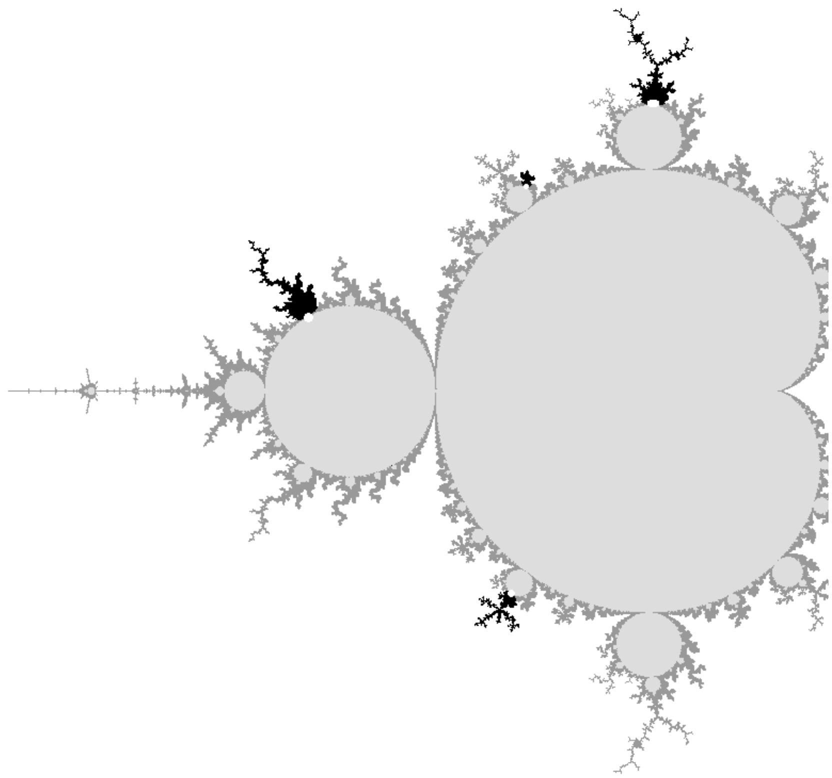

Figure 1. Truncated secondary limbs of the Mandelbrot set.

2.4. Douady-Hubbard polynomial-like maps. The main reference for the following material is [DH2]. Let $U' \Subset U$ be two topological disks. A branched covering $f : U' \to U$ is called a *DH polynomial-like map* (we will sometimes skip "DH" in case this does not cause confusion with "generalized" polynomial-like maps defined below). Every polynomial with connected Julia set can be viewed as a polynomial-like map after restricting it onto an appropriate neighborhood of the filled Julia set.

Polynomial-like maps of degree 2 are called *(DH) quadratic-like*. Unless otherwise is stated, any quadratic-like map will be normalized so that the origin 0 is its critical point.

One can naturally define the filled Julia set of f as the set of non-escaping points:

$$K(f) = \{z : f^n z \in U' : n = 0, 1, \dots\}.$$

The Julia set is defined as $J(f) = \partial K(f)$. These sets are connected if and only if non of the critical points is escaping.

The choice of the domain U' and the range U of a polynomial-like map is not canonical. It can be replaced with any other pair $V' \Subset V$ such that $f : V' \to V$ is a polynomial-like map with the same Julia set (compare [McM2], Thm. 5.11).

Given a polynomial-like map $f : U' \to U$, we can consider a *fundamental annulus* $A = U \setminus U'$. It is certainly not a canonical object but rather depending on the choice of U' and U. Let

$$\operatorname{mod}(f) = \sup \operatorname{mod} A,$$

where A runs over all fundamental annuli of f.

Two polynomial-like maps f and g are called *topologically (quasi-conformally, conformally, affinely) conjugate* if there is a choice of domains $f : U' \to U$ and $g : V' \to V$ and a homeomorphism $h : (U, U') \to (V, V')$ (qc map, conformal or affine isomorphism correspondingly) such that $h \circ f|U = g \circ h|U$.

If there is a qc conjugacy h between f and g with $\bar\partial h = 0$ almost everywhere on the filled Julia set $K(f)$, then f and g are called *hybrid* or *internally* equivalent. A *hybrid class* $\mathcal{H}(f)$ is the space of DH polynomial-like maps hybrid equivalent to f modulo affine equivalence. According to Sullivan [S1], a hybrid class of polynomial-like maps can be viewed as an infinitely dimensional Teichmüller space. In contrast with the classical Teichmüller theory this space has a preferred point:

Straightening Theorem [DH2]. *Any hybrid class $\mathcal{H}(f)$ of DH polynomial-like maps with connected Julia set contains a unique (up to affine conjugacy) polynomial.*

In particular, any hybrid class of quadratic-like maps with connected Julia set contains a unique quadratic polynomial $z \mapsto z^2 + c$ with $c = c(f) \in M$. So the hybrid classes of quadratic-like maps with connected Julia set are labeled by the points of the Mandelbrot set. In what follows we will freely identify such a hybrid class with its label $c \in M$.

Sullivan supplied any Teichmüller space of quadratic-like maps (with connected Julia set) with the following *Teichmüller metric* [S1]:

$$\operatorname{dist}_T(f, g) = \inf \log \operatorname{Dil}(h),$$

where h runs over all hybrid conjugacies between f and g, and $\operatorname{Dil}(h)$ denotes the qc dilatation of h. It is easy to see from the construction of the straightening that the Teichmüller distance from f to the quadratic $P_{c(f)} : z \mapsto z^2 + c(f)$ in its hybrid class is controlled by the modulus of f:

Proposition 2.5. *If* $\text{mod}(f) \geq \mu > 0$ *then* $\text{dist}_T(f, P_{c(f)}) \leq C$ *with a* $C = C(\mu)$ *depending only on* μ*. Moreover,* $C(\mu) \to 0$ *as* $\mu \to \infty$*.*

This is a reason why control of the moduli of polynomial-like maps is crucial for the renormalization theory (see [S2]).

Given a polynomial-like map with connected Julia set, we can define *external rays* and *equipotentials* near the filled Julia set by conjugating it to a polynomial and transferring the corresponding curves. This definition is certainly not canonical but rather depends on the choice of conjugacy. If $\text{mod}(f) > \epsilon$, then we can use a $K(\epsilon)$-qc conjugacy. In what follows we always assume that the choice of the curves is made in such a way.

2.5. Douady-Hubbard renormalization. The reverse procedure under the name of *tuning* is discussed in [DH2], [D1] and [M3]. A more general point of view (but which is equivalent to the tuning, after all) is discussed in [McM2].

Let $f : U' \to U$ be a quadratic-like map. Let $\bar{a}$ be a dividing repelling cycle, so that there are at least two rays landing at each point of $\bar{a}$. Let $\mathcal{R} \equiv \mathcal{R}(\bar{a})$ denote the configuration of rays landing at $\bar{a}$, and let $\mathcal{R}' = -\mathcal{R}$ be the symmetric configuration. Let us also consider an arbitrary equipotential E. Let now Ω be the component of $\mathbb{C} \setminus (E \cup \mathcal{R} \cup \mathcal{R}')$ containing the critical point 0. By Proposition 2.2, it is bounded by four arcs γ_i of external rays ant two pieces of the equipotential E.

Let p be the period of the above rays, and a be the point of the cycle $\bar{a}$ lying on $\partial\Omega$. Let us consider a domain $\Omega' \subset \Omega$, the component of $f^{-p}\Omega$ attached to a (see Figure 2). If $\Omega' \ni 0$ then $f^p : \Omega' \to \Omega$ is a double covering map (otherwise Ω' is a strip univalently mapped onto Ω).

A quadratic-like map f is called *DH-renormalizable* if there is a repelling cycle $\bar{a}$ as above such that $\Omega' \ni 0$, and 0 does not escape Ω' under iterates of f^p. We will also say that this renormalization is associated with the periodic point a. We call f *immediately DH renormalizable* if a is the dividing fixed point α of f.

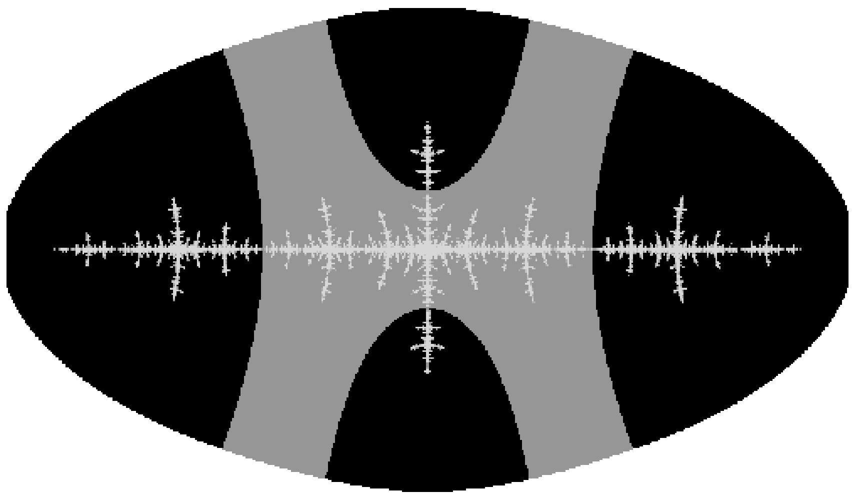

Figure 2. Renormalization domain for the Feigenbaum polynomial.

Note that the disks $\Omega', f\Omega', \dots, f^{p-1}\Omega'$ have disjoint interiors. Indeed, otherwise $f^k\Omega'$ would be inside Ω for some $k < p$. But this is impossible since the external rays which bound $f^k\Omega'$ are outside of Ω.

In the DH-renormalizable case one can extract a polynomial-like map $f^p : V' \to V$ by means of a "thickening procedure" (see [DH1] or [M2]). Namely, let us consider a little bit bigger domain $V \supset \Omega$ bounded by arcs of four external rays close to γ_i, two arcs of circles going around the point a and the symmetric point a' (i.e., $fa' = a$), and two arcs of E. Pulling V back by f^p, we obtain a domain $V' \Subset V$ such that the map $f^p : V' \to V$ is quadratic-like. This map considered up to rescaling (that is, up to affine conjugacy) is called the *DH renormalization* of f.

Let now $f : z \mapsto z^2 + c_0$ be a quadratic polynomial, $c_0 \in M$. If it is renormalizable then there is a homeomorphic copy $M_0 \ni c_0$ of the Mandelbrot set with the following properties (see [DH2, D1]). For $z \in M_0' = M_0 \setminus \{\textit{one point}\}$ the polynomial $P_c : z \mapsto z^2 + c$ is renormalizable. Moreover, there is the analytic parameter extension a_c of the periodic point a to a neighborhood of M_0' such that the above renormalization of P_c is associated to a_c. At the parameter value b removed from M_0 the periodic point a_c is becoming parabolic with multiplier one. This parameter value is called the *root* of M_0. We say that the component H_0 of M_0 corresponding to the main hyperbolic component of M "gives origin" to the copy M_0. Vice versa, any hyperbolic component H_0 of the Mandelbrot set gives origin to a copy of M. In particular, the copies corresponding to the immediate renormalization are attached to the main cardioid.

We will see below that among all renormalizations there is the *first* one, which we denote Rf (see §3.4). This renormalization corresponds to a *maximal* copy of the Mandelbrot set (that is a copy, which is not contained in any bigger copies except M itself). Let $\mathcal{M}$ denote the family of maximal Mandelbrot copies.

It may happen that Rf is also renormalizable, so that f is "twice renormalizable". In such a way we can associate to f a canonical finite or infinite sequence of renormalizations $f, Rf, R^2f, \dots$. Accordingly f can be classified as "at most finitely" or "infinitely renormalizable".

Given any sequence $\tau = \{M_0, M_1, \dots\}$ of maximal copies of M, there is an infinitely renormalizable quadratic polynomial P_b such that $c(R^m P_b) \in M_m$, $m = 0, 1, \dots$. Indeed, the sets

$$\mathcal{C}om_N(\tau) = \{b : c(R^m P_b) \in M_m,\ m = 0, 1, \dots, N\}$$

form a nest of copies of M whose intersection $\mathcal{C}om(\tau)$ consists of the desired parameter values.

We say that these infinitely renormalizable quadratics have combinatorics τ. The MLC problem for infinitely renormalizable parameter values is equivalent to the assertion that there is only one quadratic with a given combinatorics, i.e., $\mathcal{C}om(\tau)$ is a single point for any τ (see Schleicher [Sc1] for a detailed discussion of the combinatorial aspects of the MLC).

Let us say that f satisfies the *secondary limbs condition* if there is a finite family of truncated secondary limbs L_i of the Mandelbrot set such that the hybrid classes of all renormalizations $R^m f$ belong to $\cup L_i$. Let $\mathcal{SL}$ stand for the class of quadratic-like maps satisfying the secondary limbs condition.

Here are some examples of maps of class $\mathcal{SL}$:

- Maps which are at most finitely renormalizable and don't have non-repelling periodic points (Yoccoz class).
- Infinitely renormalizable maps of bounded type ("bounded type" means that there are only finitely many different Mandelbrot copies in the string $\tau = \{M_0, M_1, \dots\}$).
- Real maps which don't have non-repelling periodic points.
- Select a finite family of (non-truncated) limbs L_j of order 3 (see §2.3). If $c(R^m f) \in \cup L_j$, $m = 0, 1, \dots$, then $f \in \mathcal{SL}$. Unlike $\mathcal{SL}$ assumption which involves truncation, this property is combinatorial.

All the above combinatorial notions are readily extended to quadratic-like maps via the straightening. A quadratic-like map f is said to have *a priori bounds* if there is an $\epsilon > 0$ such that $\mathrm{mod}(R^m f) \geq \epsilon > 0$ for all the renormalizations (note that maps of the Yoccoz class satisfy this condition by logic).

2.6. Yoccoz puzzle. Let $f : U' \to U$ be a quadratic-like map with both fixed points α and β repelling. As usual, α denotes the dividing fixed point with rotation number $\rho(\alpha) = q/p$, $p > 1$. Let E be an equipotential sufficiently close to $K(f)$ (so that both E and fE are closed curves). Let $\mathcal{R}_\alpha$ denote the union of external rays landing at α. These rays cut the domain bounded by E into p closed topological disks $Y_i^{(0)}$, $i = 0, \dots, p-1$, called *puzzle pieces of zero depth* (Figure 3). The main property of this partition is that $f\partial Y_j^{(0)}$ is outside of $\cup \operatorname{int} Y_i^{(0)}$.

Let us now define *puzzle pieces* $Y_i^{(n)}$ *of depth* n as the closures of the connected components of $f^{-n} \operatorname{int} Y_k^{(0)}$. They form a finite partition of the neighborhood of $K(f)$ bounded by $f^{-n}E$. If the critical orbit does not land at α, then for every depth there is a single puzzle-piece containing the critical point. It is called *critical* and is labeled as $Y^{(n)} \equiv Y_0^{(n)}$.

Let $\mathcal{Y}_f$ denote the family of all puzzle pieces of f of all levels. It is *Markov* in the following sense:

(i) Any two puzzle pieces are either nested or have disjoint interiors. In the former case the puzzle piece of bigger depth is contained in the one of smaller depth.

(ii) The image of any puzzle piece $Y_i^{(n)}$ of depth $n > 0$ is a puzzle piece $Y_k^{(n-1)}$ of the previous depth. Moreover, $f : Y_i^{(n)} \to Y_k^{(n-1)}$ is a two-to-one branched covering or a conformal isomorphism depending on whether $Y_i^{(n)}$ is critical or not.

We say that $f^k|Y_i^{(n)}$ *l-to-one covers* a union of pieces $\bigcup_{m,j} Y_j^{(m)}$ if $f^k|\operatorname{int} Y_i^{(n)}$ is l-to-one covering map onto its image, and

$$f^k|(Y_i^{(n)} \cap J(f)) = \bigcup_{m,j} Y_j^{(m)} \cap J(f).$$

In this case $\cup Y_j^{(m)}$ is obtained from $f^k|Y_i^{(n)}$ by cutting with appropriate equipotential arcs.

On depth 1 we have $2p-1$ puzzle pieces: one central $Y^{(1)}$, $p-1$ non-central $Y_i^{(1)}$ attached to the fixed point α (cuts of $Y_i^{(0)}$ by the equipotential $f^{-1}E$), and $p-1$ symmetric ones Z_i^1 attached to α'. Moreover, $f|Y^{(1)}$ two-to-one covers $Y_1^{(1)}$, $f|Y_i^{(1)}$ univalently covers $Y_{i+1}^{(1)}$, $i = 1, \dots, p-2$, and $f|Y_{p-1}^{(1)}$ univalently covers $Y^{(1)} \cup \bigcup_i Z_i^{(1)}$. Thus $f^p Y^{(1)}$ truncated by $f^{-1}E$ is the union of $Y^{(1)}$ and $Z_i^{(1)}$ (Figure 3).

Theorem 2.6 (Yoccoz, 1990). *Assume that both fixed points of a polynomial-like map f are repelling, and that f is DH non-renormalizable. Then the following divergence property holds:*

$$\sum_{n=0}^{\infty} \operatorname{mod}(Y^{(n)} \setminus Y^{(n+1)}) = \infty.$$

Hence $\operatorname{diam} Y^{(n)} \to 0$ *as* $n \to \infty$.

Corollary 2.7. *Under the circumstances of the above theorem the Julia set $J(f)$ is locally connected.*

The reader can consult [H, L3, Mi2] for a proof (or go to Theorem II of this paper).

The Yoccoz puzzle provides us with a Markov family of puzzle pieces to play with. Two original ways of playing this game were by means of the Branner-Hubbard tableaux [BH] and by means of the Yoccoz τ-function (unpublished). Our way based on the idea of generalized renormalization is quite different.

2.7. Expanding sets. Let us consider Yoccoz puzzle pieces $Y_i^{(N)}$ of depth N, and let $\mathcal{Y}^{(N)}$ denote the family of puzzle pieces $Y_j^{(N+l)}$ such that

$$f^k Y_j^{(N+l)} \cap Y^{(N)} = \emptyset,\ k = 0 \dots, l-1.$$

Let $K^{(N)} = \{z : f^k z \notin Y^{(N)},\ k = 0, 1 \dots\}$ Recall that an invariant set K is called *expanding* if there exist constants $C > 0$ and $\rho \in (0, 1)$ such that

$$|Df^k(z)| \geq C\rho^k,\ z \in K,\ k = 0, 1, \dots$$

Lemma 2.8. *For a given N, $\operatorname{diam} Y_s^{(N+l)} \to 0$ as $Y_s^{(N+l)} \in \mathcal{Y}^{(N)}$ and $l \to \infty$. Moreover, the set $K^{(N)}$ is expanding.*

Proof. Let us consider thickened puzzle pieces $\hat{Y}_i^{(N)}$ as in Milnor [Mi2] or §2.5. Then $\operatorname{int}(f\hat{Y}_i^{(N)})$ contains $\hat{Y}_j^{(N)}$ whenever $fY_i^{(N)} \supset Y_j^{(N)}$ (recall that the $\hat{Y}^{(N)}$ are closed). Hence the inverse map $f^{-1} : \hat{Y}_j^{(N)} \to \hat{Y}_i^{(N)}$ is contracting by a factor $\lambda < 1$ in the hyperbolic metrics of the pieces under consideration.

Let $Y_s^{(N+l)} \subset Y_i^{(N)}$. It follows that the hyperbolic diameter of $\hat{Y}_s^{(N+l)}$ in $\hat{Y}_i^{(N)}$ is at most λ^l, and the statement follows. □

3. Principal nest and generalized renormalization

In the rest of the paper we will assume, unless otherwise is stated, that both fixed points of the quadratic-like maps under consideration are repelling. Up to §3.6 quadratic-like maps and renormalization are understood in the sense of Douady and Hubbard.

3.1. Principal nest. Given a set $W = \operatorname{cl}(\operatorname{int} W)$ and a point z such that $f^l z \in \operatorname{int} W$, let us define the *pull-back* of W along the $orb_l z$ as the chain of sets $W_0 = W, W_{-1} \ni f^{n-1}z, \dots, W_{-l} \ni z$ such that W_{-k} is the closure of the component of $f^{-k}(\operatorname{int} W)$ containing $f^{l-k}z$. In particular if $z \in \operatorname{int} W$ and $l > 0$ is the moment of first return of $\operatorname{orb} z$ back to $\operatorname{int} W$ we will refer to the *pull-backs corresponding to the first return of* $\operatorname{orb} z$ to $\operatorname{int} W$.

Let us consider the puzzle pieces of depth 1 as described above: $Y^{(1)}, Y_i^{(1)}$ and $Z_i^{(1)}$, $i = 1, \dots p-1$ (Figure 3). If $z \in Y^{(1)}$ then $f^p z$ is either in $Y^{(1)}$ or in one of $Z_i^{(1)}$. Hence either $f^{pk}0 \in Y^{(1)}$ for all $k = 0, 1, \dots$, or there is the smallest $t > 0$ and a ν such that $f^{tp}0 \in Z_\nu^{(1)}$. Thus either f is immediately DH renormalizable, or the critical point escapes through one of the non-critical pieces, attached to α'.

In the immediately renormalizable case the principal nest of puzzle pieces consists of just single puzzle piece $Y^{(0)}$ (which is not very informative). In the escaping case we will construct *the principal nest*

$$Y^{(0)} \supset V^0 \supset V^1 \supset \dots \tag{3.1}$$

in the following way. Let $V^0 \ni 0$ be the pull-back of $Z^{(1)}_\nu$ along the $orb_{tp} 0$. Further, let us define V^{n+1} as the pull-back of V^n corresponding to the first return of the critical point 0 back to $\operatorname{int} V^n$. Of course it may happen that the critical point never returns back to $\operatorname{int} V^n$. Then we stop, and the principal nest turns out to be finite. This case is called *combinatorially non-recurrent.* If the critical point is *recurrent* in the usual sense, that is $\omega(0) \ni 0$, it is also combinatorially recurrent, and the principal nest is infinite.

Let $l = l(n)$ be the first return time of the critical point back to $\operatorname{int} V^{n-1}$. Then the map $g_n = f^{l(n)} : V^n \to V^{n-1}$ is two-to-one branched covering. Indeed, by the Markov property of the puzzle, $f^k V^n \cap \operatorname{int} V^{n-1} = \emptyset$ for $k = 1, \dots, l-1$, so that the maps $f : f^k V^n \to f^{k+1} V^n$ are univalent for those k's.

