Weil-Petersson volumes and intersection theory on the moduli space of curves

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Contents

1 Introduction

In this paper, we establish a relationship between the Weil-Petersson volume $V_{g,n}(b)$ of the moduli space $\mathcal{M}_{g,n}(b)$ of hyperbolic Riemann surfaces with geodesic boundary components of length $b_1, \ldots, b_n$ and the intersection numbers of tautological classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves. As a result, by using the recursive formula for $V_{g,n}(b)$ obtained in [Mirz], we derive a new proof of the Virasoro constraints for a point. This result is equivalent to the Witten-Kontsevich formula [K].

Intersection theory of $\overline{\mathcal{M}}_{g,n}$. Let $\mathcal{M}_{g,n}$ be the moduli space of genus g curves with n distinct marked points and $\overline{\mathcal{M}}_{g,n}$ its Deligne-Mumford compactification. The space $\overline{\mathcal{M}}_{g,n}$ is a connected complex orbifold of dimension $3g-3+n$ [Har] . These moduli spaces are endowed with natural cohomology classes. An example of such a class is the Chern class of a vector bundle on the moduli space. There are n tautological line bundles defined on $\overline{\mathcal{M}}_{g,n}$:For each marked point i, there exists a canonical line bundle $\mathcal{L}_i$ in the orbifold sense whose fiber at the point $(C, x_1, \ldots, x_n) \in \overline{\mathcal{M}}_{g,n}$ is the cotangent space of C at x_i. The first Chern class of this bundle is denoted by $\psi_i = c_1(\mathcal{L}_i)$. Note that although the complex curve C may have nodes, x_i never coincides with the singular points.

For any set $\{d_1, \ldots, d_n\}$ of integers define the top intersection number of ψ classes by

$$\langle\, \tau_{d_1}, \ldots, \tau_{d_n} \rangle_g = \int\limits_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \quad .$$

Such products are well defined when the d_i's are non-negative integers and $\sum_{i=1}^{n} d_i = 3g-3+n$. In other cases $\langle\, \tau_{d_1}, \ldots, \tau_{d_n} \rangle_g$ is defined to be zero. Since we are in orbifold setting, these intersection numbers are rational numbers. See [Lo1] and [Har] for more details.

Introduce formal variables t_i, $i \geq 0$, and define F_g, the generating function of all top intersections of ψ classes in genus g, by

$$F_g(t_0, t_1, \ldots) = \sum_{\{d_i\}} \langle \prod \tau_{d_i} \rangle_g \prod_{r>0} t_r^{n_r}/n_r! \quad ,$$

where the sum is over all sequences of nonnegative integers $\{d_i\}$ with finitely many nonzero terms, and $n_r = \mathrm{Card}(i : d_i = r)$. The generating function

$$F = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g,$$

arises as a partition function in two-dimensional quantum gravity.

Witten [Wit1] conjectured a recursive formula for the intersections of tautological classes in the form of KdV differential equations satisfied by F. Dijgraaf, Verlinde and Verlinde [DVV] showed that Witten's conjecture implies that e^F is annihilated by a sequence of differential operators

$$L_{-1}, L_0, \ldots, L_n, \ldots$$

satisfying the Virasoro relations

$$[L_m, L_k] = (m-k)L_{m+k}.$$

For the definition of the L_i's see §6. The Virasoro constraints determine the intersection numbers of tautological line bundles in all genera.

In [K], Kontsevich introduces a matrix model as the generating function for the intersection numbers on the moduli space to prove Witten's conjecture by expressing intersection numbers in terms of sums over ribbon graphs. Also, A. Okounkov and R. Pandharipande gave a different proof by using the relation between the Gromov-Witten theory of $\mathbb{P}^1$ and Hurwitz numbers [OP]. For expository accounts of these proofs see [Lo1] and [O].

In this paper we prove that F, the generating function of the intersection numbers, satisfies the Virasoro constraints. Our proof relies on the Weil-Petersson symplectic geometry of the moduli space of curves, and results of G. McShane [M] on lengths of simple closed geodesics on hyperbolic surfaces.

Weil-Petersson geometry of $\overline{\mathcal{M}}_{g,n}$. The key tool for obtaining the recursive formula for the intersections of the tautological classes is understanding the the relationship between the tautological classes and Weil-Petersson symplectic form.

This form is the symplectic form of a Kähler, non-complete metric on the moduli space of curves introduced by A. Weil [IT]. In [Mas], Masur obtained growth estimates for the coefficients of the Weil-Petersson metric close to the boundary of the moduli space. In [Wol4], Wolpert showed that the Weil-Petersson symplectic form has a simple expression in terms of the Fenchel Nielsen twist-length coordinates of the Teichmüller space (§2). Moreover, he showed that the Weil-Petersson Kähler form ω_{WP} extends as a closed form to $\overline{\mathcal{M}}_{g,n}$, and defines a cohomology class $[\omega] \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{R})$. See §2 for more details.

Volumes of moduli spaces of bordered Riemann surfaces. The Weil-Petersson volume of the moduli space $\mathcal{M}_{g,n}$ is a finite number and its value as a function of g and n arises naturally in different contexts [KMZ] .

In order to integrate certain type of geometric functions over the moduli space [Mirz], we find it fruitful to consider more generally the moduli space $\mathcal{M}_{g,n}(b_1,\ldots,b_n)$ of bordered Riemann surfaces with the geodesic boundary components of length $b_1,..,b_n$. We calculate the Weil-Petersson volume $V_{g,n}(b)$ of the moduli space $\mathcal{M}_{g,n}(b)$ using two different methods:

(I): In [Mirz], we approach the study of the volumes of these moduli spaces via the length functions of simple closed geodesics on a hyperbolic surface and show that $V_{g,n}(b)$ is a polynomial in b. We also give an explicit recursive method for calculating these polynomials (see §5).

(II): In §4, we use the symplectic geometry of moduli spaces of bordered Riemann surfaces to calculate these volumes. This method allows us to read off the intersection numbers of tautological line bundles from the volume polynomials.