Let us call return to level $n-1$ *central* if $g_n 0 \in V^n$. In other words $l(n) = l(n+1)$. Let us say that a sequence $n, n+1, \dots, n+N-1$ (with $N \geq 1$) of levels (or corresponding puzzle pieces) of the principal nest form a *(central) cascade* if the returns to all levels $n, n+1, \dots, n+N-2$ are central, while the return to level $n+N-1$ is non-central (see Figure 4). In this case

$$g_{n+k}|V^{n+k} = g_{n+1}|V^{n+k}, \;\; k = 1, \dots, N.$$

and $g_{n+1} 0 \in V^{n+N-1} \smallsetminus V^{n+N}$. Thus all the maps $g_{n+1}, \dots, g_{n+N}$ are the same quadratic-like maps with shrinking domains of definition (see the conventions in §2.4). We call the number N of levels in the cascade its *length.* Note that a cascade of length 1 consists of a single non-central level. Let us call the cascade *maximal* if the return to level $n-1$ is non-central. Clearly the whole principal nest, except the first element $Y^{(0)}$, is the union of disjoint maximal cascades. The number of such cascades is called the *height* $\chi(f)$ of f. In other words, $\chi(f)$ is the number of different quadratic-like maps among the g_n's. (If f is immediately renormalizable set $\chi(f) = 0$.)

The annuli $A^n = V^{n-1} \smallsetminus V^n$ and their moduli $\mu_n = \operatorname{mod}(A^n)$ will also be called *principal.*

Remark 1. The notion of the principal nest admits some useful modifications. First, there is a flexibility in the choice of the puzzle piece V^0 (compare §8). Second, one can modify the nest after passing through a long central cascade (see §3.6). The latter modification is useful, e.g., for study the Hausdorff dimension of Julia sets (see Przytycki [Prz], Prado [Pra1]).

Remark 2. Given a quadratic polynomial $f : z \mapsto z^2 + c$, the principal nest determines a specific way to approximate c by superattracting parameter values. Namely, one should perturb c in such a way that the critical point becomes fixed under g_n, while the combinatorics on the preceding levels keeps unchanged see [L8]. The number of points in this approximating sequence is equal to the height $\chi(f)$. This resembles "internal addresses" of Lau and Schleicher [LSc] but turns out to be different.

3.2. Initial Markov tiling. Let P_i be a finite or countable family of topological discs with disjoint interiors, and $g : \cup P_i \to \mathbb{C}$ be a map such that the restrictions $g|P_i$ are branched coverings onto their images. This map is called *Markov* if $gP_i \supset P_j$ whenever $\operatorname{int} gP_i \cap \operatorname{int} P_j \neq \emptyset$. Let us call it a *unbranched Markov map* if all the restrictions $g|P_i$ are one-to-one onto their images.

A Markov map is called *Bernoulli* if there is a topological disc D such that $gP_i \supset D \supset \cup P_j$ for all i. Any such a D will be called a *range* of g. Similarly we can define an unbranched Bernoulli map.

We know that $f^p|Y^{(1)}$ two-to-one covers $Y^{(1)}$ and the puzzle pieces $Z_i^{(1)}$ attached to α'. If $f^p 0 \in Y^{(1)}$ (central return) then the pull back of $Y^{(1)}$ by this map is the critical piece $Y^{(1+p)}$, while each $Z_i^{(1)}$ has two univalent pull-backs $Z_j^{(1+p)}$ (we label them by j in an arbitrary way) (see Figure 3).

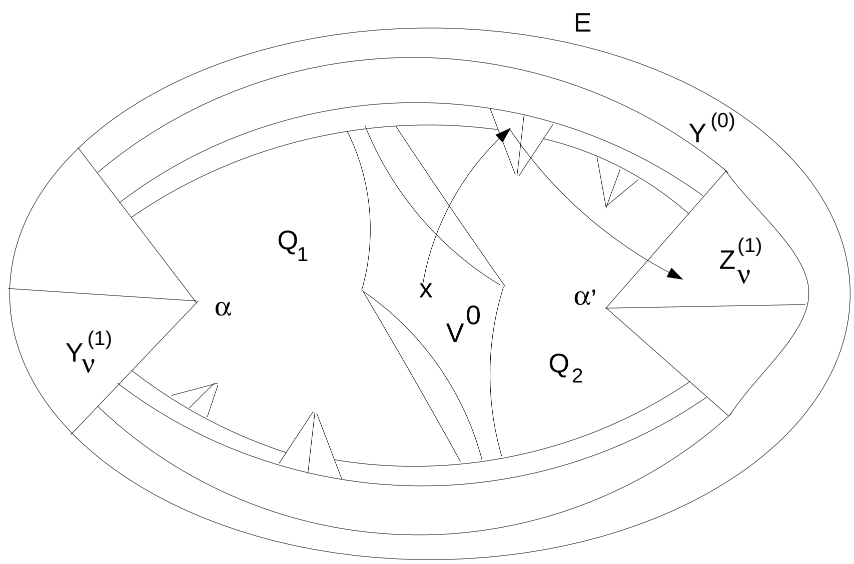

Figure 3. Initial tiling ($p = 3$, $t = 2$).

Now, $f^p|Y^{(1+p)}$ two-to-one covers all these puzzle pieces. If we again have a central return, that is $f^p 0 \in Y^{(1+p)}$, then the pull-back will give us one critical piece $Y^{(1+2p)}$, and $4(p-1)$ off-critical $Z_j^{(1+2p)}$.

Repeating this procedure t times (where $f^{tp} 0 \in Z_\nu^{(1)}$), we obtain the initial central nest

$$Y^{(1)} \supset Y^{(1+p)} \supset \ldots \supset Y^{(1+(t-1)p)}, \tag{3.2}$$

and a family of off-critical puzzle pieces $Z_j^{(1+sp)}$, $0 \le s \le t-1$. Moreover

$$f^p 0 \in Z_\nu^{(1+(t-1)p)}, \tag{3.3}$$

where $f^{(t-1)p} Z_\nu^{(1+(t-1)p)} = Z_\nu^{(1)}$.

Let us say that a set D is *tiled into pieces* $W_i \operatorname{rel} F(f)$ if the $\operatorname{int} W_i$ are disjoint, and $D \cap J(f) = \cup W_i \cap J(f)$.

Thus we have tiled $Y^{(0)} \operatorname{rel} F(f)$ into the pieces $Z_i^{(1+sp)}$, $0 \le s \le t-1$, and $Y^{(1+(t-1)p)}$. Let us look closer at this last piece. Its image under f^p two-to-one covers all above puzzle pieces of depth $1+(t-1)p$. The pull-back of $Z_\nu^{(1+(t-1)p)}$ from (3.3) gives us exactly $V^0 \ni 0$, the first puzzle piece in the principal nest (3-0). The pull-backs of the other pieces $Z_j^{(1+(t-1)p)}$ provide some off-critical pieces $Z_i^{(1+tp)}$. Finally, we have two univalent pull-backs Q_1 and Q_2 of $Y^{(1+(t-1)p)}$. Altogether these pieces tile the piece $Y^{(1+(t-1)p)} \operatorname{rel} F(f)$.

To understand how the critical point returns back to V^0 we need to tile $Q_1 \cup Q_2$ further. To this end let us iterate the unbranched Bernoulli map $f^p | Q_1 \cup Q_2$ with range $Q_1 \cup Q_2 \cup V^0 \cup Z_j^{(1+tp)}$. So take a point $z \in Q_1 \cup Q_2$ and consider its f^p-orbit until it escapes $Q_1 \cup Q_2$ (or iterate forever if it never escapes). It can escape through the piece V^0 or through a piece $Z_j^{(1+tp)}$. In any case pull the corresponding piece back to this point. In such a way we will obtain a tiling

$$Q_1 \cup Q_2 = \bigcup_{k>0} \bigcup_i X_i^k \cup \bigcup_{k>t} \bigcup_j Z_j^{(1+kp)} \cup R \ \operatorname{rel} F(f),$$

where X_i^k denote the pull-backs of V^0 under f^{kp}, Z_j^{1+kp} denote the pull-backs of the $Z_i^{(1+tp)}$ under $f^{(k-t)p}$, and R denote the residual set of non-escaping points.

Altogether we have constructed the *initial Markov tiling*:

$$Y^{(0)} \setminus R = V^0 \cup \bigcup_{k>0} \bigcup_i X_i^k \cup \bigcup_{k \ge 0} \bigcup_j Z_j^{(1+kp)} \operatorname{rel} F(f). \tag{3.4}$$

It is convenient (in order to reduce the number of iterates in what follows) to consider a Markov map

$$G : V^0 \cup \bigcup_{k,i} X_i^k \cup \bigcup_{k,j} Z_j^{(1+kp)} \to \mathbb{C} \tag{3.5}$$

defined as follows. Observe that for any j there is an i such that $f^{sp} Z_j^{(1+sp)}$ univalently covers $Z_i^{(1)}$. Moreover $f^{p-i} Z_i^{(1)}$ univalently covers $Y^{(0)}$. Let us set $G | Z_j^{1+sp} = f^{ps+(p-i)}$. The image of each piece $Z_j^{(1+sp)}$ under this map univalently covers $Y^{(0)}$. Similarly let us set $G | V^0 = f^{tp+p-\nu}$, so that the image of this piece two-to-one covers $Y^{(0)}$. Finally $G | X_i^k = f^{kp}$, so that these pieces are univalently mapped onto V^0.

3.3. A non-degenerate annulus. Yoccoz has shown that if f is non-renormalizable then in the nest $Y^{(0)} \supset Y^{(1)} \dots$ there is a non-degenerate annulus $Y^{(n)} \setminus Y^{(n+1)}$. However the modulus of this annulus is not under control. We will construct a different non-degenerate annulus whose modulus we can control.

Proposition 3.1. *Let f be a quadratic-like map which is not immediately renormalizable. Then all the principal annuli $A^n = V^{n-1} \setminus V^n$ are non-degenerate.*

Proof. Observe first that V^0 is strictly inside $Y^{(0)}$, that is, the annulus $Y^{(0)} \setminus V^0$ is non-degenerate. Indeed, V^0 is the pull-back of $Z_\nu^{(1)}$ which is strictly inside $Y^{(0)}$. As the iterates of $\partial Y^{(0)}$ stay outside $\operatorname{int} Y^{(0)}$, V^0 may not touch $\partial Y^{(0)}$.

For the same reason all other pieces $Z_j^{(1+kp)}$ and X_i^k of the initial Markov tiling (3.4) are strictly inside $Y^{(0)}$ as well.

Let us consider the orbit of the critical point 0 under iterates of the map G (see (3.5)) until it returns back to V^0. It first goes through the Z-pieces of the initial Markov tiling, then at some moment $l \geq 1$ it lands at either V^0 or some X_i^s. In the latter case, it lands at V^0 at the next moment.

Since the map $G : V^0 \cup Z_j^{(1+kp)} \to \mathbb{C}$ is Bernoulli with range $Y^{(0)}$, there is a topological disc $P \subset V^0$, such that $G^l|P$ two-to-one covers $Y^{(0)}$. Clearly V^1 is the pull-back of either V^0 or X_i^s by $G^l : P \to Y^{(0)}$. Since both V^0 and X_i^s are strictly inside $Y^{(0)}$, we conclude that $V^1 \Subset P$.

Now it is easy to see that all the annuli A^n are non-degenerate as well. Indeed, it follows that the orbit of ∂V^1 stays away from V^1. Hence V^2 cannot touch ∂V^1, for otherwise there would be a point on ∂V^1 which returns back to V^1. So A^2 is non-degenerate. Now we can proceed inductively. □

3.4. Renormalization and central cascades.

Proposition 3.2. *A quadratic-like map is renormalizable if and only if it is either immediately renormalizable, or the principal nest $V^0 \supset V^1 \supset \dots$ ends with an infinite cascade of central returns. Thus the height $\chi(f)$ is finite if and only if f is either renormalizable or combinatorially non-recurrent.*

Proof. Let the principle nest ends with an infinite central cascade $V^{m-1} \supset V^m \supset \dots$ Then the return times stabylize, $l_m = l_{m+1} = \dots \equiv l$, and $g_n|V^n = g_m|V^n$, $n \geq m$. Moreover, by Proposition 3.1, $V^m \Subset V^{m-1}$, and hence $g \equiv g_m = f^l : V^m \to V^{m-1}$ is a quadratic-like map. We conclude that $\cap V^k$ consists of all points which never escape V^m under iterates of g, that is, $\cap V^k = K(g)$. Since $0 \in \cap V^k$, $K(g)$ is connected.

Take now the non-dividing fixed point b of g. Let us show that b is dividing for the big Julia set $K(f)$. To this end let us consider the configuration of the *full* external rays whose segments bound V^n. They divide the plane into the central component Ω^n containing V^n, and a family $\mathcal{S}^n = \{S_i^n\}, \quad i \in \mathcal{I}^n$ of disjoint sectors each bounded by two external rays. Since the critical puzzle pieces V^n are symmetric (with respect to the involution $z \mapsto z'$ such that $fz' = fz$), the families $\mathcal{S}^n$ are symmetric as well. It follows that every sector has external angle less than π.

Observe that $\mathbb{C} \setminus (V^n \cup \bigcup_i S_i^n)$ does not intersect the big Julia set $J(f)$. Hence every sector S_i^n is contained in some sector $S_{\tau(i)}^{n+1}$ of the next level. Moreover, since rays are mapped to rays, $g(\partial S_i^{n+1}) = \partial S_{\kappa(i)}^n$. (Warning: however gS_i^{n+1} does not necessarily coincide with $S_{\kappa(i)}^n$ but can be the whole complex plane. This makes the argument below somewhat involved). So we have two families of maps $\tau : \mathcal{I}^n \to \mathcal{I}^{n+1}$ and $\kappa : \mathcal{I}^{n+1} \to \mathcal{I}^n$ (hopefully skipping label n in the notation of these maps will not lead to confusion).

Let us show that these two maps commute. Indeed, by definition $S_i^n \subset S_{\tau i}^{n+1}$. Let us consider a domain $D = S_{\tau i}^{n+1} \cap V^n$. Then $gD = S_{\kappa(\tau i)}^n \cap V^{n-1}$. Since ∂D contains an arc of ∂S_i^n, $\partial(gD)$ contains an arc of $g(\partial S_i^n) = \partial S_{\kappa(i)}^{n-1}$. Hence $S_{\kappa(\tau i)}^n \supset S_{\kappa(i)}^{n-1}$, so that $\kappa(\tau i) = \tau(\kappa i)$.

Let $\sigma = \tau \circ \kappa : \mathcal{I}^n \to \mathcal{I}^n$, $n \geq m$. This map commutes with τ. Let $A \subset \mathcal{I}^m$ be a set of r indices which are cyclically permuted by $\sigma : \mathcal{I}^m \to \mathcal{I}^m$. By the commutation property, the set $\tau^k A \subset \mathcal{I}^{m+k}$ is cyclically permuted by $\sigma : \mathcal{I}^{m+k} \to \mathcal{I}^{m+k}$ as well. Thus for $i \in A$ we have: $\kappa^r(\tau^r i) = \sigma^r i = i$. Applying $\tau^{(l-1)r}$ to this equation taking into account the commutation law we conclude that

$$\kappa^r(\tau^{lr})i = \tau^{(l-1)r} i, \quad l \geq 1. \tag{3.6}$$

Let $T_i^l = S_{\tau^{lr} i}^{m+lr}$, $i \in A$, $l \geq 0$. Then $T_i^0 \subset T_i^1 \subset \ldots$, and by (3.6)

$$g^r(\partial T_i^l) = \partial T_i^{l-1}. \tag{3.7}$$

Let us consider the union of these sectors, $T_i \equiv \bigcup_i T_i^l$. Clearly all the sectors T_i and the symmetric sectors T_i' have pairwise disjoint interiors. Moreover, by (3.7), the boundary of each T_i is g^r-invariant and consists of two external rays (limits of the external rays which bound T_i^l) and a piece of the Julia set $J(f)$.

By [DH1] these boundary rays land at some periodic points. Actually, they land at the same point. Indeed, otherwise the piece of the Julia set $J(f)$ contained in the ∂T_i would correspond to an invariant arc of the ideal boundary $\mathbb{T}$ of $\mathbb{C} \setminus K(f)$ ($\mathbb{T}$ is the boundary of the unit disc uniformizing $\mathbb{C} \setminus K(f)$). This arc would not coincide with the whole circle $\mathbb{T}$ since the boundary of T_i cannot contain the whole Julia set $J(f)$ (as one of the symmetric sectors $\operatorname{int} T_i'$ contains a piece of the Julia set). But such arcs don't exist.

Thus each T_i is bounded by two g^r-invariant rays landing at the same periodic point.

Observe finally that the period r of this point must be equal to 1, so that it actually coincides with the fixed point b. Indeed, by construction all the sectors T_i and the symmetric sectors T_i' have pairwise disjoint interiors. If $r > 1$ then this situation would contradict Proposition 2.2 (ii).

So the periodic point b is dividing for $J(f)$. Let $\Omega' \subset \Omega$ be the corresponding domains constructed in §2.5. Recall that Ω is bounded by the rays landing at b and b' and two equipotential arcs, and Ω' is the connected component of $(f^l|\Omega)^{-1}$ attached to b. Then $K(g) \subset \Omega$ since the external rays landing at b and b' don't cut through $K(g)$. Since $K(g)$ is connected, $\Omega' \supset K(g)$, and hence $\Omega' \ni 0$. It follows that $g : \Omega' \to \Omega$ is a double covering. Moreover, $g^n 0 \in K(g) \subset \Omega'$, $n = 0, 1, \dots$. Thus f is renormalizable.

Vice versa, assume that f is renormalizable. Let $Rf = f^l : \Omega' \to \Omega$ be the corresponding double covering.

Then the fixed point α may not lie in $\operatorname{int} \Omega'$, for otherwise $\operatorname{int} f\Omega'$ would intersect $\operatorname{int} \Omega'$. Hence α does not cut the filled Julia set $K(Rf)$. But then the preimages of α don't cut $K(Rf)$ either. Hence given a puzzle piece $Y_i^{(n)}$, either $K(Rf)$ is contained in $Y_i^{(n)}$, or $K(Rf) \cap \operatorname{int} Y_i^{(n)} = \emptyset$. In particular $V^m \supset K(Rf)$. But then $f^l 0 \in V^m$ for all m, so that the first return times to V^m are uniformly bounded. Hence this nest must end up with a central cascade. □

The above discussion shows that there is a well-defined first renormalization Rf with the biggest Julia set, and it can be constructed in the following way. If f is immediately renormalizable, then Rf is obtained by thickening $Y^{(1)} \to Y^{(0)}$. Otherwise the principal nest ends up with the infinite central cascade $V^{m-1} \supset V^m \supset \dots$. Then $Rf = g_m : V^m \to V^{m-1}$.

The internal class $c(Rf)$ of the first renormalization belongs to a maximal copy M_0 of the Mandelbrot set.

3.5. Return maps and Koebe space. Let f be a quadratic-like map, and let $V \in \mathcal{Y}_f$ be a puzzle piece.

Lemma 3.3. *Let z be a point whose orbit passes through* $\operatorname{int} V$. *Let l be the first positive moment of time for which $f^l z \in \operatorname{int} V$. Let $U \ni z$ be the puzzle piece mapped onto V by f^l. Then $f^l : U \to V$ is either a univalent map or two-to-one branched covering depending on whether U is off-critical or otherwise.*

Proof. Let $U_k = f^k U$, $k = 0, 1, \dots l$. Since $f^k z \notin \operatorname{int} V$ for $0 < k < l$, by the Markov property of the puzzle, $U_k \cap \operatorname{int} V = \emptyset$ for those k's. Hence $f : U_k \to U_{k+1}$ is univalent for $k = 1, \dots, l-1$, and the conclusion follows. □

Let $z \in \operatorname{int} V$ be a point which returns back to $\operatorname{int} V$, and let $l > 0$ be the first return time. Then there is a puzzle piece $V(z) \subset V$ containing z such that $f^l V(z) = V$. It follows that the first return map $A_V f$ to $\operatorname{int} V$ is defined on the union of disjoint open puzzle pieces $\operatorname{int} V_i$. Moreover, if

$$f^m \partial V \cap V = \emptyset, \; m = 1, 2, \dots \tag{3.8}$$

then it is easy to see that the closed pieces V_i are pairwise disjoint and are contained in $\operatorname{int} V$. Indeed, otherwise there would be a boundary point $\zeta \in \partial V$ whose orbit would return back to V, despite (3.8).

Somewhat loosly, we will call the map

$$A_V f : \bigcup V_i \to V \tag{3.9}$$

the *first return map* to V. (Warning: it may happen that a point $z \in \partial V$ returns back to V but does not belong to $\cup V_i$; it may also happen that a point $z \in \partial V_i$ returns to V earlier than prescribed by the map A_f.)

Let V_0 denote the critical ("central") puzzle piece (provided the critical point returns back to V). Now Lemma 3.3 immediately yields:

Lemma 3.4. *The first return map A_V univalently maps all the off-critical pieces V_i onto V, and maps the critical piece V_0 onto V as a double branched covering.*

Thus the first return map $A_V f$ is Bernoulli, and is unbranched on $\bigcup_{i\neq 0} V_i$.

Let us now state an important improvement of Lemma 3.3 which will provide us later on with a "Koebe space" and distortion control.

Lemma 3.5. *Let z be a point whose orbit passes through the central domain* $\operatorname{int} V_0$ *of the first return map (3.9), and $l \geq 0$ be the first moment when $f^l z \in V_0$. Then there is a puzzle piece $\Omega \ni z$ mapped univalently by f^l onto V.*

Proof. Let s be the first moment when $f^s z \in V$. Then $f^l z = (A_V f)^k (f^s z)$ for some $k \geq 0$. Moreover, $(A_V f)^r (f^s z) \notin V_0$ for $r < k$.

Since the return map is unbranched Bernoulli outside of the central piece, there is a piece $X \subset V$ containing $f^s z$ which is univalently mapped by $(A_V f)^k$ onto V. On the other hand, by Lemma 3.3 there is a domain $D \ni z$ which is univalently mapped by f^s onto V. Hence the domain $(f|D)^{-s} X$ is univalently mapped by f^l onto V. □

Let us now consider the principal nest (3.1) of f. Let

$$g_n : \cup V_i^n \to V^{n-1} \tag{3.10}$$

be the first return map to V^{n-1}, where $V_0^n \equiv V^n \ni 0$. We will call it a *principal return map.* We will also let $g_0 \equiv f$.

Corollary 3.6. *For $n \geq 2$ the pieces V_i^n are pairwise disjoint, and the annuli $V^{n-1} \setminus V_i^n$ are non-degenerate. Moreover, the map $g_n | V_i^n$ can be decomposed as $h_{n,i} \circ f$ where $h_{n,i}$ is a univalent map with range V^{n-2}.*

Proof. As $V^{n-1} \Subset V^{n-2}$ (Proposition 3.1), and

$$f^m(\partial V^{n-1}) \cap \operatorname{int} V^{n-2} = \emptyset, \; m = 1, 2, \dots,$$

condition (3.8) is satisfied for $V = V^{n-1}$, and the first statement follows.

Take a piece V_i^n, and let $g_n | V_i^n = f^l$. By Lemma 3.5 the map $f^{l-1} : f V_i^n \to V^{n-1}$ can be extended to a univalent map with range V^{n-2}, and the second statement follows as well. □

Let $\Phi : z \mapsto z^2$ be the quadratic map. Since $f : U' \to U$ is a double covering with the critical point at 0, it can be decomposed as $\Phi \circ h$ where $h : U' \to \mathbb{C}$ is a univalent map. Let $U'' = f^{-1} U$. If $\operatorname{mod}(U' \setminus U) > \epsilon > 0$ then by the Koebe Theorem, $h|U''$ has a $L(\epsilon)$-bounded distortion. Thus f is "quadratic up to bounded distortion" . Moreover, once we know that the $\operatorname{mod} A^n$ are bounded away from 0 (see Theorem II below), we can conclude by Corollary 3.6 that all the maps g_n are quadratic up to bounded distortion.

3.6. Solar system: Bernoulli scheme associated to a central cascade. In the case of a central cascade we need a more precise analysis of the Koebe space. Let us consider a central cascade $\mathcal{C} \equiv \mathcal{C}^{m+N}$:

$$V^m \supset V^{m+1} \supset \ldots \supset V^{m+N-1} \supset V^{m+N}, \tag{3.11}$$

where $g_{m+1}0 \in V^{m+N-1} \setminus V^{m+N}$. Set $g = g_{m+1}|V^{m+1}$. Then $g : V^k \to V^{k-1}$ is a double branched covering, $k = m+1, \ldots m+N$.

Let us consider the first return map $g_{m+1} : \cup V_i^{m+1} \to V^m$, see (3.10). Let us pull the pieces V_i^{m+1} back to the annuli $A^k = V^{k-1} \setminus V^k$ by iterates of g, $k = m+1, \ldots, m+N$. We obtain a family $\mathcal{W}(\mathcal{C}) \equiv \mathcal{W}^{m+N}$ of pieces W_j^k. By construction, $W_j^k \subset A^k$ and g^{k-m-1} univalently maps each W_j^k onto some $V_i^{m+1} \equiv W_i^{m+1}$.

Let us define an unbranched Bernoulli map

$$G \equiv G_{m+N} : \bigcup_{\mathcal{W}(\mathcal{C})} W_j^k \to V^m \tag{3.12}$$

as follows: $G|W_j^k = g_{m+1} \circ g^{k-m-1}$ (see Figure 4).

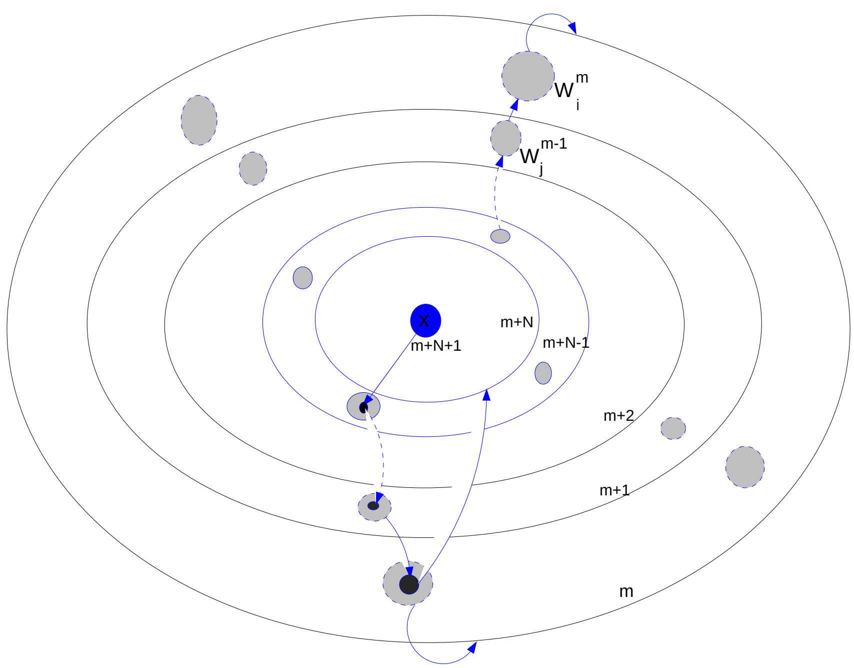

Figure 4. Solar system.

Lemma 3.7. *Let us consider the central cascade (3.11). Let z be a point whose orbit passes through V^{m+N}, and l be the first moment for which $f^l z \in V^{m+N}$. Then there is a piece $\Omega \ni z$ which is univalently mapped by f^l onto V^m.*

Proof. Let s be the first moment for which $f^s z \in V^m$. Then $f^l z = G^k(f^s z)$ where G is the Bernoulli map (3.12). Now repeat the argument of Lemma 3.5 just using G instead of the first return map. □

Corollary 3.8. *Let us consider the central cascade (3.11). Then the map $g_{m+N+1} : V^{m+N+1} \to V^{m+N}$ can be represented as $h_{m+N+1} \circ f$ where h_{m+N+1} is a univalent map with range V^m.*

Proof. Repeat the proof of Corollary 3.6 using Lemma 3.7 instead of 3.5. □

Remark. The modification of the principal nest after passing a long central cascade mentioned in Remark 1 of §3.1 is the following. Let $g_{m+1}0 \in W_j^{m+N}$. Define $\tilde{V}^{m+N+1}$ as the pull-back of W_j^{m+N} by $g_{m+1} : V^{m+N} \to V^{m+N-1}$. Then continue the nest by the first return pull-backs beginning with $\tilde{V}^{m+N+1}$ modifying it each time after passing a long cascade. Note that the construction of the first piece V^0 described in §3.2 is similar to this modification after passing the initial degenerate central cascade (3.2).

3.7. Generalized polynomial-like maps and renormalization. Let $\{U_i\}$ be a finite or countable family of topological discs with disjoint interiors strictly contained in a topological disk U. We call a map $g : \cup U_i \to U$ a *(generalized) polynomial-like map* if $g : U_i \to U$ is a branched covering of finite degree which is univalent on all but finitely many U_i.

Let us say that a polynomial-like map g is *of finite type* if its domain consists of finitely many disks U_i. In this case we define the filled Julia set $K(g)$ as the set of all non-escaping points, and the Julia set $J(g)$ as its boundary. The DH polynomial-like maps correspond to the case of a single disk U_0.

Generalized Straightening Theorem. *Any generalized polynomial-like map of finite type is qc conjugate to a polynomial with the same number of non-escaping critical points.*

Proof. For the case of two discs U_0, U_1 the proof is given in [LM]. In general let us proceed inductively in the number of the discs. Enclose two of the discs by a figure eight, and make a qc surgery which creates a new escaping critical point at the singularity of the figure eight, see [LM]. This surgery decreases by one the number of the discs. □

Let us call a generalized polynomial-like map a *generalized quadratic-like* if it has a single (and non-degenerate) critical point. In such a case we will assume, unless otherwise is stated, that 0 is the critical point, and label the discs U_i in such a way that $U_0 \ni 0$. In what follows we will deal exclusively with quadratic-like maps, namely with the principal sequence g_n of the first return maps (3.10).

Given a V_j^{n+1}, $n \geq 1$, let l be its first return time back to V^n under iterates of g_n, that is, $g_{n+1}|V_j^{n+1} = g_n^l$. Then

$$g_n^k V_j^{n+1} \subset V_{i(k)}^n,\ k = 0, 1, \dots, l,$$

with $i(0) = i(l) = 0$. Moreover, $g_n^k V_j \Subset V_{i(k)}^n$ for $k < l$. The sequence $0 = i(0), i(1), \ldots, i(l) = 0$ is called the *itinerary* of V_j^{n+1} through the domains of previous level. A piece V_j^{n+1} is called *precritical* if $g_n V_j^{n+1} = V_0^n$, so that it has the shortest possible itinerary: $l = 1$.

Let us define the n-fold *generalized renormalization* $T^n f$ of f as the first return map g_n restricted to the union of puzzle pieces V_i^n meeting the critical set $\omega(0)$, and considered up to rescaling. In the most interesting situations these maps are of finite type:

Lemma 3.9. *If f is a DH renormalizable quadratic-like map, then all the maps $T^n f$ are of finite type.*

Proof. In the tail of the principal nest the maps $T^n f$ are DH quadratic-like, and their domains consist just of one component. So we should take care only of the initial piece of the cascade.

Let us take the renormalization $Rf = f^l : V^{t+1} \to V^t$ with $t \geq n$. Since 0 is non-escaping under iterates of Rf, we have the following property: the first landing time of any point $f^k 0$ back to V^{t+1} is at most l. All the more, the landing time to the bigger domain $V^n \supset V^t$ is bounded by l. Hence the components of $f^{-t} V^{n-1}$, $t = 0, 1, \ldots, l-1$, cover the whole postcritical set. For sure, there are only finitely many these components. But the domain of $T^n f$ consists of the pull-backs of these components by $f|V^{n-1}$. □

3.8. Return graph. Let $\mathcal{I}^n$ be the family of puzzle pieces V_i^n intersecting $\omega(0)$, that is, the pieces in the domain of the generalized renormalization

$$T^n f : \bigcup_{\mathcal{I}^n} V_i^n \to V^{n-1}.$$

Let us consider a graded graph Υ_f whose vertices of level n are the pieces $V_j^n \in \mathcal{I}^n$, $n = 0, 1, \ldots$, where V_j^0 stand for the pieces of the initial tiling (3.4). Let us take a vertex $V_j^{n+1} \in \mathcal{I}^{n+1}$, and let $i(1), \ldots, i(t) = 0$ be its itinerary through the pieces of the previous level under the iterates of g_n. Then join V_j^{n+1} with V_i^n by k edges, provided the symbol i appears in the above itinerary k times. This means that the piece V_j^{n+1} under iterates of g_n passes through V_i^n k times before the first return back to V^n. Let us order the edges joining two vertices V_j^{n+1} and V_i^n so that the first edge represents the first return of V_j^{n+1} to V_i^n, the second one represents the second return, etc.

Note that for any vertex V_j^{n+1} there is exactly one edge joining it to the critical vertex V_0^n of the previous level. Note also that by Lemma 3.9 in the DH renormalizable case the number of vertices on a given level is finite. In any case there are clearly only finitely many edges leading from a V_j^{n+1} to the previous level n. Let $\tau(V_j^{n+1})$ denote the number of such edges, which is equal to the first return time of V_j^{n+1} back to V^n under iterates of g_n.

By a *path* in the graded graph Υ_f we mean a sequence of consecutively adjacent vertices $V_{i(n)}^n$, $n = l, l+1, \ldots, l+k$ up to reversing the order. So we don't endow a path with orientation, and can go along it either strictly upwards or strictly downwards.

Diverse combinatorial data can be easily read off this graph. For example, given $n \geq m$, the number of paths joining V_j^{n+1} to V_i^m is equal to the number of times which the g_m-orbit of V_j^{n+1} passes through V_i^m before the first return back to V^n. Hence the return time of V_j^{n+1} back to V^n under iterates of g_m is equal to the total number of paths in Υ_f leading from V_j^{n+1} up to level m. For $m = 0$ we obtain the return time under iterates of the original map $f = g_0$.

Assume now that the map f is DH renormalizable, and let s be a renormalization level in the principal nest, that is, $g_{s+1} : V^{s+1} \to V^s$ is a quadratic-like map with non-escaping critical point. Then there is a single vertex V^{s+1} at level $s + 1$, and below it the return graph is just the "vertical path" through the critical vertices. By the above discussion, the total number of paths in the graph Υ_f joining the top level to the bottom vertex V^{s+1} is equal to the *renormalization period* $\mathrm{per}(f)$ (i.e., the return time of V^{s+1} back to V^s under iterates of f).

It follows that the $\mathrm{per}(f)$ is bounded if and only if the DH-level s is bounded, and all the return times $\tau(V_i^{m+1})$ are bounded for $1 \leq m \leq s$ and any i. For instance, the "if" statement means: If $s \leq \bar{s}$ and $\tau(V_i^{m+1}) \leq \bar{\tau}$ for all vertices V_i^{m+1}, then $\mathrm{per}(f) \leq p(\bar{s}, \bar{\tau})$. Indeed, the total number of paths in the graph is bounded by τ^s.

Note that central cascades correspond to the vertical paths through the critical vertices. We say that a path γ passes through a central cascade (3.11) if $\gamma \ni V_j^n$ with $n \in [m + 1, m + N]$.

Let us now define one more combinatorial notion, the *rank* (compare [L4], §3). Let $D^n \subset V^{n-1}$ be a puzzle piece of the full Markov family $\mathcal{Y}_f$ (see §2.6) containing at least one piece V_i^n of level n. Let us consider the shortest path γ leading from D^n (i.e., from one of the pieces $V_i^n \subset D^n$) down to a critical piece V^{n+s}. The number of central cascades this γ passes through will be called the rank of D^n.

This notion is motivated by the following consideration. Let us consider two adjacent puzzle pieces $V_i^n \subset D^n$ & V_j^{n+1}, and an edge γ joining them. Let t be the return time represented by γ, i.e., $g_n^t V_j^{n+1} \subset V_i^n$. The piece $D^{n+1} \supset V_j^{n+1}$ in V^n such that $g_n^t D^{n+1} = D^n$ will be called the pull-back of D^n along the edge γ. More generally, let us define the pull-back of D^n along a path γ leading from D^n downwards by composing the pull-backs along the edges.

Lemma 3.10. *Let γ be the shortest path leading from D^n down to a critical piece V_0^{n+s}, and let D^{n+s} be the pull-back of D^n along this path. Then*

$$V^{n+s} \subset D^{n+s} \subset V^{n+s-1},$$

and the map $D^{n+s} \to D^n$ is a double branched covering.

Proof. Easily follows from the definitions. □

3.9. Full principal nest. Let f be a non-immediately DH renormalizable quadratic-like map. Then its principal nest

$$Y^{(0,0)} \supset V^{0,0} \supset V^{0,1} \supset \ldots \supset V^{0,t(0)} \supset V^{0,t(0)+1} \supset \ldots$$

ends up with an infinite cascade of central returns (we call this nest "short" and label it by two indices for the reason which will become clear in a moment). Let us select a level $t(0)$ of this cascade, so that the return map $Rf = g_{0,t(0)+1} : V^{0,t(0)+1} \to V^{0,t(0)}$ is DH quadratic-like. We will call such a level DH. (The particular choice of DH levels in what follows will depend on the geometry).

If Rf is non-immediately renormalizable, let us cut the puzzle piece $V^{0,t(0)+1}$ by the external rays, and construct its short principal nest:

$$Y^{(1,0)} \supset V^{1,0} \supset V^{1,1} \supset \ldots \supset V^{1,t(1)} \supset V^{1,t(1)+1} \supset \ldots$$

If Rf is DH renormalizable, then this nest also ends up with an infinite central cascade. Then select a DH level $t(1)+1$, and pass to the next short nest.

If f is infinitely DH renormalizable but none of the renormalizations are immediate, then in such a way we construct the *full principal nest*

$$\begin{array}{c} Y^{(0,0)} \supset V^{0,0} \supset V^{0,1} \supset \ldots \supset V^{0,t(0)} \supset V^{0,t(0)+1} \supset \\ Y^{(1,0)} \supset V^{1,0} \supset V^{1,1} \supset \ldots \supset V^{1,t(1)} \supset V^{1,t(1)+1} \supset \\ \ldots\ldots \\ Y^{(m,0)} \supset V^{m,0} \supset V^{m,1} \supset \ldots \supset V^{m,t(m)} \supset V^{m,t(m)+1} \supset \\ \ldots\ldots. \end{array} \tag{3.13}$$

Here $Y^{(m,0)}$ is the first critical Yoccoz puzzle piece for the m-fold DH renormalization $R^m f$, while the pieces $V^{m,n}$ form the corresponding short principal nest. Moreover, for $m > 1$, $Y^{(m,0)}$ is obtained by cutting $V^{m-1,t(m-1)+1}$ with the external rays of $R^m f : V^{m-1,t(m-1)+1} \to V^{m-1,t(m-1)}$.

The annuli $A^{m,n} = V^{m,n-1} \setminus V^{m,n}$ will be called the *principal annuli.*

3.10. Big type: special families of Mandelbrot copies. Assume that we associated to any quadratic-like map a "combinatorial parameter" $\tau(f)$, which depends only on the hybrid class $c(f)$ and is constant over any maximal copy of the Mandelbrot set. (Keep in mind the height function $\chi(f)$ or the period $\mathrm{per}(f)$.) Thus we can use the notation $\tau(M')$.

Let $\mathcal{S} \subset \mathcal{M}$ be a family of maximal copies of the Mandelbrot set. Let us call it *τ-special* if it satisfies the following property: for any truncated secondary limb L there is a τ_L such that $\mathcal{S}$ contains all maximal copies $M' \subset L$ of the Mandelbrot set with $\tau(M') \geq \tau_L$.

Let f be an infinitely renormalizable quadratic-like map. Let us say that it is of $\mathcal{S}$-type if all the internal classes $c(R^n f)$ belong to copies M' from $\mathcal{S}$.

4. Initial geometry

The goal of this section is to give a bound on the first principal modulus depending only on the choice of the secondary limbs and $\mathrm{mod}(f)$:

Theorem I. *Let f be a quadratic-like map with internal class $c(f)$ ranging over a truncated secondary limb L_b^{tr}. If $\mathrm{mod}(f) \geq \mu > 0$ then*

$$\mathrm{mod}(A^1) \geq C(\mu)\,\nu(L_b^{tr}) > 0,$$

where $C(\mu) > 0$ and $C(\mu) \nearrow 1$ as $\mu \nearrow \infty$.

4.1. Geometry of rays. Let us consider a parameter region D. Assuming that the rays $\mathcal{R}_c^{\theta,(\rho,r)}$ (see §2.2) don't bounce off the critical point for $c \in D$, let us consider their natural parametrization $\psi_c : (\rho, r) \to \mathcal{R}_c^{\theta,(\rho,r)}$. Continuous/smooth/real analytic dependence of the ray on $c \in D$ is defined as the corresponding property of the function $(t, c) \mapsto \psi_c(t)$. The same definitions are applied to equipotentials.

Let $B(a, \delta) = \{z : |z - a| < \delta\}$.

Lemma 4.1. *(i) Assume that a ray $\mathcal{R}_c^\theta$ and an equipotential E_c^ρ don't hit the critical point, $c \in D$. Then $\mathcal{R}_c^\theta$ and E_c^ρ depend real analytically on c;*
(ii) Let a_c be a repelling periodic point of P_c continuously depending on $c \in D$. Let the ray $\mathcal{R}_c^\theta$ lands at a_c. Then the closure of this ray, $\bar{\mathcal{R}}_c^\theta$, depends continuously on c.

Proof. (i) The first statement follows from the fact that the Bötcher function analytically depends on c, which is clear from the explicit formula (2.1).

(ii) Let us check continuity at some $d \in D$. By (i), we only need to check that for $r > 0$ sufficiently small, the arc $\mathcal{R}_c^{\theta,[0,r]}$ is uniformly close to a_d. Indeed, for any $\epsilon > 0$ there exist $\delta > 0$ and $r > 0$ such that

- For $|c - d| < \epsilon$, P_c univalently maps $B_c \equiv B(a_c, \delta)$ onto a strictly bigger disc;
- $\mathcal{R}_c^{\theta,[r,2r]} \subset B_c$ (this follows from (i)).

Pulling back the arc $\mathcal{R}_c^{\theta,[r,2r]}$ by $P_c|B_c$, we conclude that $\mathcal{R}_c^{\theta,[0,2r]} \subset B_c$. □

Given a configuration $\mathcal{C}_0$ of finitely many parametrized curves and point in C, let us consider the space $\mathrm{QC}(\mathcal{C}_0)$ of all configurations qc equivalent to $\mathcal{C}_0$. There is a natural Teichmüller (pseudo-) distance on this space:

$$\mathrm{dist}_T(\mathcal{C}_1, \mathcal{C}_2) = \inf \log \mathrm{Dil}(h),$$

where h runs over all qc equivalences $h : (\mathbb{C}, \mathcal{C}_1) \to (\mathbb{C}, \mathcal{C}_2)$.

We say that configurations of a certain family have *bounded geometry* if they stay bounded Teichmüller distance from a reference configuration $\mathcal{C}_0$ whose curves are smooth and intersect transversally.

Lemma 4.2. *Let a_c, $c \in D$, be a repelling periodic as in Lemma 4.1. Let us consider a configuration $\mathcal{R}(a_c)$ of finitely many rays $\mathcal{R}_c^{\theta_i}$ landing at a_c, $c \in D$. Then $\mathcal{R}(a_c)$ has bounded geometry when c ranges over any compact subset of D.*

Proof. Take a $d \in D$. For any nearby $c \in B(d, \delta)$, let us truncate the configuration $\mathcal{R}_c$ with an equipotential E_c^ρ, where ρ is selected big enough so that E_c^ρ is a Jordan curve. We obtain an inner configuration $\mathcal{R}_c^1$ and an outer one, $\mathcal{R}_c^2$. The latter one has bounded geometry over $B(d,\delta)$, as it is conformally equivalent to the configuration consisting of the circle of radius ρ and raidial rays of angles θ_i.