(I): A recursive formula for volumes. By using an identity for lengths of simple closed geodesics on a bordered Riemann surface which generalizes the result in [M], we obtain a recursive formula for $V_{g,n}(b)$ in terms of $V_{g_1,n_1}(b)'$s where $2g_1+n_1<2g+n$ (See equation 5.7).

As a result, we establish:

Theorem 1.1. *The volume $V_{g,n}(b)=\mathrm{Vol}(\mathcal{M}_{g,n}(b_1,\ldots,b_n))$ is a polynomial in $b_1,\ldots,b_n$, namely:*

$$V_{g,n}(b)=\sum_{|\alpha|\leq 3g-3+n} C_g(\alpha)\cdot b^{\,2\alpha},$$

where $C_g(\alpha)>0$ lies in $\pi^{6g-6+2n-2|\alpha|}\cdot\mathbb{Q}$.

Here the exponent $\alpha=(\alpha_1,\ldots,\alpha_n)$ ranges over elements in $(\mathbb{Z}_{\geq 0})^n$, $b^\alpha=b_1^{\alpha_1}\cdots b_n^{\alpha_n}$, and $|\alpha|=\sum\alpha_i$.

(II): Symplectic geometry of $\mathcal{M}_{g,n}(b)$. Working with $\mathcal{M}_{g,n}(b)$ allows us to exploit the existence of commuting Hamiltonian S^1-actions. The space $\mathcal{M}_{g,n}(b)$ has a natural orbifold structure. We generalize the tautological line bundle $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g,n}$ to the following circle bundle (in the orbifold sense) over $\overline{\mathcal{M}}_{g,n}(b)$:

$$\begin{array}{ccc} S^1 & \longrightarrow & \{(X,p)\mid p\in\beta_i, X\in\overline{\mathcal{M}}_{g,n}(b)\} \\ & & \downarrow \\ & & \overline{\mathcal{M}}_{g,n}(b) \end{array}$$

where S^1 acts by moving the points p on β_i. This shows that $\overline{\mathcal{M}_{g,n}}(b)$ is a reduced space. Then we can use the method of symplectic reduction, discussed in §3, to relate the volumes of moduli spaces of curves to the intersection numbers of tautological classes $\overline{\mathcal{M}}_{g,n}$(§4). Note that the picture is a bit different when $g = n = 1$ in which case all elements of $\mathcal{M}_{1,1}(b)$ have non trivial automorphisms of order 2 ; namely, every $X \in \mathcal{M}_{1,1}(b)$ comes with an elliptic involution.

When $(g, n) \neq (1, 1)$, a generic element of $\mathcal{M}_{g,n}(b)$ does not have any non trivial automorphism which leaves the boundary components setwise fixed. In this case, the coefficient $C_g(\alpha)$ in Theorem 1.1 is given by

$$C_g(\alpha) = \frac{1}{2^{|\alpha|}\, |\alpha|!\, (3g-3+n-|\alpha|)!} \int\limits_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \cdot \omega^{3g-3+n-|\alpha|}, \quad (1.1)$$

where ψ_i is the first Chern class of the i-th tautological line bundle, ω is the Weil-Petersson symplectic form, $\alpha! = \prod \alpha_i!$ and $|\alpha| = \sum \alpha_i$.

Remark. By a result of Wolpert [Wol2],

$$\kappa_1 = \frac{[\omega]}{2\pi},$$

where κ_1 is the first Mumfords tautological cohomology class on $\overline{\mathcal{M}_{g,n}}$.

Examples. Using the recursive formula in Section 5, one can show that

$$\mathrm{Vol}_{0,4}(b) = \frac{1}{2}(4\pi^2 + b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Therefore, we have

$$\mathrm{Vol}(\mathcal{M}_{0,4}) = 2\pi^2,$$

and

$$\int\limits_{\overline{\mathcal{M}}_{0,4}} \psi_1 = 1.$$

Also, we have

$$\mathrm{Vol}_{2,1}(b) = \frac{(4\pi^2 + b_1^2) \cdot (12\pi^2 + b_1^2) \cdot (6960\, \pi^2 + 384\, \pi^2 b_1^2 + 5\, b_1^4)}{2211840},$$

which implies that

$$\int\limits_{\overline{\mathcal{M}}_{2,1}} \psi_1^4 = \frac{2^4 \cdot 4! \cdot 5}{2211840} = \frac{1}{24^2 \cdot 2}.$$

Remark. It is known [IZ] that in general,

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-2} = \frac{1}{24^g \cdot g!}.$$

Also, a formula for $\mathrm{Vol}_{0,n}(0)$, the Weil-Petersson volume of $\mathcal{M}_{0,n}$, was obtained in [Zo]. Note that there is a small difference in the nolmalization of the volume form; in [Zo] the Weil-Petersson Kähler form is 1/2 the imaginary part of the Weil-Petersson pairing, while here we work with the imaginary part of the pairing. So our answers are different by a power of 2.

There is an exceptional case which arises for $g = n = 1$. In this case generic $X \in \mathcal{M}_{1,1}$ has a symmetry of order 2 which acts non trivially on the cotangent space of X at the marked point point. See [Wit1]. Therefore, the integral of ψ_1 is half of what equation 1.1 predicts. In §5, we show that:

$$\mathrm{Vol}_{1,1}(b) = b^2/24 + \pi^2/6.$$

Hence, we get

$$\mathrm{Vol}(\mathcal{M}_{1,1}) = \pi^2/6,$$

and

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{2} \times \frac{1}{12} = \frac{1}{24},$$

which agree with the known results [Har].

Note that since we are in an orbifold setting the intersection numbers of tautological classes are positive rational numbers which agrees with our result that the leading coefficients of $V_{g,n}(b)$ lie in $\mathbb{Q}_+$.

The main result. By combining equation 1.1 and the recursive formula for the $V_{g,n}(b)$'s obtained in [Mirz], we prove that the generating function for all top intersections of ψ classes in all genera satisfy the Virasoro constraints (§6).

Analogies with moduli spaces of stable bundles. The discussion above suggests some similarities between $\mathcal{M}_{g,n}$ and the variety $\mathrm{Hom}(\pi_1(S), G)/G$ of representations of the fundamental group of the oriented surface S in a compact Lie group G, up to conjugacy. This space is naturally equipped with a symplectic structure [Gol1]. For $G = \mathrm{SU}(2)$, the representation variety is identified with the moduli space of semi-stable holomorphic rank 2 vector bundles over a fixed Riemann surface.

For $\theta_1, \dots, \theta_n \in G$ let

$$R_{g,n}(\theta_1, \dots, \theta_n)$$

be the variety of representations of $\pi_1(S_{g,n})$ in $SU(2)$ such that the monodromy around β_i lies in the conjugacy class of θ_i. Here fixing the conjugacy class of the monodromy around a boundary component β corresponds to fixing the length of β in the case of $\mathcal{M}_{g,n}(b)$.

Like our argument for proving Theorem 6.1, it is possible to derive recursive formulas for intersection numbers of line bundles on $R_{g,n}$ by relating these numbers to the symplectic volume of $R_{g,n}(\theta_1, \ldots, \theta_n)$. This approach was first suggested by Witten [Wit2],and also used in [Weit].

An important difference is that the action of the mapping class does not enter in the $R_{g,n}$ case. The space $R_{g,n}$ is analogous to Teichmüller space, but it has finite volume. Also, the action of the mapping class group on $R_{g,n}(\theta)$ is ergodic [Gol2].

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2 Background material

In this section, We present some familiar concepts in a less familiar setting about the symplectic structure of the moduli space of bordered Riemann surfaces and basic hyperbolic geometry.

Recall that a *symplectic structure* on a manifold M is a non-degenerate closed 2-form $\omega \in \Omega^2(M)$. The n-fold wedge product

$$\frac{1}{n!}\,\omega \wedge \cdots \wedge \omega$$

never vanishes and defines a volume form on M.

First, we briefly summarize basic background material and constructions in Teichmüller theory of Reimann surfaces with geodesic boundary components. For further background see [IT] and [Bus].

Teichmüller Space. A point in the *Teichmüller space* $\mathcal{T}(S)$ is a complete hyperbolic surface X equipped with a diffeomorphism $f : S \to X$. The map f provides a *marking* on X by S. Two marked surfaces $f : S \to X$ and $g : S \to Y$ define the same point in $\mathcal{T}(S)$ if and only if $f \circ g^{-1} : Y \to X$ is isotopic to a conformal map. When ∂S is nonempty, consider hyperbolic

Riemann surfaces homeomorphic to S with geodesic boundary components of fixed length. Let $A = \partial S$ and $L = (L_\alpha)_{\alpha \in A} \in \mathbb{R}_+^{|A|}$. A point $X \in \mathcal{T}(S, L)$ is a marked hyperbolic surface with geodesic boundary components such that for each boundary component $\beta \in \partial S$, we have

$$\ell_\beta(X) = L_\beta.$$

Let $S_{g,n}$ be an oriented connected surface of genus g with n boundary components $(\beta_1, \ldots, \beta_n)$. Then

$$\mathcal{T}_{g,n}(L_1, \ldots, L_n) = \mathcal{T}(S_{g,n}, L_1, \ldots, L_n),$$

denote the Teichmüller space of hyperbolic structures on $S_{g,n}$ with geodesic boundary components of length $L_1, \ldots, L_n$. By convention, a geodesic of length zero is a cusp and we have

$$\mathcal{T}_{g,n} = \mathcal{T}_{g,n}(0, \ldots, 0).$$

Let $\mathrm{Mod}(S)$ denote the mapping class group of S, or the group of isotopy classes of orientation preserving self homeomorphisms of S leaving each boundary component setwise fixed. The mapping class group $\mathrm{Mod}_{g,n} = \mathrm{Mod}(S_{g,n})$ acts on $\mathcal{T}_{g,n}(L)$ by changing the marking. The quotient space

$$\mathcal{M}_{g,n}(L) = \mathcal{M}(S_{g,n}, \ell_{\beta_i} = L_i) = \mathcal{T}_{g,n}(L_1, \ldots, L_n) / \mathrm{Mod}_{g,n}$$

is the moduli space of Riemann surfaces homeomorphic to $S_{g,n}$ with n boundary components of length $\ell_{\beta_i} = L_i$. Also, we have

$$\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(0, \ldots, 0).$$

For a disconnected surface $S = \bigcup_{i=1}^{k} S_i$ such that $A_i = \partial S_i \subset \partial S$, we have

$$\mathcal{M}(S, L) = \prod_{i=1}^{k} \mathcal{M}(S_i, L_{A_i}),$$

where $L_{A_i} = (L_s)_{s \in A_i}$.

The Weil-Petersson symplectic form. By work of Goldman [Gol1], the space $\mathcal{T}_{g,n}(L_1, \ldots, L_n)$ carries a natural symplectic form invariant under the action of the mapping class group. This symplectic form is called the *Weil-Petersson symplectic form,* and denoted by ω or ω_{wp}. We investigate the volume of the moduli space with respect to the volume form induced by the Weil-Petersson symplectic form. Also, when S is disconnected, we have

$$\mathrm{Vol}(\mathcal{M}(S, L)) = \prod_{i=1}^{k} \mathrm{Vol}(\mathcal{M}(S_i, L_{A_i})).$$

When $L = 0$, there is a natural complex structure on $\mathcal{T}_{g,n}$, and this symplectic form is in fact is the Kähler form of a Kähler metric [IT].