Take a small $\epsilon > 0$. Lemma 4.1 implies that for $c \in B(d,\delta)$ with a small δ, there exists a smooth parametrized Jordan curve $\gamma \subset B(a_c,\epsilon) \equiv B_c$ enclosing a_c which transversally intersects every ray of $\mathcal{R}_c$ at a single point. It truncates $\mathcal{R}_c^1$ into the inner configuration $\mathcal{R}_c^i$ and the outer one, $\mathcal{R}_c^o$. The latter one has bounded geometry over $B(d,\delta)$ since by Lemma 4.1 it smoothly depends on c.

Let us consider $\mathcal{R}_c^i$. The parametrized curves $\gamma_{-N} = (f|B_c)^{-N}\gamma$ also intersect every ray of $\mathcal{R}_c$ at a single point. Moreover, for sufficiently big N (locally uniform), γ_{-N} lies strictly inside γ with a definite space in between.

Let us consider a configuration $\mathcal{C}_c$ consisting of the annulus bounded by γ and γ_{-N} with the arcs of the rays $\mathcal{R}^{\theta_i}$ in between (with the natural parametrization). Since this configuration smoothly depends on c (by Lemma 4.1), it has bounded geometry near d. Thus $\mathcal{C}_c$ stays bounded Teichmüller distance from a standard configuration $\mathcal{C}_0$, the round annulus $\mathbb{A}(1/2,1)$ with p equally spaced radial intervals inside.

So for c near d, there is a K-qc map $h : \mathcal{C}_c \to \mathcal{C}_0$ with locally uniform dilatation K, which conjugates f to $z \mapsto 2z$ on the inner boundaries of the configurations. Pulling this map back by the dynamics, we obtain a K-qc equivalence between the configuration $\mathcal{R}$ and a standard configuration $\mathcal{I}$ consisting of the unit circle and p equally spaced radial intervals eminating from 0.

Since the dilatation K is locally uniform, it is uniform over any compact set. □

Let the α-fixed point of f has rotation number q/p. Then there is a single periodic point $\gamma \in \operatorname{int} Y^{(1)}$ of period p. Let $\mathcal{C}(f)$ stand for the configuration of the rays landing at α, γ and the symmetric points α', γ'.

Corollary 4.3. *The configuration $\mathcal{C}(P_c)$ has a bounded geometry while c ranges over a truncated secondary limb L_b^{tr}.*

4.2. Fundamental domain near the fixed point. The goal of this subsection is to construct a combinatorially defined fundamental domain with bounded geometry near the fixed point α. It is where the secondary limbs condition comes into the scene.

Let γ and γ' be the periodic and co-periodic points defined prior to Corollary 4.3. Consider the family $\mathcal{R}(\gamma')$ of rays landing at γ'. Let $D = D_f$ be the component of $Y^{(1)} \smallsetminus \mathcal{R}(\gamma')$ attached to the fixed point α (see Figure 5). Then f^p univalently maps D onto a domain containing the component of $Y^{(0)} \smallsetminus \mathcal{R}(\gamma)$ attached to α. Note that $\partial D \cap \partial(f^p D)$ is contained in the union of two rays landing at α.

Hence there is a univalent branch of f^{-p} which fixes α and maps D inside itself. It is now easy to see that $f^{-pn}D$ shrink to α as $n \to \infty$. So we can select $Q = Q_f = D \smallsetminus f^{-p}D$ as a fundamental domain for f^p near α: any trajectory which starts near α must pass through $Q = Q_f$. Now Corollary 4.3 yields:

Lemma 4.4. *Geometry of the fundamental domain Q_f is bounded if $c(f)$ ranges over a truncated secondary limb and f has a definite modulus.*

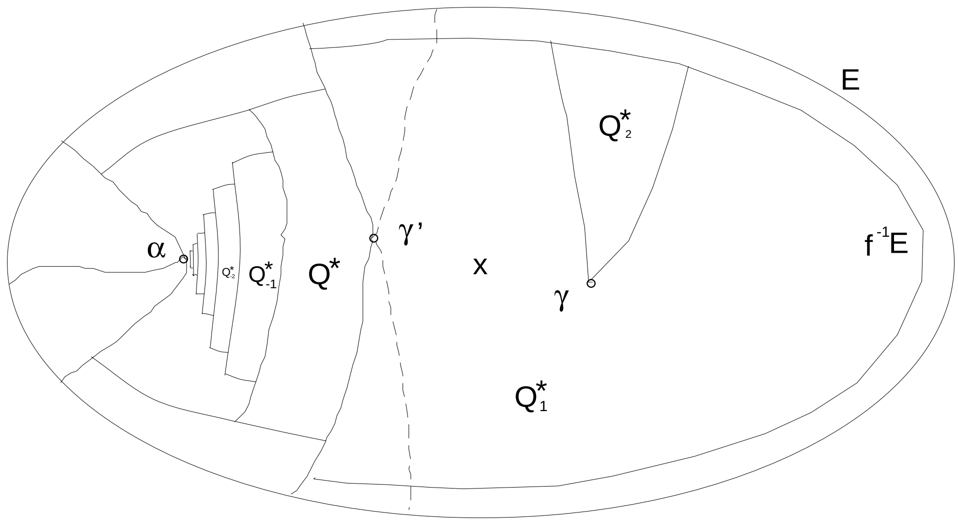

Figure 5. The fundamental domain near α.

4.3. Modulus of the first annulus.

Lemma 4.5. *Let P_c be a quadratic polynomial with c outside the main cardioid but not immediately renormalizable. If c ranges over a truncated secondary limb L_b^{tr}, then all the pieces W of the initial Markov tiling (3.4) are well inside $Y^{(0)}$:* $\mathrm{mod}(Y^{(0)} \setminus W) > \nu(L_b^{tr}) > 0$.

Proof. Let $Y^{(0)}$ be bounded by the equipotential $E \equiv E^1$ of level 1 (together with two rays). Let $U^r \supset J(f)$ be the domain bounded by the equipotential E^r.

Take a little $\epsilon > 0$. Then there exist N and $\delta > 0$ such that the distance from $U^{1/2^N} \setminus B(\alpha, \epsilon)$ to $\partial Y^{(0)}$ is at least δ (for all $c \in L_b^{tr}$).

The statement is obviously true for all the pieces W of depth $\leq N$.

Any other piece W is contained in $U^{1/2^N}$. Hence if $\mathrm{dist}(W, \alpha) > \epsilon$ then $\mathrm{dist}(W, \partial Y^{(0)}) \geq \delta$. As $\mathrm{diam}\, W$ is uniformly bounded, we conclude that W is well inside $Y^{(0)}$.

Assume now that $\mathrm{dist}(W, \alpha) < \epsilon$. Then W intersects the domain $D = D_f$. Since $\partial D \cap \partial W = \emptyset$, $W \subset D$. Let us consider the iterates $f^{pk}W$, $k = 0, 1, \dots$ until the last moment l such that $f^{pl}W \subset D$. At this moment $f^{pl}W$ must intersect the fundamental domain Q. Since their boundaries don't intersect, we conclude that $f^{pl}W \subset Q$.

Let us consider domain $Q^* = Q \cap U^{1/2^p} \subset Q$ obtained by truncating Q with the equipotential $f^{-p}E \equiv E^{1/2^p}$. This domain has a bounded geometry since the fundamental domain Q does (Lemma 4.4). Hence Q^* is well inside f^pD. Moreover, $f^{pl}W \subset Q^*$ since all the puzzle pieces of (3.4) which belong to D are enclosed by the equipotential $f^{-p}E$ (see Figure 5). Hence $f^{pl}W$ is well inside f^pD as well.

We conclude that there is always a definite space around $f^{pl}W$ in f^pD. Pulling this space back by iterates of the univalent branch $f^{-p} : f^pD \to D$, we obtain a definite space around W in D. $\square$

We are now ready to prove the theorem stated in the beginning of this section:
Proof of Theorem I. Assume first that $f = P_c$ is a polynomial. Let us go through the proof of Proposition 3.1. We found an l and a puzzle piece $P \subset V^0$ such that $G^l P$ two-to-one covers $Y^{(0)}$, where G is the Markov map (3.5). Moreover, $G^l 0 \in W$ where $W = V^0$ or $W = X_i^s$. Then V^1 is the pull-back of W by $G^l|P$. But by Lemma 4.5 W is well inside $Y^{(0)}$. Hence V^1 is well inside V^0.

If f is quadratic-like then its straightening yields the desired estimate by Proposition 2.5. The constant $C(\mu)$ can certainly be selected so that it is monotone in μ. □

5. Bounds on the moduli and distortion

In this section we introduce the asymmetric moduli and prove that they don't decrease under the generalized renormalization. This yields *a priori* bounds on the principal moduli and distortion. The precise formulation (Theorem II) is given at the end of the section. Note that already this result yields the Yoccoz divergence property (Theorem 2.6).

5.1. First estimates. Let $\mathcal{V}^n \subset \mathcal{Y}_f$ stand for the family of all pieces V_i^n of level n.

Let us start with a lemma which partly explains the importance of the principal nest: the principal moduli control the distortion of the first return maps (see the Appendix for the definition of distortion). Let us consider the decomposition:

$$g_n|V^n = h_n \circ f, \tag{5.1}$$

where h_n is a diffeomorphism of fV^n onto V^{n-1}.

Lemma 5.1. *Let $D \in \mathcal{Y}_f$ be a puzzle piece such that $f^l D = V^n$, while $f^k D \cap V^n = \emptyset$, $k = 0, \dots, l-1$. If $\mu_n \geq \bar{\mu}$ then the distortion of f^l on D is $O(\exp(-\mu_{n-1}))$ with a constant depending only on $\bar{\mu}$. Hence the distortion of h_n is $O(\exp(-\mu_{n-2}))$.*

Proof. This follows from Lemma 3.5, Corollary 3.6 and the Koebe Theorem. □

Let us fix a level $n > 0$, denote $V^{n-1} = \Delta$, $V_i = V_i^n$, $g = g_n$, $A = A^n = \Delta \setminus V_0$, $\mu = \mu_n$, and mark the objects of the next level $n+1$ with prime. Thus $\Delta' \equiv V \equiv V_0$, and $g' : \cup V_i' \to \Delta'$. (We restore the index n whenever we need it).

Lemma 5.2. *Let $D' \subset \Delta'$ be a puzzle piece such that $g^k D' \subset V_{i(k)}$, $k = 1, \dots, l$, with $i(k) \neq 0$ for $0 < k < l$. Then*

$$\operatorname{mod}(\Delta' \setminus D') \geq \frac{1}{2} \sum_{k=1}^{l} \operatorname{mod}(\Delta \setminus V_{i(k)}).$$

Proof. Let us consider the following nest of topological disks:

$$\Delta' \equiv W_1 \supset \dots \supset W_l \supset W_{l+1} \supset D',$$

where W_{k+1} is defined inductively as the pullback of $V_{i(k)}$ under $g^k : W_k \to \Delta$, $k = 1, \dots l$. Since $\deg(g^k : W_k \to \Delta) = 2$,

$$\operatorname{mod}(W_k \setminus W_{k+1}) = \frac{1}{2} \operatorname{mod}(\Delta \setminus V_{i(k)}), \quad (1 \leq k \leq l).$$

But by the Grötzsch inequality

$$\operatorname{mod}(\Delta' \setminus D') \geq \sum_{k=1}^{l} \operatorname{mod}(W_k \setminus W_{k+1}),$$

and the desired estimate follows. □

Corollary 5.3. *Given a puzzle piece* V_j', *we have*

$$\operatorname{mod}(\Delta' \setminus V_j') \geq \frac{1}{2}\mu.$$

Moreover, if the return to level n *is non-central, that is* $g0 \in V_i$ *with an* $i \neq 0$, *then*

$$\operatorname{mod}(\Delta' \setminus V_j') \geq \frac{1}{2}(\mu + \operatorname{mod}(\Delta \setminus V_i)).$$

So, a definite principal modulus on some level produces a definite space around all the puzzle pieces of the next level.

5.3. Isles and asymmetric moduli. Let $\{V_i\}_{i \in \mathcal{I}} \subset \mathcal{V}^n$ be a finite family of disjoint puzzle pieces consisting of at least two pieces (that is $|\mathcal{I}| \geq 2$) and containing a critical puzzle piece V_0. Let us call such a family *admissible.* We will freely identify the label set $\mathcal{I}$ with the family itself.

Given a puzzle piece D, let $\mathcal{I}|D$ denote the family of puzzle pieces of $\mathcal{I}$ contained in D. Let D be a puzzle piece containing at least two pieces of family $\mathcal{I}$. For $V_i \subset D$ let

$$R_i \equiv R_i(\mathcal{I}|D) \subset D \setminus \bigcup_{j \in \mathcal{I}|D} V_j$$

be an annulus of maximal modulus enclosing V_i but not enclosing other pieces of the family $\mathcal{I}$. Such an annulus exists by the Montel Theorem (see Figure 6). We will briefly call it the *maximal annulus* enclosing V_i in D (rel the family $\mathcal{I}$).

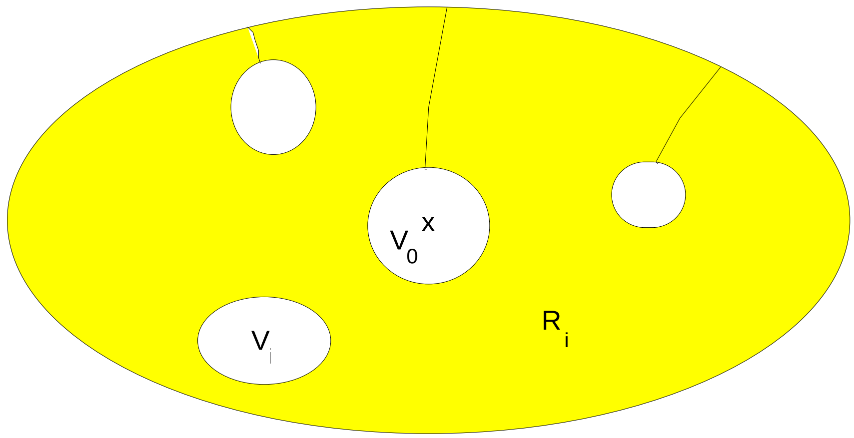

Figure 6. Annulus R_i.

Let us define *the asymmetric modulus of the family $\mathcal{I}$ in D* as

$$\sigma(\mathcal{I}|D) = \sum_{i\in\mathcal{I}} \frac{1}{2^{1-\delta_{i0}}} \bmod R_i(\mathcal{I}|D),$$

where δ_{ji} is the Kronecker symbol. So the critical modulus is supplied with weight 1, while the off-critical moduli are supplied with weights 1/2 (if D is off-critical then all the weights are actually 1/2).

For $D = V^{n-1}$, let $\sigma_n(\mathcal{I}) \equiv \sigma(\mathcal{I}|V^{n-1})$. The asymmetric modulus of level n is defined as follows:

$$\sigma_n = \min_{\mathcal{I}} \sigma_n(\mathcal{I}),$$

where $\mathcal{I}$ runs over all admissible subfamilies of $\mathcal{V}^n$.

The principal moduli μ_n and the asymmetric moduli σ_n are the main geometric parameters of the renormalized maps g_n. Again, in what follows the label n will be suppressed as long as the level is not changed.

Let $\{V_i'\}_{i\in\mathcal{I}'}$ be an admissible subfamily of $\mathcal{V}'$. Let us organize the pieces of this family in *isles* in the following way. A puzzle piece $D' \subset \Delta'$ is called an *island* (for family I') if

- D' contains at least two puzzle pieces of family $\mathcal{I}'$;
- There is a $t \geq 1$ such that $g^k D' \subset V_{i(k)}$, $k = 1, \dots t-1$, with $i(k) \neq 0$, while $g^t D = \Delta$.

Given an island D', let $\phi_{D'} = g^t : D' \to \Delta$. This map is either a double covering or a biholomorphic isomorphism depending on whether D' is critical or not. In the former case, $D' \supset V_0'$ (for otherwise $D' \subset V_0'$ contradicting the first part of the definition of isles).

We call a puzzle piece $V_j' \subset D'$ *$\phi_{D'}$-precritical* if $\phi_{D'}(V_j') = V_0$. There are at most two precritical pieces in any D'. If there are actually two of them, then they are off-critical and symmetric with respect to the critical point 0. In this case D' must also contain the critical puzzle piece V_0'.

Let $\mathcal{D}' = \mathcal{D}(\mathcal{I}')$ be the family of isles associated with $\mathcal{I}'$. Let us consider the asymmetric moduli $\sigma(\mathcal{I}'|D')$ as a function on this family. This function is clearly monotone:

$$\sigma(\mathcal{I}'|D') \geq \sigma(\mathcal{I}'|D'_1) \quad \text{if} \quad D' \supset D'_1, \tag{5.2}$$

and superadditive:

$$\sigma(\mathcal{I}'|D') \geq \sigma(\mathcal{I}'|D'_1) + \sigma(\mathcal{I}'|D'_2),$$

provided D'_i are disjoint subisles in D'.

Let us call an island D' *innermost* if it does not contain any other isles of the family $\mathcal{D}(\mathcal{I}')$. As this family is finite, innermost isles exist.

5.2. Non-decreasing of the moduli.

Lemma 5.4. *Let $\mathcal{I}'$ be an admissible family of puzzle pieces. Let D' be an innermost island associated to the family $\mathcal{I}'$, and let $\mathcal{J}' = \mathcal{I}'|D$. For $j \in \mathcal{J}'$, let us define $i(j)$ by the property $\phi_{D'}(V'_j) \subset V_{i(j)}$, and let $\mathcal{I} = \{i(j) : j \in \mathcal{J}'\} \cup \{0\}$. Then $\{V_i\}_{i\in\mathcal{I}}$ is an admissible family of puzzle pieces, and*

$$\sigma(\mathcal{I}'|D') \geq \frac{1}{2}\left((|\mathcal{J}'| - s)\mu + s \mod R_0 + \sum_{j\in\mathcal{J}',\, i(j)\neq 0} \mod R_{i(j)}\right), \tag{5.3}$$

where $s = \#\{j : i(j) = 0\}$ is the number of $\phi_{D'}$-precritical pieces, and R_i are the maximal annuli enclosing V_i in Δ rel $\mathcal{I}$.

Proof. Let $\phi \equiv \phi_{D'}$. Let us show first that the family $\mathcal{I}$ is admissible. This family is finite since $\mathcal{J}' \subset \mathcal{I}'$ is finite. The critical puzzle piece belongs to $\mathcal{I}$ by definition. So the only property to check is that $|\mathcal{I}| \geq 2$. But otherwise $\mathcal{J}'$ would consist of two precritical puzzle pieces. But then D' would be critical, and thus should have also contained the critical piece V'_0, which is a contradiction.

Let us observe next that

$$\mathrm{mod}(V_{i(j)} \smallsetminus \phi V'_j) \geq \mu \quad \text{if} \quad i(j) \neq 0. \tag{5.4}$$

Indeed, in this case $g^m(\phi V'_j) = V_0$ for some $m > 0$. Let $W \subset V_{i(j)}$ be the pull-back of Δ under g^m. Then the annulus $W \smallsetminus \phi V'_j$ is univalently mapped by g^m onto the annulus $\Delta \smallsetminus V_0$. Hence $\mathrm{mod}(W \smallsetminus \phi V'_j) = \mathrm{mod}(\Delta \smallsetminus V_0) = \mu$, and (5.4) follows.

Given an $i \in I$, let us consider a topological disk $Q_i = R_i \cup V_i \subset \Delta$ ("filled annulus R_i"). By the Grötzsch inequality and (5.4),

$$\mathrm{mod}(Q_{i(j)} \smallsetminus \phi V_j) \geq \mathrm{mod}\, R_{i(j)} + (1 - \delta_{0,i(j)})\mu. \tag{5.5}$$

For a $j \in J'$, let us consider an annulus $B'_j \subset D'$, the component of $\phi^{-1} R_{i(j)}$ enclosing V'_j. This annulus does not enclose any other pieces $V'_k \in \mathcal{J}'$, $k \neq j$. Indeed, otherwise the inner component of $\mathbb{C} \smallsetminus B'_j$ would be an island contained in D', despite the assumption that D' is innermost.

Let us now consider a topological disk P'_j obtained by filling the annulus B'_j. Then

$$\text{mod}\, R'_j \geq \text{mod}(P'_j \setminus V'_j), \tag{5.6}$$

where $R'_j \subset D'$ is the maximal annulus enclosing V'_j rel $\mathcal{J}'$. Moreover $\phi : P'_j \to Q_{i(j)}$ is univalent or double covering depending on whether $j \neq 0$ or $j = 0$. Hence

$$\text{mod}(P'_j \setminus V'_j) \geq \frac{1}{2^{\delta_{j0}}} \text{mod}(Q_{i(j)} \setminus \phi V_j). \tag{5.7}$$

Inequalities (5.5)-(5.7) yield

$$\text{mod}\, R'_j \geq \frac{1}{2^{\delta_{j0}}} (\text{mod}\, R_{i(j)} + (1 - \delta_{0,i(j)})\mu). \tag{5.8}$$

Summing up estimates (5.8) over $\mathcal{J}'$ with weights $1/2^{1-\delta_{j0}}$, we obtain the desired inequality. □

Corollary 5.5. *For any island D' of the family $\mathcal{D}'$ the following estimates hold:*

$$\sigma(\mathcal{I}'|D') \geq \frac{1}{2}\mu \quad \textit{and} \quad \sigma(\mathcal{I}'|D') \geq \sigma(\mathcal{I}) \geq \sigma.$$

Proof. By monotonicity (5.2), it is enough to check the case of an innermost island D'. Let us use the notations of the previous lemma. Since the family $\mathcal{I}$ is admissible, it contains an off-critical piece. Hence $|\mathcal{J}'|$ is always strictly greater than the number s of precritical pieces in D', and (5.3) implies the first of the above inequality.

Furthermore, as $\mu \geq \text{mod}(R_0)$ and $|\mathcal{J}'| \geq 2$, the right-hand side in (5.3) is bounded from below by

$$\frac{1}{2}\left(|\mathcal{J}'|\ \text{mod}(R_0) + \sum_{i \in \mathcal{I}, i \neq 0} \text{mod}(R_i) \right) \geq \sigma(\mathcal{I}).$$

(Note that $\sigma(\mathcal{I})$ makes sense since $\mathcal{I}$ is admissible). Finally $\sigma(\mathcal{I}) \geq \sigma$, and the second inequality follows. □

Let us fix a "big" integer quantifier $N_* > 0$. We say that a level n is in the "tail of a cascade" if all levels $n-1, n-N_*$ belong to a cascade (note that level $n-1$ itself may be non-central). Cascades of length at least N_* we call "long".