The Fenchel-Nielsen coordinates. A *pants decomposition* of S is a set of disjoint simple closed curves which decompose the surface into pairs of pants. Fix a system of pants decomposition of $S_{g,n}$, $\mathcal{P} = \{\alpha_i\}_{i=1}^{k}$, where $k = 6g-6+2n$. For a marked hyperbolic surface $X \in \mathcal{T}_{g,n}(L)$, the *Fenchel-Nielsen coordinates* associated with $\mathcal{P}$, $\{\ell_{\alpha_1}(X), \ldots, \ell_{\alpha_k}(X), \tau_{\alpha_1}(X), \ldots, \tau_{\alpha_k}(X)\}$, consists of the set of lengths of all geodesics used in the decomposition and the set of the *twisting* parameters used to glue the pieces. We have an isomorphism

$$\mathcal{T}_{g,n}(L) \cong \mathbb{R}_{+}^{\mathcal{P}} \times \mathbb{R}^{\mathcal{P}}$$

by the map

$$X \to (\ell_{\alpha_i}(X), \tau_{\alpha_i}(X)).$$

By work of Wolpert, over Teichmüller space the Weil-Petersson symplectic structure has a simple form in Fenchel-Nielsen coordinates [Wol1].

Theorem 2.1 (Wolpert)**.** *The Weil-Petersson symplectic form is given by*

$$\omega_{wp} = \sum_{i=1}^{k} d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}.$$

Twisting. For any simple closed geodesic α on $X \in \mathcal{T}_{g,n}(L)$ and $t \in \mathbb{R}$, we can deform the hyperbolic structure by a right twist as follows. We cut the surface along α and reglue back after twisting distance t to the right. The hyperbolic structure of the complement of the cut extends to a hyperbolic structure of the new surface. Let us denote the new surface by $tw_{t\alpha}(X)$. The resulting continuous path in Teichmüller space is the Fechel-Nielsen deformation of X along α which is generated by the Fenchel-Nielsen vector field. For $t = \ell_\alpha(X)$, we have

$$tw_{t\alpha}(X) = \phi_\alpha(X),$$

where $\phi_\alpha \in \mathrm{Mod}(S_{g,n})$ is a right *Dehn twist* about α. The vector field generated by twisting around α is symplectically dual to the exact one form $d\ell_\alpha$. As a consequence of Theorem 2.1 (See also [Wol1]), we have

Corollary 2.2. *The right twist flow $tw_{t\alpha}$ is the Hamiltonian flow of the length function with respect to the Weil-Petersson symplectic form.*

Compactification of the moduli space. Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford compactification of the moduli space obtained by adjoining curves with simple closed geodesics of length zero or hyperbolic surfaces with *nodes* [Har].

By work of Wolpert [Wol4], the Weil-Petersson symplectic form extends smoothly to the boundary with respect to the Fenchel-Nielsen coordinates. This form is closed and everywhere nondegenerate and therefore defines a symplectic form on $\overline{\mathcal{M}}_{g,n}(L)$. In [Wol3] Wolpert showed that $\omega/\pi^2 \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, and by multiplying $[\omega]/\pi^2$ by some integer, we get a positive line bundle over $\overline{\mathcal{M}}_{g,n}$ which implies that $\overline{\mathcal{M}}_{g,n}$ is a projective algebraic variety. See [Wol3] for more details.

In a similar way, we can compactify the space $\mathcal{M}_{g,n}(L)$ by allowing $\ell_\gamma = 0$ for a simple closed geodesic γ inside the surface. When $L \neq 0$, the moduli space $\mathcal{M}_{g,n}(L)$ does not have a natural complex structure. But it has a natural real-analytic structure from the Fenchel-Nielson coordinates [Wol4]. As it was pointed out to the author by the referee, the approach of describing stable noded curves in terms of hyperbolic surfaces first appeared in a paper by Bers [Bers].

Orbifold structure of the moduli space. Since the action of the mapping class group on Teichmüller space could have fixed points, the space $\mathcal{M}_{g,n}(L)$ is not a manifold. The orbifold points of the moduli space correspond to Riemann surfaces where the automorphism group is non trivial. Since a complete hyperbolic surface can only have finitely many automorphisms, the moduli space is a nice orbifold.

For example, a Riemann surface $X \in \mathcal{M}_{0,n}$ does not have non trivial automorphisms. Therefore, the moduli space $\mathcal{M}_{0,n}$ is a manifold. In general, the moduli space $\overline{\mathcal{M}}_{g,n}(L)$ is a compact orbifold and the Deligne-Mumford compactification locus, $\overline{\mathcal{M}}_{g,n}(L) - \mathcal{M}_{g,n}(L)$, is a union of finitely many lower dimensional suborbifolds intersecting transversely [Har].

In order to apply results that are known for manifolds (e.g. Corollary 3.3), it is important to show that by considering a finite covering of $\overline{\mathcal{M}}_{g,n}(L)$, we can assume that the moduli space is a smooth manifold. We can consider

$$\mathcal{T}_{g,n}(L)/H$$

where H is a torsion free subgroup of $\mathrm{Mod}_{g,n}$.

More precisely, each finite quotient group G of the mapping class group determines a Galois cover $\mathcal{M}_{g,n}(L)[G] \to \mathcal{M}_{g,n}(L)$ and as proved in [Lo2] and [BP], we have

Theorem 2.3. *There exists a finite group G such that $\overline{\mathcal{M}}_{g,n}(L)[G]$ is a smooth manifold, and the compactification locus is a union of codimensional two submanifolds.*

This theorem allows us to use results of the next section on symplectic reduction for studying the moduli space of curves.