Theorem II. *Given a generalized quadratic-like map g_1, we have the following bounds of the geometric parameters within its principal nest:*

- *The asymmetric moduli σ_n grow monotonically and hence stay away from 0 on all levels: $\sigma_n \geq \bar{\sigma} > 0$.*
- *The principal moduli μ_n stay away from 0 (that is, $\mu_n \geq \bar{\mu} > 0$) everywhere except for the case when $n-1$ is in the tail of a long cascade (the bound $\bar{\mu}$ depends on the choice of N_*).*

• *The off-critical puzzle pieces V_i^n are well inside V^{n-1} (that is,* $\text{mod}(V^{n-1} \setminus V_i^n \geq \bar{\mu} > 0$*) except for the case when V_i^n is precritical and $n-2$ is the last level of a long cascade.*

• *The distortion of h_n from (5.1) is uniformly bounded on all levels by a constant $\bar{K}$.*

All the bounds depend only on the first principal modulus μ_1 and (as $\bar{\mu}$ is concerned) on the choice of N_*.

Proof. The first assertion follows from the second inequality of Corollary 5.5. Together with Corollary 5.3 it implies the second one (note that the second inequality of this corollary implies that $\mu' \geq \sigma/2$ in the non-central case). One more application of Corollary 5.3 yields the next assertion.

Let us check the last statement. If $n-2$ is not in the tail of a central cascade, then $\mu_{n-1} \geq \bar{\mu}$ by the second statement, and the desired follows from Lemma 5.1.

Let $n-2$ be in the tail of a central cascade $V^m \supset \ldots \supset V^{n-2} \supset \ldots$. If this is not the last level of this cascade then $g_n|V^n = g_{m+2}|V^n$, so that h_n is just a restriction of the map h_{m+2} with bounded distortion.

Finally, if $n-2$ is the *last* level of a central cascade, then by Corollary 3.8 h_n can be extended to a univalent map with range V^m, and the Koebe Theorem implies the distortion bound. □

Theorems I and II imply:

Corollary 5.6. *Let f be a renormalizable quadratic-like map whose internal class $c(f)$ belongs to a truncated secondary limb L. Then*

$$\text{mod}(Rf) \geq \nu_L(\text{mod}(f)) > 0.$$

Remark. Though we believe that Theorem II is still true for higher degree complex unimodal polynomials $z \mapsto z^d + c$, $c \in \mathbb{C}$, the above argument does not work. However, it is worthwhile to notice that the following estimate is still valid: $\mu_{n(k)+2} \geq \phi(\mu_{n(k-1)+2})$, where $n(k)$ is the subsequence of non-central levels and $\phi > 0$ is a function depending only on d (which can be easily written down explicitly). In particular, in the renormalizable case,

$$\text{mod}(Rf) \geq \nu(\text{mod}(f), \chi(f)) > 0,$$

where the function ν depends on d and the choice of truncated secondary limbs.

6. Linear growth of the moduli

In this section we will prove the central result of the paper:

Theorem III. *Let $n(k)$ counts the non-central levels in the principal nest $\{V^n\}$. Then*

$$\text{mod}\, A^{n(k)+2} \geq Bk,$$

where the constant B depends only on the first modulus $\mu_1 = \text{mod}\, A^1$.

6.1. Proof of Theorem III. This proof will occupy the rest of this section. Our goal is to prove that $\sigma' \geq \sigma + a$ with a definite $a > 0$ (that is, dependent only on $\operatorname{mod} A_0$) at least on every other level, except for the tails of long cascades and a couple of the following levels. (Theorem II shows the reason why these tails play a special role: In the tails the principal moduli become tiny which slows down the growth rate of asymmetric moduli.)

Clearly it is enough to show that for any innermost island D'

$$\sigma(\mathcal{I}'|D') \geq \sigma + a \tag{6.1}$$

with a definite $a > 0$. The analysis will be split into a tree of cases.

6.2. Let D' contain at least three puzzle pieces.

Proposition 6.1. *If an innermost island D' contains at least three puzzle pieces V'_j, $j \in \mathcal{J}'$, then*

$$\sigma(\mathcal{J}'|D') \geq \sigma(I) + \frac{1}{2}\mu.$$

Proof. Let us split off $(1/2)\mu$ in (5.3) and estimate all other μ's by $\operatorname{mod}(R_0)$. This estimates the right-hand side by

$$\frac{1}{2}\mu + \frac{|\mathcal{J}| - 1}{2}\operatorname{mod}(R_0) + \frac{1}{2}\sum_{i \in \mathcal{I}, i \neq 0}\operatorname{mod}(R_i),$$

which immediately yields what is claimed. □

Hence under the circumstances of Proposition 6.1 we observe a definite growth of the asymmetric modulus provided level $n - 1$ is not in the tail of a long cascade. Indeed then by Theorem II μ is bounded away from 0, and (6.1) follows.

6.3. Let D' contain two puzzle pieces. The further analysis needs some preparation in the geometric function theory summarized in Appendix A.

Assume the island D' contains two puzzle pieces V'_j, $j \in \mathcal{J}'$. Let $\phi \equiv \phi_{D'}$ and let $\phi V'_j \subset V_i$ with $i = i(j)$. Fix a quantifier $L_* > 0$. When we say that something is "big", this means that it is at least $C(L_*)$ where $C(L_*) \to \infty$ as $L_* \to \infty$. Similarly "small" means an upper bound by $\epsilon(L_*) \to 0$ as $L_* \to \infty$. The sign $\approx$ will mean an equality up to a small (in the above sense) error, while the sign $\succ$ will mean the inequality up to a small error.

Case (i). *Assume that there is a non-critical puzzle piece $V_{i(j)}$ whose Poincaré distance in Δ from the critical point is less than L_*.* Then by Lemma A.1

$$\mu \geq \operatorname{mod}(R_0) + \alpha \tag{6.2}$$

with a definite $\alpha = \alpha(L_*) > 0$. But observe that when we passed from Lemma 5.4 to Corollary 5.5 we estimated μ by $\operatorname{mod}(R_0)$. Using the better estimate (6.2), we obtain a definite increase of σ.

Case (ii). *Assume now that the hyperbolic distance in Δ from any non-critical puzzle piece $V_{i(j)}$ to the critical point is at least L_*.* Assume also that levels $k \in$

$[n-3, n]$ do not belong to the tail of a long cascade (for the sake of linear growth it is enough to prove definite growth on such levels). Then V_0 may not belong to any non-trivial island together with some off-critical piece $V_{i(j)}$. Indeed, by Theorem II all puzzle pieces of level $n-1$ are well inside V^{n-2}. But then by Lemma 5.2 all non-trivial isles of level n are well inside of $V^{n-1} \equiv \Delta$. (The quantifier L_* should be chosen bigger than the a priori bound on the hyperbolic diameters of the isles).

Subcase (ii-a). *Assume that both $V_{i(j)}$ are non-critical.* Then by Corollary 5.5 $\sigma(\mathcal{J}'|D')$ is estimated by $\sigma_n(\mathcal{I})$ where the family $\mathcal{I}$ consists of three puzzle pieces: two pieces $V_{i(j)}$ and the central puzzle piece V_0.

If the puzzle pieces $V_{i(j)}$, $j \in \mathcal{J}'$, don't belong to the same non-trivial island, then by Proposition 6.1 $\sigma(\mathcal{I}) \geq \sigma_{n-1} + a$ with a definite $a > 0$, and we are done.

Otherwise the puzzle pieces $V_{i(j)}$ belong to an island W. Since by Lemma 5.2 W is well inside of Δ, it stays on the big Poincaré distance from the critical point (namely, on distance $L_* - O(1)$). Hence $\operatorname{mod}(R_0) \approx \mu$, and

$$\sigma(\mathcal{I}) \geq \sigma(\mathcal{I}|W) + \operatorname{mod}(R_0) \succ \sigma_{n-1} + \mu,$$

where $\mu \equiv \mu_n$ is bounded away from 0, since level $n-1$ is not in the tail of a long cascade. So we have gained some extra growth, and can pass to the next case.

Below we will restore labels n and $n+1$ since many levels will be involved in the consideration.

Subcase (ii-b). *Let one of the puzzle pieces $V^n_{i(j)}$ be critical.* So we have the family $\mathcal{I}^n$ of two puzzle pieces V^n_0 and V^n_1. Remember that we also assume that *the hyperbolic distance between these pieces is at least L_**. Hence, V^{n-1} is the only island containing both of them, so that $g_{n-1}V^n_0$ and $g_{n-1}V^n_1$ belong to different puzzle pieces of level $n-1$. For the same reason we can assume that one of these puzzle pieces is critical. Denote them by V^{n-1}_0 and V^{n-1}_1. Thus one of the following two possibilities on level $n-2$ can occur:

1) *Fibonacci return* when $g_{n-1}V^n_0 \subset V^{n-1}_1$ and $g_{n-1}V^n_1 = V^{n-1}_0$ (see Figure 7);

2) *Central return* when $g_{n-1}V^n_0 = V^{n-1}_0$ and $g_{n-1}V^n_1 \subset V^{n-1}_1$.

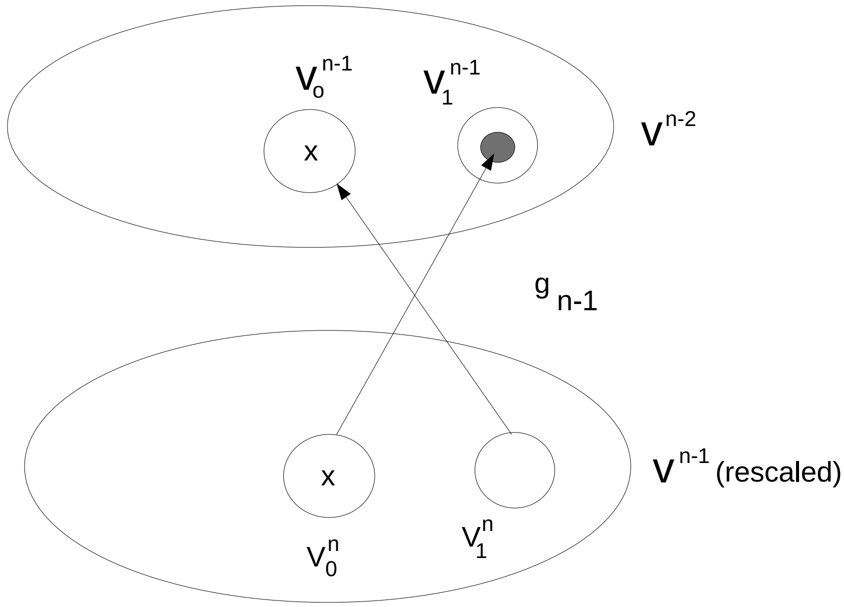

Figure 7. Fibonacci scheme.

We can assume that one of these schemes occur on several previous levels $n-3, n-4, \ldots$ as well (otherwise we gain an extra growth by the previous considerations). To fix the idea, let us first consider the following particular case, which plays the key role for the whole theorem.

6.4. Fibonacci cascades. Assume that on both levels $n-2$ and $n-3$ the Fibonacci returns occur. Let us look more carefully at the estimates of Lemma 5.4. In the Fibonacci case we just have:

$$\mathrm{mod}(R_1^n) \geq \mathrm{mod}(R_0^{n-1}), \tag{6.3}$$

$$\mathrm{mod}(R_0^n) \geq \frac{1}{2}\,\mathrm{mod}(Q_1^{n-1} \setminus g_{n-1}V_0^n), \tag{6.4}$$

where $Q_i^n = V_i^n \cup R_i^n$. Applying g_{n-2} we see that

$$\mathrm{mod}(Q_1^{n-1} \setminus g_{n-1}V_0^n) \geq \mathrm{mod}(Q_0^{n-2} \setminus V_0^{n-1}).$$

But since V_1^{n-2} is hyperbolically far away from the critical point (the assumption of Case (ii) is still effective),

$$\mathrm{mod}(Q_0^{n-2} \setminus V_0^{n-1}) \approx \mathrm{mod}(V_0^{n-3} \setminus V_0^{n-1}).$$

By the Grötzsch Inequality there is an $a \geq 0$ such that

$$\mathrm{mod}(V_0^{n-3} \setminus V_0^{n-1}) = \mu_{n-1} + \mu_{n-2} + a. \tag{6.5}$$

Clearly

$$\mu_{n-1} \geq \mathrm{mod}(R_0^{n-1}).$$

Furthermore, let $P_1^{n-1} \subset V^{n-2}$ be the pull-back of Q_0^{n-2} by g_{n-2}. Since ∂P_1^{n-1} is hyperbolically far away from V_1^{n-1}, we have:

(6.6)

$$\mu_{n-2} \geq \mathrm{mod}(R_0^{n-2}) = \mathrm{mod}(P_1^{n-1} \setminus V_1^{n-1}) \approx \mathrm{mod}(V^{n-2} \setminus V_1^{n-1}) \geq \mathrm{mod}(R_1^{n-1}).$$

Combining estimates (6.4) through (6.6) we get

$$\mathrm{mod}(R_0^n) \succ \frac{1}{2}(\mathrm{mod}(R_0^{n-1}) + \mathrm{mod}(R_1^{n-1}) + a). \tag{6.7}$$

We see from (6.3) and (6.7) that we need to check that the constant a in (6.5) is definitely positive. Assume that this is not the case, that is, for any $\delta > 0$ we can find a level n in the Fibonacci cascade as above such that $a < \delta$. Set $\Gamma_n = \partial V^n$. Then by the Definite Grötzsch Inequality (see Appendix A), the width(Γ_{n-2}) in the annulus $T = V^{n-3} \setminus V^{n-1}$ is at most $\xi(\delta)$ with $\xi(\delta) \to 0$ as $\delta \to 0$. Since Γ_{n-2} is well inside of T, we conclude by the Koebe Distortion Theorem that Γ_{n-2} is contained in a narrow neighborhood of a curve γ with a bounded geometry. Hence there is a $k = k(\delta) \to 0$ as $\delta \to 0$ and $\epsilon = \epsilon(\delta, k) > 0$ such that the curve Γ_{n-2} is not (k, ϵ)-pinched.

On the other hand, the hyperbolic distance from the puzzle piece V_1^{n-1} to the critical point 0 in V^{n-2} is at least L_*. Hence by Lemma A.4 it must be located in the Euclidean sense very close to Γ_{n-2} relative to the Euclidean distance to the critical point (that is, the relative distance is at most $\beta(\delta) \to 0$ as $\delta \to 0$). Hence the critical value $g_{n-1}0$ is also very close to Γ_{n-2} relative to the distance to the critical point, that is

$$\frac{\mathrm{dist}(g_n 0, \Gamma_{n-2})}{\mathrm{dist}(g_n 0, 0)} \leq \epsilon(L_*),$$

where $\epsilon(L_*) \to 0$ as $L_* \to \infty$.

By the last statement of Theorem II, g_{n-1} is a quadratic map up to a bounded distortion. Hence the curve Γ_{n-1} which is the pull-back of Γ_{n-2} by g_{n-1} must have a huge eccentricity around the critical point. But then by Lemma A.2 the width of Γ_{n-1} in $V^{n-2} \setminus V^n$ is also big, which by the above considerations gives a definite linear growth on the next level.

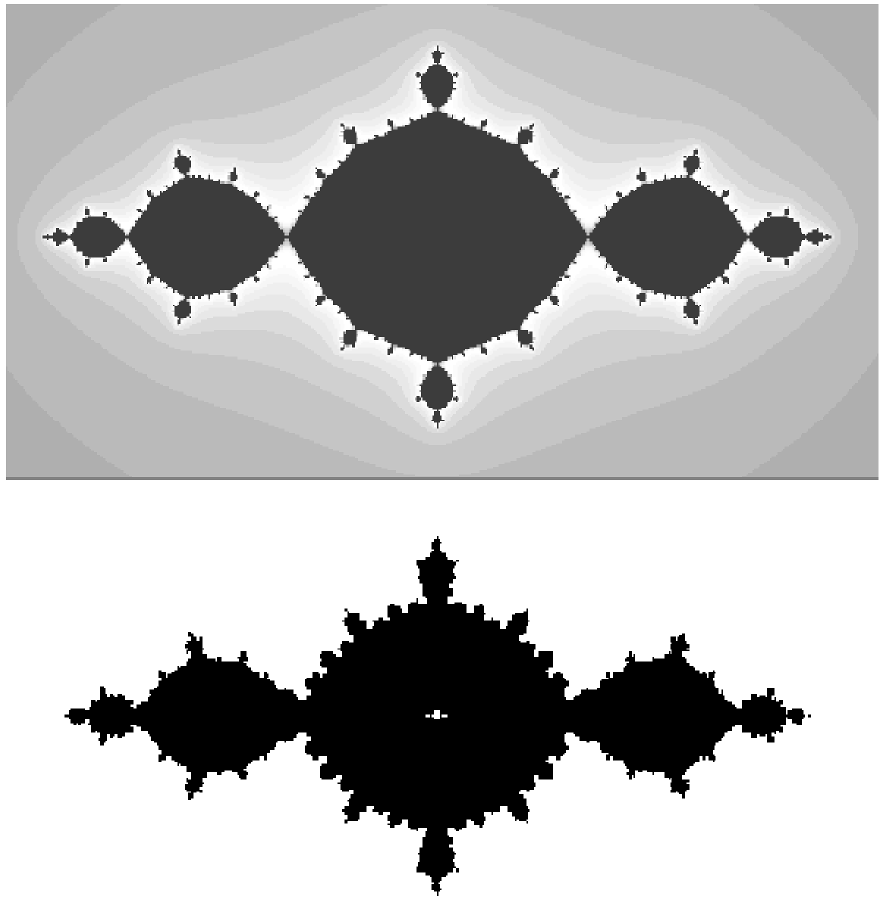

Figure 8. Fibonacci puzzle piece (below) vs the Julia set of $z \mapsto z^2 - 1$ (above).

Remark. The actual shape of a deep level puzzle piece for the Fibonacci cascade is shown on Figure 8. There is a good reason why it resembles the filled Julia set for $z \mapsto z^2 - 1$ (see [L5]). As the geodesic in V_0^{n-1} joining the puzzle pieces V_0^n and V_1^n goes through the pinched region, the Poincaré distance between these puzzle pieces is, in fact, big.

It is time now to look closer at central cascades.

6.5. Central cascades. Let $N \geq 2$, $n = m + N$, and let us consider a nest $\mathcal{C}^{m+N}$ of puzzle pieces

$$V^m \supset V^{m+1} \supset \ldots \supset V^{m+N-1} \supset V^{m+N} \supset D^{m+N} \tag{6.8}$$

satisfying the following properties (see Figure 9):

- The return on level $m-1$ is non-central: $g_m 0 \notin V_0^m$;
- Central returns occur on levels $m, m+1, \ldots m+N-2$, that is $g_{m+1} 0 \in V^{m+N-1}$;

• D^{m+N} is an island with a family $\mathcal{I}^{m+N+1}$ of two puzzle pieces inside, V_0^{m+N+1} and V_1^{m+N+1}. Let $\phi \equiv \phi_{m+N} \equiv \phi_{D^{m+N}}$ denote the corresponding double covering $D^{m+N} \to V^{m+N-1}$;

• One of the puzzle pieces $\phi_{m+N} V_0^{m+N+1}$, $\phi_{m+N} V_1^{m+N+1}$ is critical.

We would like to analyze when

$$\sigma(\mathcal{I}^{m+N+1} | D^{m+N}) \geq \sigma_{m+1} + a \tag{6.9}$$

with a definite $a > 0$. To this end we need to pass from level $m + N$ all way up to level m.

Let $V_*^{m+N+1} \subset D^{m+N}$ be a non-precritical piece of the family $\mathcal{I}^{m+N+1}$, and

$$\phi V_*^{m+N+1} \subset V_1^{m+N} \subset W_i^{m+N}$$

for some $i \neq 0$. Then the return map $g_{m+N+1} : V_*^{m+N+1} \to V^{m+N}$ can be decomposed as $G^l \circ \phi$ for an appropriate $l \geq 1$, where $G : \cup W_i^k \to V^m$ is the Bernoulli map (3.12) associated with the central cascade. Since G has range V^m,

$$\mathrm{mod}(W_i^{m+N} \setminus \phi V_*^{m+N+1}) \geq \mathrm{mod}(V^m \setminus V^{m+N}). \tag{6.10}$$

Let $\Gamma^k = \partial V^k$, and

$$w_k = \mathrm{width}(\Gamma^k | V^{k-1} \setminus V^{k+1}).$$

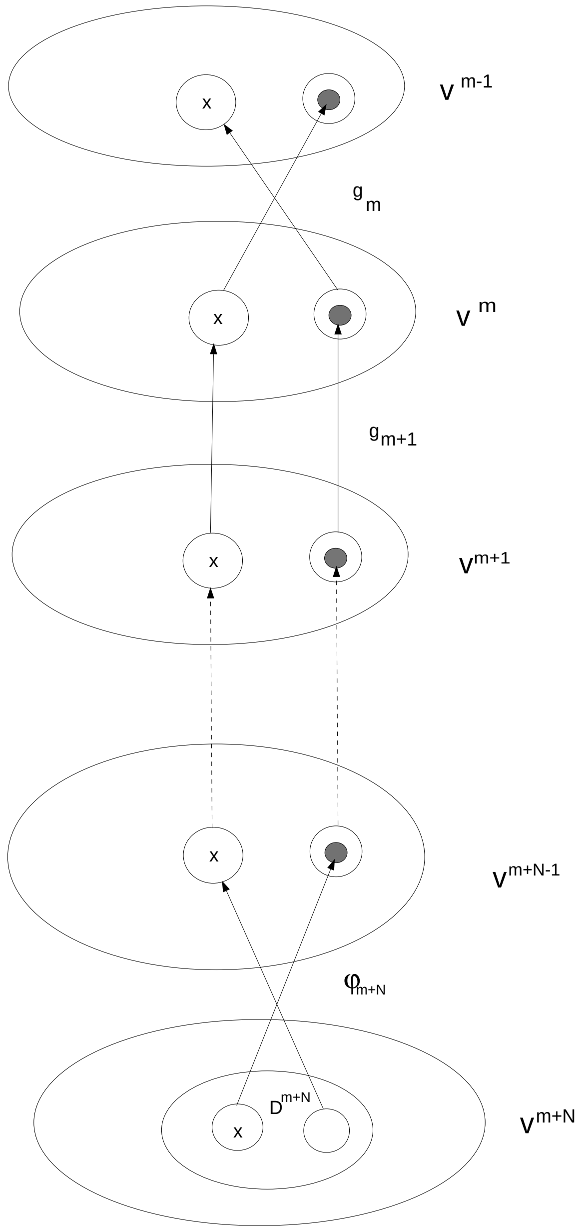

Figure 9. Central cascade
(with Fibonacci returns on the top and the bottom).