Coverings and volume forms of the $\mathcal{M}_{g,n}(L)$'s. Let $\gamma_1, \gamma_2, \dots \gamma_k$ be a set of disjoint simple closed curves on $S_{g,n}$, and $\Gamma = (\gamma_1, \dots, \gamma_k)$. Then any $g \in \mathrm{Mod}_{g,n}$ acts on Γ by

$$g \cdot \Gamma = (g \cdot \gamma_1, \dots, g \cdot \gamma_k).$$

Let $\mathcal{O}_\Gamma$ be the set of homotopy classes of elements of the set $\mathrm{Mod} \cdot \Gamma$. Consider $\mathcal{M}_{g,n}(L)^\Gamma$ defined by the following space of pairs:

$$\{(X, \eta) |\ X \in \mathcal{M}_{g,n}(L)\ ,\ \eta = (\eta_1, \dots, \eta_k) \in \mathcal{O}_\Gamma, \eta_i\text{'s are closed geodesics on } X\}.$$

Let $\pi^\Gamma : \mathcal{M}_{g,n}(L)^\Gamma \to \mathcal{M}_{g,n}(L)$ be the projection map defined by

$$\pi^\Gamma(X, \eta) = X.$$

Let $\phi_\gamma \in \mathrm{Mod}_{g,n}$ denote the Dehn twist along γ. Then

$$G_\Gamma = \bigcap_{i=1}^{s} \mathrm{Stab}(\gamma_i) \subset \mathrm{Mod}(S_{g,n})$$

is generated by the ϕ_{γ_i}'s and elements of the mapping class group of $S_{g,n}(\gamma)$, and

$$\mathcal{M}_{g,n}(L)^\Gamma = \mathcal{T}_{g,n}(L)/G_\gamma.$$

As the Weil-Petersson symplectic structure on Teichmüller space is invariant under the action of the mapping class group, it induces a symplectic structure on $\mathcal{M}_{g,n}(L)^\Gamma$ which is the same as the form $\pi^{\Gamma *}(\omega_{wp})$.

3 Symplectic reduction

In this section we recall some basic facts on symplectic geometry of symplectic quotients [Ki] and Chern-Weil theory of principle circle bundles[MS]. For an interesting exposition of general ideas surrounding symplectic quotients and some applications see [G].

Principal S^1-bundles. Let P and M be smooth manifolds, $\pi : P \to M$ map of P onto M and S^1 act on P. Then (P, S^1, M) is a *Principal* S^1 *bundle* if

1. S^1 acts freely on P.
2. $\pi(p_1) = \pi(p_2)$ if and only if there exists $g \in S^1$ such that $p_1 \cdot g = p_2$.
3. P is locally trivial over M.

In fact, the set of principal circle bundles is an Abelian group.
A connection on a principal S^1 bundle is a smooth distribution H on P such that

1. $T_pP = H_p \bigoplus V_p$, $V_p = \ker \pi_*$, and
2. $g^* H_p = H_{p\cdot g}$.

Vectors in H_p are called *horizontal*. For $v \in T_pP$, we denote the horizontal part by Hv. A connection is uniquely determined by an invariant 1-form A such that $A(X) = 1$, where X is the vector field generating the S^1 action. We can choose the one form defined by

$$A(v) = \frac{< v, X >}{< X, X >},$$

where $<,>$ is an S^1 invariant metric on P.

On the other hand, for any p-form ω on P define $D\omega$ by

$$D\omega(v_1, \dots, v_{p+1}) = d\omega(Hv_1, \dots, Hv_{p+1}).$$

If A is the connection form of H, $\Phi = D(A)$ is called the *curvature form* of H. Then we have

Lemma 3.1. *There exists a unique closed* 2*-form* Ω *on* M *such that* $\Phi = \pi^*\Omega$. *Moreover, the cohomology class of* Ω *is independent of the choice of the connection form, and*

$$c_1(P) = [\Omega] \in H^2(M, \mathbb{Z}).$$

For more details see [McD] and [MS].

Moment map. Let (M, ω) be a symplectic manifold. The *Hamiltonian* vector field ξ_H generated by the function $H : M \to \mathbb{R}$ is the vector field determined by

$$\omega(\xi_H, .) = dH(.).$$

Suppose that a compact Lie group G with Lie algebra g acts smoothly on M and preserves the symplectic form ω. This action gives rise to an infinitesimal action of g that associate to every $\xi \in g$ a vector field $\xi^{\#}$. Then the *moment map* $\mu : M \to g^*$ is defined by

$$d\mu(Y)(X) = \omega_m(X^{\#}, Y),$$

where Y is a vectorfield on M. In other words, the map $\mu_\xi : M \to \mathbb{R}$ defined by the pairing

$$\mu_\xi(m) = \mu(m) \cdot \xi$$

is a Hamiltonian function for the vector field on M induced by ξ. Assume that the map μ is proper. Because the moment map μ is G invariant, G acts on each level set of the moment map. The *reduced space* is the quotient

$$M_a = \mu^{-1}(a)/G$$

for any $a = (a_1, \ldots, a_n)$ in the image of μ. The space M_a inherits a symplectic form ω_a from the symplectic structure on M.

Example. Let

$$\pi : M_r = \mu^{-1}(0) \times [-r, r] \to \mu^{-1}(0)$$

be the projection map. If A is an S^1 invariant connection on $\mu^{-1}(0)$, then we can define an S^1 invariant 2 form, w_r by

$$\omega_r = \pi^*\omega + d(tA).$$

When r is small ω_r is a symplectic form on M_r, and the S^1 action on M_r is the Hamiltonian flow of the moment map defined by

$$\mu_r(x, t) = t.$$

Remark. If 0 is a regular value of μ, by the coisotropic embedding theorem there is a neighborhood of $\mu^{-1}(0)$ on which the symplectic form is given as in the above example [G]. This is a generalization of Darboux's theorem stating that symplectic manifolds do not have any local invariants([McD]).

Variation of the reduced form and volume. When a is close to 0, M_a is diffeomorphic to M_0. It is important to know how the symplectic geometry of M_a varies when one varies a.