For $k \in [m+1, m+N]$ let V_1^k denote the puzzle piece of level k containing $g_{m+1}^{m+N-k} \phi V_*^{m+N+1}$, and $\mathcal{I}^k$ denote the family of two puzzle pieces: V_0^k and V_1^k. Moreover, let $R_i^k \subset V^{k-1}$ denote an annulus of maximal modulus going around V_i^k but not going around the other piece of family $\mathcal{I}^k$, $i = 0, 1$.

By the Definite Grötzsch Inequality and the second part of Theorem II, there is an $a = a(w_{m+1})$ such that

$$(6.11) \qquad \mathrm{mod}(V^m \setminus V^{m+N}) \geq \sum_{k=m+1}^{m+N} \mathrm{mod}\, A^k + a =$$

$$\sum_{k=0}^{N-1} \frac{1}{2^k} \mathrm{mod}(A^{m+1}) + a \geq (2 - \frac{1}{2^{N-1}}) \mathrm{mod}\, R_0^{m+1} + a.$$

Let S_0^{m+N} and S_1^{m+N} denote the pull-backs of the annuli R_0^{m+1} and R_1^{m+1} by the map $g_{m+1}^{N-1} : V^{m+N-1} \to V^m$. Then

$$(6.12) \qquad \mathrm{mod}\, S_0^{m+N} \geq \frac{1}{2^{N-1}} \mathrm{mod}\, R_0^{m+1} \quad \text{and} \quad \mathrm{mod}\, S_1^{m+N} \geq \mathrm{mod}\, R_1^{m+1}.$$

Note that the inner boundary of S_1^{m+N} coincides with the outer boundary of $W_i^{m+N} \setminus \phi V_*^{m+N+1}$. Let Q_1^{m+N} denote the union of these two annuli. This annulus goes around ϕV_*^{m+N+1} but not around V_0^{m+N}. Now estimates (6.10), (6.11), (6.12) yield

$$\mathrm{mod}\, S_0^{m+N} + \mathrm{mod}\, Q_1^{m+N} \geq 2\, \mathrm{mod}\, R_0^{m+1} + \mathrm{mod}\, R_1^{m+1} + a \geq 2\sigma(\mathcal{I}^{m+1}) + a.$$

Finally, pulling S_0^{m+N} and Q_1^{m+N} back by $\phi = \phi_{m+N}$ to the island D^{m+N} we obtain:

$$\sigma(\mathcal{I}^{m+N+1} | D^{m+N}) \geq \frac{1}{2}(\mathrm{mod}\, S_0^{m+N} + \mathrm{mod}\, Q_1^{m+N}) \geq \sigma(\mathcal{I}^{m+1}) + a/2.$$

So we come up with the following statement:

Statement 6.2. *There is an increasing function $a : \mathbb{R}_+ \to \mathbb{R}_+$, $a(0) = 0$, such that for the cascade $\mathcal{C}^{m+N}$ estimate (6.9) holds with $a = a(w_{m+1})$, where $w_{m+1} =$* $\mathrm{width}(\Gamma^{m+1} | V^m \setminus V^{m+2})$.

Let us fix a quantifier w_* which distinguishes "small width" w from a "definite" one. For further analysis let us go several levels up. Let $m - 1 - l$ be the highest non-central level preceding $m - 1$, $l \geq 1$. We are going to study when

$$(6.13) \qquad \sigma(\mathcal{I}^{m+N+1} | D^{m+N}) \geq \sigma_{m-l} + a$$

with a definite $a > 0$. We cannot now assume that l is bounded, so we face a possibility of a long cascade $\mathcal{C}^{m-1}$: $V^{m-l} \supset \ldots \supset V^{m-1}$. Set $g = g_{m-l+1}$; then $g0 \in V^{m-1}$.

♣ *Assume first that $m - 2$ is not the last level of a long cascade* (in particular, this is the case when central return occurs on level $m - 2$, that is, $l \geq 2$). Then by the third part of Theorem II all non-central pieces of level m are well inside V^{m-1}:

$$\mathrm{mod}(V^{m-1} \setminus V_j^m) \geq \bar{\mu},\ j \neq 0$$

. Hence $g_m V_0^{m+1}$ and $g_m V_1^{m+1}$ belong to different pieces of level m. Indeed, otherwise the hyperbolic distance between V_0^{m+1} and V_1^{m+1} in V^m would be bounded by a constant $L(\bar\mu)$. But according to our assumption this distance is at least L_*. So this situation is impossible if L_* was a priori selected bigger than $L(\bar\mu)$.

For the same reason the pieces $g^k\circ g_m V_i^{m+1}$, $i=0,1$, also belong to different pieces V_j^{m-k} for $0\le k\le l-3$. Indeed, assume they belong to the same piece V_j^{m-k}. Clearly this piece is non-central, that is $j\neq 0$. Then it is contained in a piece W_j^{m-k} of the Bernoulli family $\mathcal{W}(\mathcal{C}^{m-1})$ associated to the central cascade $\mathcal{C}^{m-1}$. Hence $g_m V_0^{m+1}$ and $g_m V_1^{m+1}$ belong to W_j^m, the pull-back of W_j^{m-k} by g^k. As $\mathrm{mod}(W_j^m\smallsetminus g_m V_i^{m+1})\ge\bar\mu$, the hyperbolic distance between V_0^{m+1} and V_1^{m+1} in V^m is at most $L(\bar\mu)$ contradicting our assumptions.

Let us show now that (6.13) holds if both $g_m V_i^{m+1}$ are non-central. Indeed let us then consider the family $\mathcal{I}^m$ of three pieces: two pieces of level m containing $g_m V_i^{m+1}$ and the central piece V^m. Let $\mathcal{I}^{m-k}$ denote the family of puzzle pieces of level $m-k$ containing the pieces of $g^k\mathcal{I}^m$. By the previous two paragraphs, $\mathcal{I}^{m-k}$ consists of three puzzle pieces. Then by Corollary 5.5 and Proposition 6.1,

$$\sigma(\mathcal{I}^{m+1})\ge\sigma(\mathcal{I}^m)\ge\ldots\ge\sigma(\mathcal{I}^{m-l+2})\ge\sigma(\mathcal{I}^{m-l+1})+\frac{1}{2}\bar\mu,$$

and we are done.

Thus let us assume that the Fibonacci return occurs on level $m-1$. In this case let $\mathcal{I}^{m-k}$ denote the family of two puzzle pieces V_0^{m-k} and V_1^{m-k} containing $g^k\circ g_m V_i^{m+1}$, $i=1,2$, $k\le l-1$.

Note that in order to have (6.13) it is enough to have a definite increase of the $\sigma(\mathcal{I}^{m-k})$ in the beginning of the cascade $\mathcal{C}^{m-1}$. By Statement 6.2 applied to this cascade this is the case if $\mathrm{width}(\Gamma^{m-l+1}|V^{m-l}\smallsetminus V^{m-l+2})\ge w_*$. So assume that the opposite inequality holds. Similarly, we can assume that the hyperbolic distance from V_1^{m-l+2} to 0 in V^{m-l+1} is at least L_* (for otherwise we are fine: see Case (i) above).

It follows from Lemma A.4 from Appendix A that the piece V_1^{m-l+2} stays Euclidean distance at most $\epsilon\,\mathrm{diam}\,\Gamma^{m-l+1}$ from Γ^{m-l+1} where $\epsilon=\epsilon_{\bar\mu}(w_*,L_*)\to 0$ as $w_*\to 0$, $L_*\to\infty$ (for a fixed $\bar\mu>0$). Hence the Euclidean distance from V_1^{m-l+2} to Γ^{m-l+1} is relatively small as compared with its distance to Γ^{m-l} and Γ^{m-l+2}. More precisely, there is a $\delta=\delta_{\bar\mu}(w_*,L_*)$ with the same properties as ϵ above such that for any $z\in V_1^{m-l+2}$,

$$\mathrm{dist}(z,\Gamma^{m-l+1})\le\delta\ \mathrm{dist}(z,\partial(V^{m-l}\smallsetminus V^{m-l+2})\qquad(6.14)$$

Take $z_0\in V_1^{m-l}$, and let $r=\mathrm{dist}(z_0,\partial(V^{m-l}\smallsetminus V^{m-l+2})$. Note that the disk $B(z_0,r)$ can be univalently pulled by g^{l-3} to the annulus $V^{m-3}\smallsetminus V^{m-1}$. By the Koebe Distortion Theorem and (6.14), for any $\zeta\in V_1^{m-3}$

$$\mathrm{dist}(\zeta,\Gamma^{m-2})\le C\delta\,\mathrm{dist}(\zeta,\partial(V^{m-3}\smallsetminus V^{m-1}))\le C\delta\ \mathrm{dist}(\zeta,0)$$

with an absolute constant C. All the more,

$$\mathrm{dist}(\zeta,\Gamma^{m-2})\le C\delta\ \mathrm{diam}\,V^{m-2},$$

so that Γ^{m-2} has a big eccentricity about V_1^{m-2} (that is, this eccentricity is at least $e(w_*, L_*)$, where $e(w_*, L_*) \to \infty$ as $w_* \to 0$, $L_* \to \infty$).

Pulling Γ^{m-2} back by $g_{m+1} \circ g_m \circ g_{m-1}$, we conclude that Γ^{m+1} has a big eccentricity about 0. Hence it has big width in the annulus $V^m \setminus V^{m+2}$, and Statement 6.2 yields the desired.

Let us summarize the information which will be useful in what follows:

Statement 6.3. *If the width w_{m-l+1} is at most w_* and the Poincaré distance from V_1^{m-l+2} to 0 in V^{m-l+1} is at least L_*, then the eccentricity Γ^m about the origin is at least $e(w_*, L_*)$, where $e(w_*, L_*) \to \infty$ as $w_* \to 0$ and $L_* \to \infty$.*

♣ *Let us assume now that $m-2$ is the last level of a long cascade $\mathcal{C}^{m-2}$:*

$$V^{m-2-t} \supset \dots V^{m-2}, \quad t \geq N_*.$$

Then non-central return occurs on level $m-2$. We will show that

$$\sigma(\mathcal{I}^{m+N+1} | D^{m+N}) \geq \sigma_{m-2-t} + a \tag{6.15}$$

with a definite $a > 0$.

Let $D^m \subset V^m$ be an island containing V_0^{m+1} and V_1^{m+1}, and $\phi_m : D^m \to V^{m-1}$ be the corresponding two-to-one map. Note that in the case under consideration this island may be non-trivial and still the Poincaré distance between V_0^{m+1} and V_1^{m+1} be big (since the precritical puzzle pieces in V^{m-1} are not well inside V^{m-1}). Moreover the map ϕ_m is not necessarily bounded perturbation of the quadratic map. These are the circumstances which make this case special.

As $m-2$ is a non-central level, $\mu_{m+1} \leq \bar{\mu}$, and by the previous considerations we are done unless

• The return on level $m-1$ is Fibonacci, that is $\phi_m V_1^{m+1} = V_0^m$ and $\phi_m V_0^{m+1} \subset V_1^m$ for some puzzle piece V_1^m;

• The hyperbolic distance between the puzzle pieces V_0^m and V_1^m is at least L_*.

• The return on level $m-2$ is also Fibonacci: $g_{m-1} V_1^m = V_0^{m-1}$ and $g_{m-1} V_0^m = V_1^{m-1}$ for some puzzle piece V_1^{m-1}.

Let V_0^k and V_1^k be the pieces containing the corresponding push forwards of V_0^{m-1} and V_1^{m-1} along the cascade $\mathcal{C}^{m-2}$, $m-1 \leq k \leq m-t-1$. Then (6.15) follows unless:

• The width w_{m-t-3} is at most w_*, and the distance between V_0^{m-t-4} and V_1^{m-t-4} in V^{m-t-3} is at least L_*.

But then by Statement 6.3 applied to the cascade $\mathcal{C}^{m-2}$ the eccentricity of Γ^{m-1} about 0 is at least $e = e(w_*, L_*)$. As g_m is a bounded perturbation of the quadratic map, by Lemma A.5 the curve Γ^m is $(0.1, \epsilon)$-pinched, where $\epsilon = \epsilon_{\bar{\mu}}(e) \to 0$ as $e \to \infty$. (Note that the pinched region is not necessarily around V_1^m, since ϕ_m may differ from g_m). Applying Lemma A.5 again, we conclude that the curve Γ^{m+1} is $((10C)^{-1}, C\sqrt{\epsilon})$-pinched. By Lemma A.3 Γ^{m+1} has a definite width inside $V^m \setminus V^{m+2}$. Now Statement 6.2 yields (6.15). Theorem III is proven.

7. Big type yields big space

Below we will analize a variety of combinatorial factors (like height, return time, length of a central cascade) which yield a big modulus of the renormalized map. Altogether they are quite close to a "big renormalization period", except that "parabolic or Siegel cascades" may interfere. This is summarized in Theorem IV′ stated at the end of the section which loosely says that if the periods of $R^n f$ are sufficiently big and there are no "parabolic" or "Siegel cascades" in the principal nests then there are a priori bounds.

7.1. Big height yields big modulus. Let us start with a quick consequence of Theorem III. We refer to §3.10 for the terminology used below:

Theorem IV. *For any Q, there is a χ-special family $\mathcal{S}$ of the Mandelbrot copies with the following property. Let f be an infinitely renormalizable quadratic of $\mathcal{S}$-type. Then* $\operatorname{mod}(R^m f) \geq Q$, $m = 0, 1, \dots$.

Proof. Let us fix a $Q > 0$. Take a truncated secondary limb $L \equiv L_b^{tr}$, and find $q = C(Q)\,\nu(L)$ from Theorem I. Note that $q \geq \nu(L)/2$ for sufficiently big Q (independently of L). Let us now select all copies M' of the Mandelbrot set with the height $\chi(M') \geq Q/B$, where $B = B(q)$ is the constant from Theorem III. Taking the union of all these copies over all truncated limbs, we obtain a special family $\mathcal{S}$.

Let us now consider an infinitely renormalizable quadratic-like map f of $\mathcal{S}$-type with $\operatorname{mod}(f) \geq Q$ (to start, take a quadratic polynomial). Then by Theorem I, $\operatorname{mod}(A^1) \geq q$. Hence by Theorem III, $\operatorname{mod}(Rf) \geq B\,\chi(f) \geq Q$.

By induction, $\operatorname{mod}(R^n f) \geq Q$ for all n. □

7.2. Big return time implies big modulus. The simplest possible way to create a big modulus is the following. By Lemma 2.8, if the return time l of the critical point to a puzzle piece $Y^{(k)}$ of a given depth k is big then the pull-back $Y^{(k+l)}$ of $Y^{(k)}$ along the $\operatorname{orb}_l(0)$ has a small diameter (uniformly over a truncated primary limb), so that $\operatorname{mod}(Y^{(k)} \setminus Y^{(k+l)})$ is big. We can now start a principal nest from $Y^{(k)}$. By Theorem II (§5) we will observe big moduli on all levels down except those in the tails of cascades. In particular, $\operatorname{mod}(Rf)$ is also big if f is renormalizable.

Below we describe more involved situations creating a big modulus. We will rely on the combinatorial considerations of §3.8. Let $\mathcal{I}^n \subset \mathcal{V}^n$ stand for the family of puzzle pieces V_i^n intersecting $\omega(0)$. Consider an edge γ^{n+1} of the return graph with vertices at $V_j^{n+1} \in \mathcal{I}^{n+1}$ and $V_i^n \in \mathcal{I}^n$, and let t be the corresponding landing time, so that $g_n^t V_j^{n+1} \subset V_i^n$. Then we will use the notation $\operatorname{mod}(\gamma)$ for $\operatorname{mod}(V_i^n \setminus g_n^t V_j^{n+1})$. If $i \neq 0$ then $\operatorname{mod}(\gamma^n) \geq \mu_n$.

Lemma 7.1. *Let $D^n \subset V^{n-1}$ be a puzzle piece containing at least one piece of $\mathcal{I}^n$. Let Γ be a path leading from D^n down to some critical piece V^{n+t}, and let D^{n+t} be the pull-back of D^n along this path. Then*

$$\operatorname{mod}(D^{n+t} \setminus V^{n+t}) \geq \frac{\bar{\mu}}{2}\operatorname{rank}(D^n).$$

Proof. Let $\{\gamma^{n+1}, \dots, \gamma^{n+t}\}$ be the edges of Γ. By Lemma 3.10, all these edges except the last one represent univalent maps, and the last one represents a double covering. Hence

$$\operatorname{mod}(D^{n+t} \setminus V^{n+t}) \geq \frac{1}{2} \sum \operatorname{mod}(\gamma^{n+k}) \geq \frac{1}{2} \sum_{k=1}^{t} \mu_{n+k},$$

and Theorem II (together with the definition of the rank) completes the proof. □

Lemma 7.2. *Assume that $n-2$ is not in the tail of a long central cascade. Assume that for a puzzle piece $V_j^{n+1} \in \mathcal{I}^{n+1}$, $r \equiv \tau(V_j^{n+1}) \geq R$. Then there is a level m such that $\mu_m \geq L(R)$, where $L(R) \to \infty$ as $R \to \infty$.*

Proof. Let $M > 0$. We need to find a level m with $\mu_m \geq M$, provided r is sufficiently big. If $\operatorname{rank}(V_j^{n+1}) > N \equiv 2M/\bar{\mu}$, then Lemma 7.1 yields the desired. So let us assume that

$$\operatorname{rank}(V_j^{n+1}) \leq N$$

Let $0 = i(0), i(1), \dots, i(r) = 0$ be the itinerary of V_j^{n+1} through the pieces of the previous level. Let us consider a nest of puzzle pieces

$$V^n \equiv U_r \supset U_{r-1} \supset \dots \supset U_0 \equiv V_j^{n+1}, \tag{7.1}$$

where $g_n^{r-k} U_k = V_{i(r-k)}^n$. Then

$$U_{k+1} \setminus U_k \geq \bar{\mu}/2, \quad k = 0, \dots r-1. \tag{7.2}$$

Let us pull the pieces U_k, $k < r$, down along a path Γ joining V_j^{n+1} with a critical vertex V^{n+t}. Denote the corresponding pull-backs by U_k^{n+s}. If these pull-backs turn out to be double branched then by (7.2)

$$\mu_{n+t} \geq \frac{1}{2} \operatorname{mod}(U_{r-1} \setminus U_0) \geq \frac{1}{4}(r-1)\bar{\mu},$$

which is greater than M for sufficiently big r. Otherwise let us consider the first level $n+s$ where U_{r-1}^{n+s} hits the critical point. Let us find such an l that $0 \in U_l^{n+s} \setminus U_{l-1}^{n+s}$.

If $r - 1 - l > N$ then by (7.2) $\mu_{n+s} \geq M$, and we are done. Otherwise

$$\operatorname{mod}(U_{l-1}^{n+s} \setminus U_0^{n+s}) \geq (r - N - 2)\bar{\mu}/2.$$

Then let us repeat the same procedure with U_{l-1}^{n+s} instead of U_{r-1}. Note that $\operatorname{rank}(U_{l-1}^{n+s}) < \operatorname{rank}(U_{r-1})$, since the pull-back of U_{r-1} through the top central cascade is univalent. Hence this procedure can be repeated at most N times, and the principal modulus at the end will be at least M, provided $(r - N(N+2))\bar{\mu}/2 > M$. □

7.3. Parabolic and Siegel cascades. We will show that we usually observe a big principal modulus after just one long central cascade. Let us consider a central cascade (3.11): The double covering $g_{m+1} : V^{m+1} \to V^m$ can be viewed as a small perturbation of a quadratic-like map g_* with a definite modulus and with non-escaping critical point.

To make this precise, let us consider the space $\mathcal{Q}$ of quadratic-like maps modulo affine conjugacy supplied with the *Carathéodory topology* (see [McM2]). Convergence in this topology means Carathéodory convergence of the domains and uniform convergence of the maps on compact subsets. Given a $\mu > 0$, let $\mathcal{Q}(\mu)$ denote the set of quadratic-like maps $g \in \mathcal{Q}$ with $\operatorname{mod}(g) \geq \mu$. By Theorem II, the return maps $g_{m+1} : V^{m+1} \to V^m$ of the principal nest belong to $\mathcal{Q}(\bar{\mu})$.

Compactness Lemma (see [McM2]**).** *The set $\mathcal{Q}(\mu)$ is Carathéodory compact.*

Let $\mathcal{Q}_N$ (resp. $\mathcal{Q}_N(\mu)$) denote the space of quadratic-like maps $g : U' \to U$ from $\mathcal{Q}$ (resp. $\mathcal{Q}(\mu)$) such that $g^n 0 \in U$, $n = 0, 1, \dots N$.

As $\bigcap_N \mathcal{Q}_N(\mu) = \mathcal{Q}_\infty(\mu)$, for any neighborhood $\mathcal{U} \supset \mathcal{Q}_\infty(\mu)$, there is an N such that $\mathcal{Q}_N(\mu) \subset \mathcal{U}$. In this sense any double map $g \in \mathcal{Q}_N(\mu)$ is close to some quadratic-like map g_* with connected Julia set. In particular, this concerns the return map g_{m+1} generating a cascade (3.11) of big length N. Moreover, since g_{m+1} has an escaping fixed point, the neighborhood of g_* containing g_{m+1} also contains a quadratic-like map with hybrid class $c(g_*) \in \partial M$.

If we have a sequence of maps $f_n \in \mathcal{Q}_N$ converging to a map $g_* \in \mathcal{Q}_\infty$, we also say that the *$f_n$-central cascades converge to g_*.*

Let us say that the principal nest is minor modified if a piece V^m is replaced by a piece $\tilde{V}^n \subset V^n$ such that $\operatorname{cl} V_i^{n+1} \subset \tilde{V}^n$ for all pieces $V_i^{n+1} \in \mathcal{I}^{n+1}$.