When $G = T_n = S_1^n$, the action of G on the level set $\mu^{-1}(a)$ gives rise to n circle bundles, $\mathcal{C}_1, \ldots, \mathcal{C}_n$ defined over M_a.

$$\begin{array}{ccccc} T^n & \longrightarrow & \mu^{-1}(a) & \longrightarrow & M \\ & & \big\downarrow & & \\ & & M_a = \mu^{-1}(a)/T^n & & \end{array}$$

Let $\phi_i = c_1(\mathcal{C}_i)$. Let v_j be the vector field corresponding to the action of the jth copy of S^1. Fix a connection α on $\mu^{-1}(0)$; that is a S^1-invariant action one form such that we have

$$\alpha(v_j) = 1.$$

The following result shows that w_a varies linearly in a ([G]) :

Theorem 3.2 (Normal form theorem)**.** *The space (M_a, w_a) is symplectomorphic to M_0 equipped with the symplectic form $w_0 + a\Omega$, where Ω is the curvature form of the connection α.*

For $a = (a_1, \ldots, a_n)$ with $|a| \leq \epsilon$, M_a and M_0 are diffeomorphic. Since $c_1(\mathcal{C}) = [\Omega]$, under this diffeomorphism the cohomology classes of the symplectic forms are related by :

$$[w_a] = [w] + \sum_{i=1}^{n} a_i \cdot [\phi_i],$$

where $\phi_i = c_1(\mathcal{C}_i)$.

Remark. This theorem is closely related to a version of the Duistermaat-Heckman theorem asserting that the pushforward of the symplectic measure by the moment map for a torus action is a piecewise polynomial. For more details see [G].

Now by integrating w_a over the space M_a, we get:

Corollary 3.3. *Let 0 be a regular value of the proper moment map $\mu : M \to \mathbb{R}^n$ of the Hamiltonian action of T^n on M. Then for sufficiently small $\epsilon > 0$ and $a \in \mathbb{R}^n_+$ with $|a| \leq \epsilon$, the volume of $M_a = \mu^{-1}(a)/T^n$ is a polynomial in $a_1, \ldots, a_n$ of degree $m = \dim(M_a)/2$ given by*

$$\sum_{\substack{\alpha \\ |\alpha| \leq m}} C(\alpha) \cdot a^\alpha,$$

where

$$\alpha!\,(m - |\alpha|)!\,C(\alpha) = \int_{M_0} \phi_1^{\alpha_1} \cdots \phi_n^{\alpha_n} \cdot \omega^{m-|\alpha|}.$$

Here the exponent $\alpha = (\alpha_1, \ldots, \alpha_n)$ ranges over elements in $\mathbb{Z}^n_{\geq 0}$, $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n}$, $|\alpha| = \sum_{i=1}^{n} \alpha_i$ and $\alpha! = \prod_{i=1}^{n} \alpha_i!$.

4 Volumes of moduli spaces of bordered Riemann surfaces

In this section we establish a relationship between the volume polynomials and intersection numbers of tautological classes over moduli space.
Collar curves. Define the function $S(x)$ by

$$S(x) = \operatorname{arcsinh}\left(\frac{1}{\sinh(x/2)}\right).$$

For a simple closed geodesic γ on a hyperbolic surface X, there is a collar neighborhood of width $S(\ell_\gamma(X))$ which is an embedded annulus. Also, two simple closed geodesics are disjoint if and only of their collars are disjoint [Bus]. Therefore, there exists a continuous function $F : \mathbb{R}_+ \to \mathbb{R}_+$ such that

- For each boundary component β_i of $X \in \mathcal{T}_{g,n}(L)$, there is a curve $\widetilde{\beta_i}$ of constant curvature of length $F(\ell_{\beta_i}(X))$ inside the collar neighborhood of β_i, and
- $\lim\limits_{x \to 0} F(x) = 1/4$.

As $\ell_i \to 0$, $\widetilde{\beta_i}$ tends to a horocycle of length $1/4$ around the corresponding puncture. When $\ell_{\beta_i}(X) > 0$, there is a canonical bijection between $\widetilde{\beta_i}$ and β_i.
Geometric circle bundles. The orientation on $S_{g,n}$ defines a canonical orientation on its boundary components as follows. Let β_i be a boundary component of $X \in \mathcal{T}_{g,n}(L)$, $x \in \beta_i$, and N_x an outward vector normal to β_i at x. Then a tangent vector v_x to β_i is positive iff the pair (v_x, N_x) has positive orientation with respect to the orientation of X.

Now let $\gamma_i : [0, L_i] \to \beta_i$ be an oriented arc length parameterization of β_i. For any $t \in [0, L_i]$ define $\xi^t : \beta_i \to \beta_i$ by

$$\xi^t(\gamma_i(s)) = \gamma_i(s + t \cdot L_i).$$

As $\xi^{t+1} = \xi^t$, ξ defines an S^1-action on β_i.

Let $\widetilde{\beta_i}$ a curve parallel to the boundary component β_i on $X \in \mathcal{T}_{g,n}(L)$. The advantage of working with the paralled curve instead of the boundary component is that $\widetilde{\beta_i}$ has positive length even when the geodesic length of β_i is zero in which case $\widetilde{\beta_i}$ is a horocycle around the puncture p_i. Otherwise, there is a canonical one-to-one map between $\widetilde{\beta_i}$ and β_i. Also $\widetilde{\beta_i}$ is disjoint

from $\widetilde{\beta}_j$ when $i \neq j$.
Define $\mathcal{S}_i(\mathcal{T}_{g,n}(L))$ by

$$\mathcal{S}_i(\mathcal{T}_{g,n}(L)) = \{(X,p) \mid p \in \widetilde{\beta}_i,\ X \in \mathcal{T}_{g,n}(L)\} \to \mathcal{T}_{g,n}(L)\}.$$

On the other hand, the mapping class group $\mathrm{Mod}_{g,n}$ acts on $\mathcal{S}_i(\mathcal{T}_{g,n}(L))$. Since the stabilizer of any point is finite, the quotient space $\mathcal{S}_i(\mathcal{M}_{g,n}(L))$ is a circle bundle over $\mathcal{M}_{g,n}(L)$ in the orbifold sense. Also, the circle bundle can be similarly defined over $X \in \overline{\mathcal{M}_{g,n}}(L)$ where the length of a simple closed geodesic inside the surface can be zero. It is essential that the parallel curve $\widetilde{\beta}_i$ is always disjoint from the possible singular points of $X \in \overline{\mathcal{M}}_{g,n}(L)$. Therefore, we have