Lemma 7.3. *Let g_* be a quadratic-like map with $c(g_*) \in \partial M$ which does not have neither parabolic points, nor Siegel disks. Let g_{m+1} be the return map of the principal nest generating cascade (*3.11*). Take an arbitrary big $M > 0$. If g_{m+1} is sufficiently close to g_* (depending on a priori bound $\bar{\mu}$ from Theorem II) then the principal nest can be minor modified in such a way that $\tilde{\mu}_n \geq M$ for some $n > m + N$.*

Proof. Take a big number $e > 0$.

By the above assumptions, the Julia set $J(g_*)$ has empty interior. If g_{m+1} is sufficiently close to g_* then $\Gamma^{m+N-1} = \partial V^{m+N-1}$ is close in the Hausdorff metric to the Julia set $J(g_*)$. Hence Γ^{m+N-1} has an eccentricity at least e with respect to any point $z \in V^{m+N-1}$.

As the g_m are purely quadratic up to bounded distortion (Theorem II), the curves Γ_{m+N}, Γ_{m+N+1} and Γ_{m+N+2} also have big eccentricity with respect to any enclosed point. Moreover, by the same theorem, there is a definite space in between these two curves. Hence by Lemma A.2, $\operatorname{mod}(V^{m+N+1} \setminus V^{m+N+3})$ is at least $M(e)$ where $M(e) \to \infty$ as $e \to \infty$.

Let us assume that the non-central return occurs on level $m + N + 1$: $g_{m+N+2} 0 \in V_i^{m+N+2}$ with $i \neq 0$. As the map $g_{m+N+2} : V_i^{m+N+2} \to V^{m+N+1}$ is quadratic up to bounded distortion, the curve $\Gamma_i^{m+N+2} = \partial V_i^{m+N+2}$ has a big eccentricity e' about any enclosed point (that is, e' can be made arbitrary big by a sufficiently big choice of e, depending on a priori bound $\bar{\mu}$). By Lemma A.2

$$\operatorname{mod}(V^{m+N+1} \setminus g_{m+N+2}V^{m+N+3}) \geq M(e),$$

where $M(e) \to \infty$ as $e \to \infty$. Hence $\operatorname{mod} A^{m+N+3} \geq M(e)/2$, and we are done.

Let the central return occur on level $m+N+1$ but this is not yet a DH-renormalizable level. Then the corresponding central cascade is finite. Let $m+N+T$ be the last level of this cascade. Then by Lemma A.2 and Statement 6.2, $\mu_{m+N+T+2} \geq M(e)$, where $M(e) \to \infty$ as $e \to \infty$.

Assume finally that $m+N+1$ is a DH-renormalizable level. Then let us take a horizontal curve $\Gamma \subset A^{m+N+2}$ which divides this annulus into two subannuli of moduli at least $\bar{\mu}/2$. Let $\Gamma' \subset A^{m+N+3}$ be its pull-back by g_{m+N+2}, and $\tilde{A}$ be the annulus bounded by Γ and Γ'. Then by Lemma A.2 $\operatorname{mod}(\tilde{A}) \geq M(e)$ with $M(e)$ as above. As this is a minor modification of the nest, we are done. □

7.4. Variation. Let us now improve Theorem IV by taking into account not only the height but also the other factors yielding big space.

Theorem IV′. *Let $f \in \mathcal{SL}$ be an infinitely renormalizable quadratic polynomial, and let $P_m : z \mapsto z^2 + c_m$ be the straightened $R^m f$. Assume that*

- *The set $\mathcal{A} \subset \mathcal{Q}$ of accumulation points of the central cascades of P_m (of lengths growing to ∞) does not contain parabolic or Siegel maps;*
- $\operatorname{per}(R^m f) \geq p$.

Then $\liminf_{n\to\infty} \operatorname{mod}(R^n f) \geq Q(p)$, where the function $Q(p)$ depends on the choice of the limbs and the accumulation set $\mathcal{A}$, and $Q(p) \to \infty$ as $p \to \infty$.

Proof. By Theorem II the top modulus of the central cascades of P_m is bounded from below by some $\bar{\mu}$. Hence the set $\mathcal{A} \subset \mathcal{Q}(\bar{\mu})$ is compact. By Lemma 7.3, for any Q there is a neighborhood $\mathcal{U} \supset \mathcal{A}$ such that: If $f \in \mathcal{U}$ is renormalizable then $\operatorname{mod} Rf > Q$.

As $\mathcal{A}$ is the accumulation set for the central cascades of the P_m, there is an N such that all but finitely many of these cascades of length $\geq N$ belong to $\mathcal{U}$. Hence if the principal nest of P_m contains a cascade of length $\geq N$ then $\operatorname{mod}(R(P_m)) \geq Q$ (for sufficiently big m).

Further, by Theorems I and III, there is a χ such that if the height $\chi(P_m) \geq \chi$ then $\operatorname{mod}(R(P_m)) \geq Q$. Let us also find a T such that if for some cascade the return time from Lemma 7.2 is at least T, then $\operatorname{mod}(R(P_m)) \geq Q$.

It is easy to see that there is a p such that: If $\operatorname{per}(P_m) \geq p$ then either P_m has a central cascade of length at least N, or $\chi(P_m) \geq \chi$, or one of the above return times is at least T. In any case $\operatorname{mod}(R(P_m)) \geq Q$.

Now the same argument as for Theorem IV yields a priori bounds. □

8. Geometry of quasi-quadratic maps

In this section we discuss real unimodal maps of Epstein class. We introduce a notion of essential period, and prove that the $\operatorname{mod} Rf$ is big if and only if the corresponding essential period is big. This discussion naturally continues [L4].

We assume that the reader is familiar with some basics of one-dimensional dynamics including the real Koebe Principle (see the book of de Melo & van Strien [MS] for the reference).

8.1. Essential period. Below we will adjust the combinatorial discussion of §3 to the reall line (see [L4] for details). Let $I' \subset I$ be two nested intervals. A map $f : (I', \partial I') \to (I, \partial I)$ is called *quasi-quadratic* if it is S-unimodal and has quadratic-like critical point $0 \in \operatorname{int} I'$.

Let us also consider a more general class $\mathcal{A}$ of maps $g : \cup J_i \to J$ defined on a finite union of disjoint intervals J_i strictly contained in an interval J. Moreover, $g|J_i$ is a diffeomorphism onto J for $i \neq 0$, while $g|J_0$ is unimodal with $g(\partial J_0) \subset \partial J$. We also assume that the critical point $0 \in J_0$ is quadratic-like, and that $Sg < 0$. Maps of class $\mathcal{A}$ are real counterparts of generalized quadratic-like maps of finite type. To simplify the exposition, let us also assume $g|J_0$ is symmetric, i.e., $g(x) = g(-x)$. Then $g|J_0 = h \circ \Phi$, where $\Phi(x) = x^2$ and h is a diffeomorphism of an appropriate interval $K \supset \phi(J_0)$ onto J. By definition, this map belongs to *Epstein class* $\mathcal{E}$ (see [E, S2, L4]) if the inverse branches $f^{-1} : J \to J_i$ for $i \neq 0$ and $h^{-1} : J \to K$ admit the analytic extension to the slit complex plane $\mathbb{C} \setminus (\mathbb{R} \setminus J)$ (such functions are called Herglotz).

Let $I^0 = [\alpha, \alpha']$ be the interval between the dividing fixed point α and the symmetric one. Let $\mathcal{Y} \equiv \mathcal{Y}_f$ denote the full Markov family of pull-backs of the interval I^0. Given a critical interval $J \in \mathcal{M}$ (that is, $J \ni 0$), we can define a (generalized) renormalization $T_J f$ on J as the first return map to J restricted to the components of its domain meeting the post-critical $\omega(0)$. If f admits a unimodal renormalization $Rf \equiv T_J f$ for some J, then there are only finitely many such components, so that we have a map of class $\mathcal{A}$. Moreover, if f is a map of Epstein class or a quadratic-like map, the renormalizations $T_J f$ inherit the corresponding property.

Let $I^0 \supset I^1 \supset \ldots \supset I^{t+1}$ be the real principal nest of intervals until the next quadratic-like level (that is, I^{n+1} is the pull-back of I^n corresponding to the first return of the critical point). Let us use the same notation $g_n : \cup I^n_j \to I^{n-1}$ for the real generalized renormalizations as we used for the complex ones.

For $\sigma \in (0,1)$, let $\mathcal{E}_\sigma$ stand for the space of quasi-quadratic maps $f : I \to I$ of Epstein class with $|I| \leq \sigma |I|$. In this section we will assume that $f \in \mathcal{E}_\sigma$. All the bounds below depend on σ but become absolute after skipping first $k(\sigma)$ central cascades.

Theorem 8.1 (Martens [Ma]**).** *The following real bounds hold:* • *I^{m+1} is well inside I^m unless I^m is in the tail of a long central cascade;*
• *The return maps $g_m : I^m \to I^{m-1}$ can be decomposed as $h_m \circ \Phi$ where $h_m : L_m \to I^{m-1}$ is a diffeomorphism of an interval L_m onto I^{m-1} with bounded distortion.*

(See also Guckenheimer & Johnson [GJ] for related earlier results on bounds and distortion.)

Let us look closer at real cascades of central returns. The return to level $n-1$ is called *high* or *low* if $g_n I^n \supset I^n$ or $g_n I^n \cap I^n = \emptyset$ correspondingly. Let us classify a central cascade $\mathcal{C} \equiv \mathbb{C}^{m+N}$

$$(8.1) \qquad I^m \supset \ldots \supset I^{m+N}, \;\; g_{m+1}0 \in I^{m+N-1} \setminus I^{m+N},$$

as *Ulam-Neumann* or *saddle-node* according as the return to the level $m+N-1$ is high or low. In the former case the map $g_{m+1} : I^{m+1} \to I^m$ is combinatorially close to the Ulam-Neumann map $z \mapsto z^2 - 2$, while in the latter it is close to the saddle-node map $z \mapsto z^2 + 1/4$. There is a fundamental difference between these two types of cascades.

Remark. Unlike the complex situation, on the real line we observe only two types of the cascades. The reason is that there are only two boundary points in the "real Mandelbrot set" [-2, 1/4] (compare §7.3).

Consider the return graph Υ (see §3.8). Let $\Upsilon(I^n)$ stand for the part of this graph growing up from the vertex I^n (i.e., restrict Υ to the set of vertices I^k_j, $k \leq n$, which can be joined with I^n).

Let us consider the orbit $J_k \equiv f^k I^n$, $k = 0, \ldots, l(n)$, of I^n until its first return to I^{n-1}, i.e., $f^{l(n)} I^n \subset I^{n-1}$. Let us watch how this orbit passes through a saddle-node cascade (8.1). Let us say that a level $m+s$ of the cascade is "branched" if for some interval $J_k \subset I^m \setminus I^{m+1}$ we have: $g_{m+1} J_k \subset I^{m+s-1} \setminus I^{m+s}$ (note that this can be expressed in terms of branching of the graph $\Upsilon(I^n)$).

Let us eliminate from each saddle-node cascade of the graph $\Upsilon(I^n)$ the maximal string of levels $m+d, \ldots, m+N-d$ which don't contain branched vertices of the graph. Call the remained graph $\Upsilon_e(I^n)$. Let us define the *essential return time* $l_e(I^n)$ as the number of paths in $\Upsilon_e(I^n)$ joining I^n with the top level. The essential period $\mathrm{per}_e(f)$ of a renormalizable map f is defined as the essential return time of an interval I^n of the renormalizable level.

Let us define the *scaling factors*

$$\lambda_n \equiv \lambda_n(f) = \frac{|I^n|}{|I^{n-1}|}.$$

Let us call the geometry of f *essentially K-bounded* until the next renormalization level if the scaling factors λ_n bounded below by K^{-1}, while the configurations $(I^{n-1} \setminus I^n, I^n_k)$ have K-bounded geometry (that is, all the intervals I^n_j, $j \neq 0$, and all the components of $I^{n-1} \setminus \cup I^n_k$ ("gaps") are K-commensurable). Remark that the scaling factors λ_n are allowed to be close to 1.

8.2. Complex bounds. Theorem V. *Assume that f admits a unimodal renormalization. Then:*

- *If $\mathrm{per}_e(f)$ is sufficiently big then the unimodal renormalization Rf admits a quadratic-like extension to the complex plane. Moreover, $\mathrm{mod}(Rf) \geq \mu(\mathrm{per}_e(f))$ where $\mu(p) \to \infty$ as $p \to \infty$.*

• *Real geometry of f is essentially K-bounded until the next renormalization level, with $K = K(\mathrm{per}_e(f))$.*

In [LY] complex bounds have been proven for infinitely renormalizable maps with essentially bounded combinatorics (which means that the essential periods $\mathrm{per}_e(R^m f)$ are uniformly bounded). This yields the Complex Bounds Theorem stated in the Introduction.

Remark. To get a bound for $\mathrm{mod}(R^m f)$ we never go beyond level m, so that our bounds are still valid for m times renormalizable maps.

Given an interval I, let $|I|$ denote its length, and let $D(I)$ denote the Euclidian disk based upon I as a diameter.

The rest of the section will be occupied with the proof of Theorem V. It relies on the following geometric fact:

Schwarz Lemma. *Let I and J be two real intervals. Let $\phi : \mathbb{C} \setminus (\mathbb{R} \setminus I) \to \mathbb{C} \setminus (\mathbb{R} \setminus J)$ be an analytic map which maps I to J. Then $\phi(D(I)) \subset D(J)$.*

Proof. Just notice that $D(I)$ is the hyperbolic r-neighborhood of I in the slit plane $\mathbb{C} \setminus (\mathbb{R} \setminus I)$ (with r independent of I). Since analytic maps are hyperbolic contractions, the statement follows. □

Lemma 8.2. *If a scaling factor λ_n is sufficiently small then the generalized renormalization $g_{n+1} : \cup I_j^{n+1} \to I^n$ admits a (generalized) polynomial-like extension to the complex plane, $g_{n+1} : \cup V_j^{n+1} \to V^n$. Moreover, $\mathrm{mod}(V^n \setminus V^{n+1}) \geq \mu(\lambda) \to \infty$ as $\lambda \to 0$.*

Proof. Let us select V^n as the Euclidean disc $D \equiv D(I^n)$. Let us pull it back by the inverse branches of g_{n+1}. We obtain domains V_j^{n+1} based on the intervals I_j^{n+1}. Moreover, $g_{n+1} : V_j^{n+1} \to D$ is a double branched covering for $j = 0$ and is univalent otherwise.

By the Schwarz Lemma, $V_j^{n+1} \subset D(I_j^{n+1}) \Subset D$ for $j \neq 0$.

Let us estimate the size of V_0^{n+1}. Let t be the first return time of the critical point back to I^n under iterates of g_n. Let $J \ni g_n(0)$ be the interval which is monotonically mapped onto I^n under g_n^{t-1}. Then by the real Koebe Principle,

$$|J|/\,\mathrm{dist}(J, \partial I^{n-1}) = O(\lambda_n). \tag{8.2}$$

Let us consider the decomposition

$$g_{n+1}|I^{n+1} = g_n^{t-1} \circ h \circ \Phi,$$

where $h : (K, L) \to (I^{n-1}, J)$ is a diffeomorphism of an appropriate interval K onto I^{n-1}. Using the real Koebe Principle once more, we derive from (8.2) that

$$|L|/\,\mathrm{dist}(L, \partial K) = O(\lambda_n). \tag{8.3}$$

By the Schwarz Lemma, the pull-back U of D by the inverse branch of $g^{t-1} \circ h : L \to I^n$ is contained in $D(L)$. Hence $V^{n+1} \subset \Phi^{-1} D(L)$ and by (8.3)

$$\mathrm{diam}\, V^{n+1}/|I^n| = O(\sqrt{\lambda_n}).$$

It follows that V^{n+1} lies well inside $D = D(I^n)$, and we have a generalized polynomial-like map with desired properties. $\square$

In the following two lemmas we analyse the geometry of long central cascades. Let us call a unimodal map saddle-node or Ulam-Neumann if it is topologically conjugate to $z \mapsto z^2 + 1/4$ or $z \mapsto z^2 - 2$ correspondingly.

Lemma 8.3. *Let us consider an Ulam-Neumann cascade* 8.1. *If it is sufficiently long then the generalized renormalization g_{m+N+1} admits a polynomial-like extension to the complex plane with a definite modulus. Moreover,* $\operatorname{mod} g_{m+N+1} \to \infty$ *as* $N \to \infty$.

Proof. Take the Euclidean disk $D = D(I^{m+N})$ and pull it back by the inverse branches of g_{m+N+1}. We obtain domains V_j^{m+N+1} based upon the intervals I_j^{m+N+1}. By the Schwarz lemma, all the off-critical domains V_j^{m+l+1}, $j \neq 0$, are contained in the round discs $D(I_j^{m+N+1})$, and hence are strictly contained in D.

Let us estimate the size of the central domain $V \equiv V_0^{m+N+1}$. By Theorem 8.1, I^{m+1} is well inside I^m. If the scaling factor λ_{m+1} is small then the statement follows from Lemma 8.2 and Theorem II. So we can assume that I^{m+1} is commensurable with I^m. By compactness argument, if the cascade is long enough then the map $g \equiv g_{m+1} : I^{m+1} \to I^m$, with the domain rescaled to the unit size, is C^1-close to an Ulam-Neumann map. It follows that $|I^{m+k} \setminus I^{m+N}|$ decrease with k at a uniformly exponential rate. Hence for a sufficiently long cascade, $I^{m+N-1} \setminus I^{m+N}$ is ϵ-tiny as compared with I^{m+N}.

Let $g(0) \in I \equiv I_j^{m+N}$, $j \neq 0$. Let U be the pull-back of $D(I^{m+N-1})$ by the inverse branch of $g_{m+N}^{-1} : I^{m+N-1} \to I$ extended to the complex plane. By the Schwarz Lemma, $U \subset D(I)$.

Furthermore, by Theorem 8.1, there is an interval $L \supset \Phi(I^{m+N})$ such that $g|I^{m+N} = h \circ \Phi$, where $h : L \to I^{m+N-1}$ is a diffeomorphism of bounded distortion. Hence the image $\Phi I^{m+N} = h^{-1} g I^{m+N} \supset h^{-1} I^{m+N}$ occupies at least $(1 - O(\epsilon))$-portion of L.

It follows that $h_{-1} U \subset D(h^{-1} I)$ is of size $O(\epsilon)$ as compared with $|L|$. Hence $V \subset \Phi^{-1} D(h^{-1} I)$ is of size $O(\sqrt{\epsilon})$ as compared with $|I^{m+N}|$, and the lemma follows. $\square$

Lemma 8.4. *All saddle-node patterns* (8.1) *of the same length with commensurable I^m and I^{m+1} are κ-qs equivalent, with an absolute κ.*

Proof. Let $g : I' \to [0,1]$ be a quasi-quadratic map of Epstein class (and perhaps escaping critical point): $g \in \mathcal{E}$. By definition, $g = h \circ \Phi$ with a diffeomorphism h whose inverse admits the analytic extension to $\mathbb{C} \setminus [0,1]$. Let us supply this space with the Montel topology on the h^{-1}.

Take a $\delta \in (0, 1/2)$. The set of maps $g \in \mathcal{E}$ with $\delta \leq |I'| \leq 1 - \delta$ is compact. Hence given a long saddle-node cascade (8.1), the map G obtained from $g_{m+1} : I^{m+1} \to I^m$ by rescaling I^m to the unit size must be close to a saddle-node quasi-quadratic map. Hence we can reduce G to a form $z \mapsto z + \epsilon + \psi(z)$ where $\psi(z) > 0$ is uniformly comparable with z^2 (here the fixed point of the nearby saddle-node map is selected as the origin). Moreover, we will see in a moment that ϵ is determined, up to a bounded error, by the length of the cascade.

Take a big $a > 0$. When $|z| < a\sqrt{\epsilon}$, the step $G(z) - z$ is of order ϵ. Otherwise $\psi(z)$ dominates over ϵ, and in the chart $\zeta = 1/z$ the step is of order 1. It follows that the qs class of the cascade is determined by ϵ, which in turn is related to the length of the cascade by $N \asymp 1/\sqrt{\epsilon}$. □

The following lemma refines Lemma 7.2 in the case of real cascades.

Lemma 8.5. *Let us consider a cascade (8.1). Let $t = t_{m+N}$ be the maximal return time of the intervals $g_{m+N+1} I_i^{m+N+1} \subset I^{m+N-1}$ back to I^{m+N} under iterates of the Bernoulli map G_{m+N}, see (3.12). Then there exists a level l such that $\lambda_l \leq \lambda(t)$ where $\lambda(t) \to 0$ as $t \to \infty$.*

Proof. If some interval I_j^{m+1} is not well inside I^m then by Theorem 8.1 and Lemma 8.4, the level m follows a long Ulam-Neumann cascade. Then the scaling factor λ_{m+1} is small. It follows that λ_{m+N+1} is small as well (see [L4], §2, for the estimate of λ_{m+N+1} via λ_{m+1}).

If all the intervals I_j^{m+1} are well inside of I^m then repeat the argument of Lemma 7.2 on the real line using negative Schwarzian in place of conformality and the Bernoulli map G_{m+N} in place of g_n. □

Let $\chi(m)$ stand for the height of I^m, that is, the number of central cascades preceding it.

Lemma 8.6. *If the height $\chi(f)$ is sufficiently big then there is an interval $J \in \mathcal{M}$ such that the generalized renormalization $T_J f$ admits a polynomial-like extension to the complex plane with a definite principle modulus μ. Moreover, J lies on a bounded height, i.e., $J \supset I^m$ with a bounded $\chi(m)$.*

Proof. Take small $\epsilon > 0$ and $\delta > 0$, and consider the inequality

$$\lambda_l > (1 - \delta)\lambda_{l-1}. \tag{8.4}$$

If (8.4) fails to happen on the first $s = \log \epsilon / \log(1 - \delta) + 1$ levels then we come up with an ϵ-small scaling factor, and Lemma 8.2 yields the desired, provided ϵ is small enough.

Otherwise a desired interval J exists by [L4], §4 (provided δ is small enough). □

Lemma 8.7. *Assume f is renormalizable. If $\text{per}_e(f)$ is sufficiently high, then the renormalization Rf is polynomial-like. Moreover, $\text{mod}(Rf) > \mu(\text{per}_e(f))$ where $\mu(p) \to \infty$ as $p \to \infty$.*

Proof. Big essential period amounts to one of the following circumstances:
(i) The height of $\chi(f)$ is big; or
(ii) For some cascade (8.1) (maybe of length 1) the return time t_{m+N} of Lemma 8.5 is big; or
(iii) There is a long Ulam-Neumann cascade; or

(iv) There is a saddle-node cascade (8.1) and an interval I_j^{m+1} which lands deep inside the cascade under one iterate of g_m.

Assume that (i) occurs. Then the statement follows from Lemma 8.6 and Theorem III.

If (ii) happens then by Lemma 8.5 we observe a small scaling factor on some level, and Lemma 8.2 yields the statement.

If (iii) occurs then the desired follows from Lemma 8.3 and Theorem II.

Assume finally that (iv) happens. Let $J \equiv I_j^{m+1}$. Then $g_{m+1}J \subset I^{m+i} \setminus I^{m+i+1}$ for $d \le i \le N-d$, with big d. Then by Lemma 8.4 $I^{m+i} \setminus I^{m+i+1}$ is tiny in I^m. It follows that J is tiny as compared with the $\text{dist}(J, \partial I^m)$. By [L4], this produces a small scaling factor several levels down (if $\text{rank}(J)$ is big, use Lemma 3.6 of [L4]; otherwise use Lemma 2.12 of that paper). Now Lemma 8.2 and Theorem II complete the proof. □

Lemma 8.8. *If* $\text{per}_e(f)$ *is bounded, then the geometry of* f *is essentially bounded until the next renormalization level.*

Proof. Assume that the geometry is bounded on level $n-1$, and let us see what happens on the next level. Given an $x \in \omega(c) \cap (I^{n-1} \setminus I^n)$, let $J(x)$ denote the pull-back of I^n corresponding to the first landing of $\text{orb}(x)$ at I^n. As the landing time under iterates of g_n is bounded, $J(x)$ is commensurable with I^{n-1}.

To create the intervals I_j^{n+1}, we should pull all intervals $J(x)$ back by $g_n : I^n \to I^{n-1}$. As g_n is a quasi-quadratic map, all non-central intervals I_j^{n+1} and the gaps in between are commensurable with I^n.

The only possible problem is that the central interval I^{n+1} may be tiny in I^n. This may happen only if the critical value $g_n 0 \in J(x)$ is very close to the $\partial J(x)$. Let l be such that $g_n^l J(x) = I^n$. Since $g_n^l : J(x) \to I^n$ is qs, $g_{n+1} 0 = g_n^{l+1}$ turns out to be very close to ∂I^n ("very low return"). But then $g_{n+1}0$ belongs to some non-central interval I_j^{n+1} whose Poincaré length in I^n is definite (as we have shown above). This is a contradiction.

So when we pass from one level to the next, the geometric bounds change gradually. But the same is also true when we pass through a saddle-node cascade (8.1). Let us consider the Bernoulli map $G : \cup K_j^{m+i} \to I^m$ associated with this cascade (see §3.6), where the $K_j^{m+i} \subset I^{m+i-1} \setminus I^{m+i}$ are the pull-back of the I_k^{m+2}.

Observe that for $i < N$ the transit maps

$$g_{m+1}^{i-2} : I^{m+i-1} \setminus I^{m+i} \to I^{m+1} \setminus I^{m+2}$$

have bounded distortion, as its Koebe space spreads over the appropriate components of $I^m \setminus I^{m+3}$. Moreover, the passages from the level m to $m+1$ and from $m+N-2$ to $m+N-1$ have bounded distortion by Theorem 8.1 and Lemma 8.4.

Hence if the geometry of the configuration $(I^m \setminus I^{m+1}, \{I_i^{m+2}\})$ on level m is bounded, then the geometry of the configuration $(I^{m+N-1} \setminus I^{m+N}, K_j^{m+N})$ is bounded as well. Moreover, by Lemma 8.4, I^{m+N} is commensurable with $I^{m+N-1} \setminus$

I^{m+N}. Thus the configuration $(I^{m+N-1}, \{K_i^{m+N}\})$ of level $m+N-1$ has bounded geometry.

Let us now define the intervals $J(x)$, $x \in \omega(c) \cap I^{m+N-1}$, as the pull-backs of I^{m+N} corresponding to the first landing of $\mathrm{orb}(x)$ at I^{m+N}. Then it follows from the bounded geometry on level $m+N-1$ together with the bounded return G-times and the landing depths that the configuration of the intervals $J(x)$ has bounded geometry in I^{m+N-1}.

Let us now pull these intervals back to the next level $m+N$. Then the same argument as in the beginning of the proof shows that the geometry on level $m+N$ is still bounded. □

Now Theorem V follows from the last two lemmas.

Remark. Theorem V is still valid for higher degree real unimodal polynomials $z \mapsto z^d + c$, $c \in \mathbb{R}$, except for the growing of $\mu(p)$. The same proof works, with the following adjustment of logic. Proof of Lemma 8.6 shows that generalized quadratic-like maps with a definite modulus can be created on a sequence of levels m_i with bounded $\chi(m_{i+1}) - \chi(m_i)$. Together with the Remark at the end of §5 this implies that all the generalized renormalizations $T^{n(k)+1}f$ have a definite principal modulus (where $n(k)$ counts the non-central levels). In particular, $\mathrm{mod}(Rf)$ is definite.

Part II. Rigidity and local connectivity

9. Space between Julia bouquets

In this section we will prove local connectivity of the Julia sets satisfying the secondary limbs condition with *a priori* bounds (Theorem VI).

9.1. Space and unbranching. Let J_i^m denote the little Julia sets of level m, that is, $J^m \equiv J_0^m = J(R^m f)$ and $J_i^m = f^i J^m$, $i = 0, \dots, r_m - 1$. They are organized in the pairwise disjoint bouquets $B_j^m = B_j^m(f)$ of the Julia sets touching at the same periodic point. Namely, if level $m-1$ is immediately renormalizable with period l then each B_j^m consists of l little Julia sets J_i^m touching at their β-fixed points. Otherwise the bouquets B_j^m just coincide with the little Julia sets J_j^m. By $B^m \equiv B_0^m$ we will denote the *critical* bouquet containing the critical point 0. Let $\mathbb{J}^m = \mathbb{J}^m(f) = \cup_i J_i^m = \cup_j B_j^m$. Finally let K_i^m be little filled Julia sets.

We will use the notation F_m for the quadratic-like map f^{r_m} near any little Julia set J_i^m (it should be clear from the context which one is considered). In particular, $F_m = R^m f$ near the critical Julia set $J^m \ni 0$.

Recall that $\mathcal{Q}(\mu)$ stands for the space of quadratic-like maps f with $\mathrm{mod}(f) \geq \mu > 0$ supplied with the Carathéodory topology (see §7.3). Take a little copy $M' \subset M$ of the Mandelbrot set truncated at the root. Let $\mathcal{Q}(\mu, M')$ denote the subspace of $\mathcal{Q}(\mu)$ consisting of renormalizable quadratic-like maps f whose hybrid class belongs to M'.

Let us have a family $\mathcal{F}$ of sets $X_a \subset \mathbb{C}$ depending on some parameter a ranging over a topological space $\mathcal{T}$. This dependence is said to be (sequentially) *upper semi-continuous* if for any $a(i) \to a$, the Hausdorff limit of $X_{a(i)}$ is contained in X_a. For example it is easy to see that the filled Julia set $K(f)$ of a quadratic-like map f depends upper semi-continuously on f. Let us say that a family $\mathcal{F}$ of sets $X_f \subset \mathbb{C}$ is (upper) *semi-compact* if any sequence X_n of these sets contains a subsequence $X_{n(i)}$ converging in Hausdorff topology to a subset of some $X \in \mathcal{F}$.

Lemma 9.1. *The little filled Julia sets $K_i^1(f)$ form a semi-compact family of sets as f ranges over the space $\mathcal{Q}(\mu, M')$.*

Proof. By the Compactness Lemma (see §7.3), the space $\mathcal{Q}(\mu, M')$ is compact. Moreover the quadratic-like map F_1 depends continuously on $f \in \mathcal{Q}(\mu, M')$ near any K_i^1. In turn, the little filled Julia sets K_i^1 depend upper semi-continuously on F_1. □

Lemma 9.2. *Let f be a quadratic-like map of class $\mathcal{SL}$ with complex a priori bounds. Then there is a definite space in between its bouquets B_j^m.*

Proof. Let us take a bouquet B^m. Let $\mathcal{I}^m$ stand for the set of indices j such that $B_j^{m+1} \subset B^m$. We will show first that there is a definite annulus

$$T^m \subset \mathbb{C} \setminus \bigcup_{j \in \mathcal{I}^m} B_j^{m+1},$$

which goes around B^{m+1} but does not go around other bouquets B_j^{m+1}, $j \in \mathcal{I}^m$.

If $R^m f$ is not immediately renormalizable, then this follows from Theorem II (ii). So assume that $R^m f$ is immediately renormalizable.

If $B^m = J^m$, then it is nothing to prove as there is only one bouquet B^{m+1} inside B^m. Otherwise there are only finitely many renormalization types producing the bouquet B^m (which correspond to the little Mandelbrot sets attached to the main cardioid and belonging to the selected secondary limbs). By Lemma 9.1, the bouquets B_j^{m+1} contained in B^m belong to a compact family of sets. As they don't touch each other, there is a definite space in between them.

Let $N(L, \epsilon)$ denote an $(\epsilon \operatorname{diam} L)$-neighborhood of a set L (that is, the set of points on distance at most $\epsilon \operatorname{diam} L$ from L). We have shown that there is an $\epsilon > 0$ such that the neighborhood $N(B^{m+1}, \epsilon)$ does not intersect other bouquets B_j^{m+1} contained in the same B^m. In particular, $N(B^1, \epsilon)$ does not intersect any other B_j^1 (as all of them are contained in $B^0 \equiv J(f)$).

Let us show by induction that

$$N(B^m, \epsilon) \cap B_k^m = \emptyset, \; k \neq 0 \tag{9.1}$$

Assuming this for m, we should show that

$$N(B^{m+1}, \epsilon) \cap B_j^{m+1} = \emptyset, \; j \neq 0. \tag{9.2}$$

As we already know (9.2) for $j \in \mathcal{I}^m$, let $j \notin \mathcal{I}^m$. Then $B_j^{m+1} \subset B_k^m$ for some $k \neq 0$, while $N(B^{m+1}, \epsilon) \subset N(B^m, \epsilon)$, and (9.2) follows from (9.1).

What is left, is to show that there is a definite space around any bouquet B_j^{m+1} (not only around the critical one). But there is an iterate f^l which univalently maps

B_j^{m+1} onto B^{m+1}. Pulling back the space around B^{m+1} we obtain the desired space about B_j^{m+1}. □

An infinitely renormalizable map f is said to satisfy an *unbranched a priori bounds* condition (see [McM3]) if for infinitely many levels m, there is a definite space in between J^m and the rest of the postcritical set, $\omega(0) \setminus J^m$.

Lemma 9.3. *A map $f \in \mathcal{SL}$ with a priori bounds satisfies an unbranched a priori bounds condition.*

Proof. We will show that the unbranched condition can fail only if the level m is not immediately renormalizable, while $m-1$ is immediately renormalizable. As the complimentary sequence of levels is infinite, the lemma will follow.

If $R^{m-1}f$ is not immediately renormalizable then the bouquet B^m coincides with the little Julia set J^m. By Lemma 9.2, there is a definite space in between J^m and $\mathbb{J}^m \setminus J^m$. As $\omega(0) \setminus J^m \subset \mathbb{J}^m \setminus J^m$, the unbranched condition holds on level m.

Assume now that both levels $m-1$ and m are immediately renormalizable. Then we will show that there is a definite space in between J^m and $\mathcal{B}^{m+1} \equiv \cup_{j\neq 0} B_j^{m+1}$.

By Lemma 9.2, there is a definite space in between $B^m \supset J^m$ and $\mathcal{B}^{m+1} \setminus B^m$. So we should check that there is a definite space in between J^m and $\mathcal{B}^{m+1} \cap B^m$ (that is, the union of non-critical bouquets B_j^{m+1} contained in B^m). But J^m does not touch any such B_j^{m+1}. Indeed, the only point where they can touch could be the β-fixed point β_m of J^m. But one can easily see that the little Julia sets of level $m+1$ never contain β_m. By Lemma 9.1 there is a desired space.

Finally, as $\omega(0) \setminus J^m \subset \mathcal{B}^{m+1}$, the statement follows. □

Remark. If $R^m f$ is not immediately renormalizable, while $R^{m-1}f$ is immediately renormalizable, then the unbranched condition can fail. Indeed in this case there are several Julia sets J_i^m which touch at the common fixed point $\beta_m \in J^m$. But the postcritical set $\omega(0) \cap J_i^m$ can come arbitrarily close to β_m (when $R^m f$ is a small perturbation of a map whose critical orbit eventually lands at β_m).

9.2. Local connectivity of Julia sets. Using Sullivan's *a priori* bounds Hu and Jiang [HJ] proved that the Feigenbaum quadratic polynomial has locally connected Julia set. Then a more general result of this kind was worked out: Any infinitely renormalizable quadratic map with unbranched *a priori* bounds has locally connected Julia set (see [J, McM3]). Together with Lemma 9.3 this yields:

Theorem VI. *Let $f \in \mathcal{SL}$ be an infinitely renormalizable quadratic polynomial with a priori bounds. Then the Julia set $J(f)$ is locally connected. In particular, all maps from Theorems IV and IV′ of §7 have locally connected Julia sets.*

Proof. I learned the argument given below from J. Kahn (Durham 93).

A priori bounds imply that the "little" Julia sets J^m shrink down to the critical point. Indeed let $f_m \equiv R^m f \equiv f^{r_m} : U'_m \to U_m$ where $\mathrm{mod}(U_m \setminus U'_m) \geq \epsilon > 0$, with an ϵ independent of m. Clearly U_m does not cover the whole Julia set.

Let $\Gamma_m \subset U_m \setminus U'_m$ be a horizontal curve in the annulus $U_m \setminus U'_m$ which divides it into two sub-annuli of modulus at least $\epsilon/2$, and $\Gamma'_m \subset U'_m$ be its pull-back by f_m.

By the Koebe Theorem, these curves have a bounded eccentricity about 0 (with a bound depending on ϵ). Since the inner radius of curve Γ'_m about 0 tends to 0 as $m \to \infty$ (it follows from the fact that the sufficiently high iterates of any disk intersecting $J(f)$ cover the whole $J(f)$), the $\operatorname{diam} \Gamma'_m \to 0$ as well. All the more, $\operatorname{diam}(J_m) \to 0$ as $m \to \infty$.

Let us take a $\delta > 0$, and find an m such that J_m is contained in the $\mathbb{D}_\delta$.

Let us now inscribe into $\mathbb{D}_\delta$ a domain bounded by equipotentials and external rays of the original map f. Let β_m denote the non-dividing fixed point of the Julia set J^m, and $\beta'_m = -\beta_m$ be the symmetric point. Let us consider a puzzle piece $P^{m,0} \ni 0$ bounded by any equipotential and four external rays of the *original map* f landing at β_m and β'_m. This is a "degenerate" domain of the renormalized map F_m (see §2.5). By definition of the renormalized Julia set, the preimages $P^{m,k} \equiv F_m^{-k} P^{m,0}$ shrink down to J^m. Hence there is a puzzle piece $P^{m,l}$ contained in the $\mathbb{D}_\delta$. As $J(f) \cap P^{m,l}$ is clearly connected, the Julia set $J(f)$ is locally connected at the critical point.

Let us now prove local connectivity at any other point $z \in J(f)$. This is done by a standard spreading of the local information near the critical point around the whole dynamical plane. Let us consider two cases.

Case (i). Let the orbit of z accumulates on all Julia sets J^m. Let m be an unbranched level. Then there is an $l = l(m)$ such that the puzzle piece $P^{m,l}$ is well inside $\mathbb{C} \setminus (\omega(0) \setminus J^m)$.

Take now the first moment $k = k(m) \geq 0$ such that $f^k z \in P^{m,l}$. Let us consider the pull-backs $Q^{m,l} \ni z$ of $P^{m,l}$ along the orbit $\operatorname{orb}_k(z)$. By Lemma 3.3, this pull-back is univalent. Moreover, it allows a univalent extension to a definitely bigger domain.

By the Koebe Theorem, $Q^{m,l}$ has a bounded eccentricity about z. Since the inner radius of this domain about z tends to 0 as $m \to \infty$, the $\operatorname{diam} Q^{m,l} \to 0$ as well. As $Q^{m,l} \cap J(f)$ are connected, the Julia set is locally connected at z.

Case (ii). Assume now that the orbit of z does not accumulate on some J^m. Hence it accumulates on some point $a \notin \omega(0)$. Let us consider the puzzle associated with the periodic point β_m (so that the initial configuration consists of a certain equipotential and the external rays landing at β_m). Since the critical puzzle pieces shrink to J^m, the puzzle pieces $Y_i^{(l)}$ of sufficiently big depth l containing a are disjoint from $\omega(0)$ (there are several such pieces if a is a preimage of β_m). Take such an l, and let X be the union of these puzzle pieces. It is a closed topological disk disjoint from $\omega(0)$ whose interior contains a.

Consider now the moments $k_i \to \infty$ when the orbit of z lands at $\operatorname{int} X$, and pull X back to z. By the same Koebe argument as in case (i) we conclude that these pull-backs shrink to z. It follows that $J(f)$ is locally connected at z. □

9.3. Standard neighborhoods. In this section we will construct some special fundamental domains near little Julia bouquets. Let us consider first the non-immediately renormalizable case when the construction can be done in a particularly nice geometric way.

Lemma 9.4. *Let f be m times renormalizable quadratic map. Assume that the space in between the little Julia sets J_i^m is at least $\mu > 0$. Then there are disjoint fundamental annuli A_i^m around little Julia sets J_i^m, with* $\operatorname{mod} A_i^m \geq \nu(\mu) > 0$.

Proof. Let us consider the Riemann surfaces $S = \mathbb{C} \setminus \mathbb{J}^m$ and $S' = \mathbb{C} \setminus f^{-1}\mathbb{J}^m \subset S$. Then $f : S' \to S$ is a double covering. Let us uniformize S, that is represent it as the quotient $\mathcal{H}^2/\Gamma$ of the hyperbolic plane modulo the action of a Fuchsian group. In this conformal representation S admits a compactification $S \cup \partial S$ to a bordered Riemann surface, with the components ∂S_i^m of the "ideal boundary" ∂S corresponding to the little Julia sets J_i^m.

Let $\hat{S} = S \cup \partial S \cup \bar{S}$ be the double of S, that is $(\mathbb{C} \setminus \Lambda(\Gamma))/\Gamma$, where $\Lambda(\Gamma) \subset S^1$ is the limit set of Γ. The boundary components ∂S_i^m are geodesics in $\hat{S}$. Moreover, these geodesics have hyperbolic length bounded by a constant $L = L(\mu)$ independent of m.

Let $\sigma : S \to S$ be the natural anti-holomorphic involution of S. Let $\bar{S}' = \sigma S'$ and $\hat{S}' = S' \cup \partial S \cup \bar{S}' \subset \hat{S}$ be the double of S' inside S. Then f admits an extension to a holomorphic double covering $\hat{f} : \hat{S}' \to \hat{S}$ commuting with the involution σ. Its restriction $\hat{f}|\partial S_0^m \to \partial S_1^m$ is a double covering, while the restrictions to the other boundary components $\partial S_i^m \to \partial S_{i+1}^m$ are diffeomorphisms.

Let $C_i^m(r) \supset \partial S_i^m$ stand for the hyperbolic r-neighborhood of the geodesic ∂S_i^m. By the Collar Lemma (see [Ab]), there is an $r = r(L)$ (independent of the particular Riemann surface and geodesics) such that the collars $C_i^m \equiv C_i^m(r)$ are pairwise disjoint. Moreover, $\operatorname{mod}(C_i^m) \geq \mu(L) > 0$.

Let us now take such a collar $C = C_i^m$, and let $\gamma = \partial S_i^m$. Let $C' \subset S' \cap C$ be the component of $\hat{f}^{-p}C$ containing γ (where p is the period of the little Julia sets). Then $\hat{f}^p : C' \to C$ is a double covering preserving γ. As we have in the hyperbolic metric of S:

$$\int_\gamma \|D\hat{f}^p\| = 2l(\gamma),$$

there is a point $z \in \gamma$ such that $\|D\hat{f}^p(z)\| \geq 2$. This easily implies that $\|D\hat{f}^{-p}(\zeta)\| \leq q(a) < 1$ if the hyperbolic distance between $\hat{f}^p z$ and ζ does not exceed a. In particular, $\|D\hat{f}^{-p}\|(\zeta) \leq q = q(L, r) < 1$ for all $\zeta \in C$.

It follows that C' is contained in the hyperbolic r/q-neighborhood of γ, and hence $\operatorname{mod}(C \setminus C') \geq \rho(r, q) = \rho(\mu)$. Let now $A_i^m = (C \setminus C') \cap S$. □

Note that in the above lemma we don't assume *a priori* bounds but just a definite space between the Julia sets (which thus implies *a priori* bounds). Assuming *a priori* bounds, let us now give a different construction which works in the immediately renormalizable case as well.

Let us consider a bouquet $B_j^m = \cup_i J_i^m$ of level m, where J_i^m touch at point α_{m-1}. Let $b_i^m \in J_i^m$ be the points F_m-symmetric to α_{m-1}, that is, $F_m b_i^m = \alpha_{m-1}$ ("co-fixed points"). Let us consider the domain Υ_j^m bounded by the pairs of rays landing at these points (defined via a straightening of F_{m-1}), and p_m arcs of equipotentials. Let us then thicken this domain near the points b_i^m as described in §2.5 (that is, replace the rays landing at b_i^m by nearby rays and little circle arcs around b_i^m). Denote the thickened domains by U_j^m (see Figure 10). We also require that these domains are naturally related by dynamics so that $f\Upsilon_j^m = \Upsilon_k^m$ and $fU_j^m = U_k^m$ whenever $fB_j^m = B_k^m$ and B_j^m is non-critical. Let us call U_j^m a *standard neighborhood* of the bouquet B_j^m. Let $\mathbb{U}^m = \cup U_j^m$.