Lemma 4.1. *For any $1 \leq i \leq n$ and $L \in (\mathbb{R}_+)^n$, $(\mathcal{S}_i(L), S^1, \overline{\mathcal{M}}_{g,n}(L))$ is a principal circle bundle over $\overline{\mathcal{M}}_{g,n}(L)$ in the orbifold sense.*

Tautological classes. Now we consider the case when the length of all boundary components is zero. Since $\overline{\mathcal{M}}_{g,n}$ is an orbifold, the first Chern class of the circle bundle $\mathcal{S}_i$ defines an element of the cohomology class of the moduli space

$$[c_1(\mathcal{S}_i)] \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

In this part, we will relate the first Chern class of $\mathcal{S}_i$ to the tautological class

$$\psi_i = c_1(\mathcal{L}_i).$$

Each $X \in \mathcal{T}_{g,n}$ naturally gives rise to a complex 1-manifold via its uniformization. In fact, there is a unique compact complex curve C and finitely many points $p_1, \ldots, p_n$ on C such that X is conformally equivalent to $C - \{p_1, \ldots, p_n\}$.

Note that each cusp neighborhood of X is conformally equivalent to a punctured disk [Bus]. Around each boundary component p_i, we consider the parallel curve $\widetilde{\beta}_i$ as defined earlier in this section. Let $\Delta \subset \mathbb{C}$ be the unit disk. Then any element of the tangent space at the origin corresponds to a point on $\widetilde{\beta}_i$, the horocycle around the origin with respect to the hyperbolic structure on $\Delta - \{0\}$. But the orientation we put on β_i earlier in this section is different from the one induced by the orientation on tangent vectors at x.

On the other hand as $\mathcal{L}_i$ is a complex bundle, the underlying real vector bundle has a canonical orientation. Therefore the duality between the tangent and cotangent space at p_i will give us an orientation reversing isomorphism between the line bundle $\mathcal{L}_i$ and the circle bundle $\mathcal{S}_i$ with reverse orientation. Therefore, we can establish the following result:

Theorem 4.2. *For any $1 \leq i \leq n$, we have:*

$$[c_1(\mathcal{S}_i)] = [\psi_i] \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

where ψ_i is the ith tautological class over $\overline{\mathcal{M}}_{g,n}$.

Remark. From now on, we only deal with the circle bundle $\mathcal{S}_i$ and forget about the complex structure of $\mathcal{L}_i$. Later, we will use the Chern-Weil description of Characteristic classes in terms of the curvature form for calculating the intersection numbers. See Appendix C of [MS] for more details.
Moduli space of bordered Riemann surfaces. Now we consider the moduli spaces of bordered Riemann surfaces with marked points (without fixing the lengths of the boundary components)

$$\widehat{\mathcal{M}_{g,n}} = \{(X, p_1, \ldots, p_n) \mid p_i \in \widetilde{\beta}_n, X \in \overline{\mathcal{M}}_{g,n}(L_1, \ldots, L_n), L_i \geq 0\}.$$

Define the map $\ell : \widehat{\mathcal{M}_{g,n}} \to \mathbb{R}^n_+$ by

$$\ell(X, p_1, \ldots, p_n) = (\ell_{\beta_1}(X), \ldots, \ell_{\beta_n}(X)).$$

On the other hand, we have a natural $T^n = S^n_1$ action on the space $\widehat{\mathcal{M}_{g,n}}$ as follows. For each $1 \leq i \leq n$, S^1_i acts by moving p_i on the curve $\widetilde{\beta}_i$, that is

$$\xi^t_i(X, p_1, \ldots, p_n) = (X, p_1, \ldots, \xi^t(p_i), \ldots, p_n).$$

The goal of this part is to show that the Weil-Petersson symplectic form defines a symplectic form on $\widehat{\mathcal{M}_{g,n}}$ with respect to which the T^n action is the Hamiltonian flow of the function $\ell^2/2$. The key tool is the extension of the Weil-Petersson symplectic form to the compactification locus of the moduli space.
Extension of the Weil-Petersson symplectic form to $\overline{\mathcal{M}}_{g,n}(b)$. As we mentioned in §2, the moduli space $\overline{\mathcal{M}}_{g,n}(b)$ has a natural real analytic structure arising from the Fenchel-Nielsen coordinates [Wol4].

By work of Wolpert [Wol4], Weil-Peterssen symplectic form has a smooth extension ω^{FN} to $\overline{\mathcal{M}}_{g,n}(L)$ (§2). Using the extension of the Weil-Petersson symplectic form, we can define a T^n invariant symplectic form on $\widehat{\mathcal{M}_{g,n}}$.
Remark. There is a different method for extending the Weil-Peterssen symplectic form to $\overline{\mathcal{M}}_{g,n}$ by using a closed current ω^C relative to the complex structure of $\mathcal{M}_{g,n}$. But the complex structure and the Fenchel Nielsen coordinates do not have the same smooth structure on $\overline{\mathcal{M}}_{g,n}$. In [Wol4], Wolpert showed that ω^{FN} and ω^C determine the same cohomology class.

Theorem 4.3. *The orbifold $\widehat{\mathcal{M}_{g,n}}$ has a natural T^n-invariant symplectic structure such that*

1. *The map*
$$\ell^2/2 = (\ell_{\beta_1}(X)^2/2, \ldots, \ell_{\beta_n}(X)^2/2).$$
is the moment map for the action of T^n on $\widehat{\mathcal{M}_{g,n}}$.

2. *The canonical map*
$$s : \ell^{-1}(L_1, \ldots, L_n)/T \to \overline{\mathcal{M}}_{g,n}(L_1, \ldots, L_n)$$
is a symplectomorphism.

Note that the restriction of this symplectic form to $\overline{\mathcal{M}}_{g,n}(0, \ldots, 0)$ is just the Weil-Petersson symplectic form.

Proof. Let $S_{g,2n}$ be a surface of genus g with $2n$ boundary components $\beta_1, \ldots, \beta_{2n}$. We fix n simple closed curves $\gamma_1, \ldots, \gamma_n$ on $S_{g,2n}$ such that γ_i bounds a pair of pants with β_{2i-1} and β_{2i}, and let $\Gamma = (\gamma_1, \ldots, \gamma_n)$. Let $\mathcal{O}_\Gamma$ be the set of homotopy classes of elements of the set $\mathrm{Mod}_{g,2n} \cdot \Gamma$. We consider $\overline{\mathcal{M}_{g,2n}}^{\Gamma}$ defined by:

$$\{(X, \eta) |\ X \in \overline{\mathcal{M}}_{g,2n}\ ,\ \eta = (\eta_1, \ldots, \eta_n) \in \mathcal{O}_\Gamma, \eta_i\text{'s are closed geodesics on } X\}.$$

Note that by Wolpert's result, the symplectic form induced by the Weil-Petersson form on $\mathcal{M}^{\Gamma}_{g,2n}$ extends to $\overline{\mathcal{M}_{g,2n}}^{\Gamma}$. See §2 for more details.

On the other hand, each boundary of a pair of pants has two canonical points corresponding to the other two boundary components, the end points of the length minimizing geodesics connecting to the other boundaries.

Therefore we get a map
$$f : \widehat{\mathcal{M}_{g,n}} \to \overline{\mathcal{M}_{g,2n}}^{\Gamma},$$
where for $X \in \overline{\mathcal{M}}_{g,n}(L_1, \ldots, L_n)$, $f(X, p_1, \ldots, p_n)$ is a surface of genus g with $2n$ punctures that we get by gluing n pairs of pants $\Sigma_1, \ldots, \Sigma_n$ with boundary lengths $(L_i, 0, 0)$ to boundary components of X so that the point p_i is adjacent to the canonical point on the boundary of Σ_i corresponding to β_{2i-1}.

Then f is a diffeomorphism and defines a symplectic form on $\widehat{\mathcal{M}_{g,n}}$ coming from the Weil-Petersson symplectic form on $\overline{\mathcal{M}_{g,2n}}^{\Gamma}$. Now the result is immediate by using Corollary 2.2.

□

Theorem 4.4. *The coefficients of the volume polynomial*

$$\mathrm{Vol}(\mathcal{M}_{g,n}(L_1,\ldots,L_n)) = \sum_{|\alpha|\leq 3g-3+n} C_g(\alpha)\cdot L^{2\alpha}$$

are given by

$$C_g(\alpha_1,\ldots,\alpha_n) = \frac{2^{m(g,n)|\alpha|}}{2^{|\alpha|}\;|\alpha|!\;(3g-3+n-|\alpha|)!}\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}\cdots\psi_n^{\alpha_n}\cdot\omega^{3g-3+n-|\alpha|},$$

where ψ_i is the first Chern class of the i-th tautological line bundle and ω is the Weil-Petersson symplectic form. Here $m(g,n)=\delta(g-1)\times\delta(n-1)$, $\alpha! = \prod \alpha_i!$ and $|\alpha| = \sum_{i=1}^n \alpha_i$.

Proof. Note that by Theorem 2.3, we can assume that the moduli space is a manifold. Using Theorem 4.3, the result is an immediate Corollary of Theorem 4.2 and Theorem 3.3 for $\mu = \ell^2/2$. See the introduction for the exceptional case when $g=n=1$. □

5 A recursive formula for Weil-Petersson volumes

In this section we state a recursive formula for the $V_{g,n}(L)$'s obtained in [Mirz]. The recursive formula (equation 5.7) relates the volume polynomial $V_{g,n}(L)$ to the volume polynomials of the moduli spaces of Riemann surfaces that we get by cutting one pair of pants from $S_{g,n}$.

An identity for the lengths of simple closed geodesics. Our point of departure for calculating these volume polynomials is an identity [M] for the lengths of simple closed geodesics on a punctured hyperbolic Riemann surface.

Theorem 5.1 (Generalized McShane identity for bordered surfaces)**.** *For any $X \in \mathcal{T}_{g,n}(b_1,\ldots,b_n)$ with $3g-3+n>0$, we have*

$$\sum_{(\alpha_1,\alpha_2)} \mathcal{D}(b_1,\ell_{\alpha_1}(X),\ell_{\alpha_2}(X)) + \sum_{i=2}^{n}\sum_{\gamma}\mathcal{R}(b_1,b_i,\ell_\gamma(X)) = b_1. \qquad (5.1)$$

Here the first sum is over all unordered pairs of simple closed geodesics (α_1,α_2) bounding a pair of pants with β_1, and the second sum is over simple closed geodesics γ bounding a pair of pants with β_1 and β_i.

In fact we have

$$\mathcal{D}(x,y,z) = 2\ \log\left(\frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{\frac{-x}{2}} + e^{\frac{y+z}{2}}}\right), \tag{5.2}$$

and

$$\mathcal{R}(x,y,z) = x - \log\left(\frac{\cosh(\frac{y}{2}) + \cosh(\frac{x+z}{2})}{\cosh(\frac{y}{2}) + \cosh(\frac{x-z}{2})}\right). \tag{5.3}$$

Define $H : \mathbb{R}^2 \to \mathbb{R}$ by

$$H(x,y) = \frac{1}{1+e^{\frac{x+y}{2}}} + \frac{1}{1+e^{\frac{x-y}{2}}}.$$

It is easy to check that:

$$\frac{\partial}{\partial x}\mathcal{D}(x,y,z) = H(y+z,x), \tag{5.4}$$

and

$$\frac{\partial}{\partial x}\mathcal{R}(x,y,z) = \frac{1}{2}(H(z,x+y) + H(z,x-y)). \tag{5.5}$$

In [Mirz], we also develop a method to integrate the generalized identity over certain coverings of $\mathcal{M}_{g,n}(b_1,\ldots,b_n)$. As a result, we obtain a recursive formula for the $V_{g,n}(b)$'s without having to find a fundamental domain for the action of the mapping class group on the Teichmüller space [Mirz].

Calculation of $V_{1,1}(L)$. Before stating the recursive formula we sketch the main idea of the calculation of the $V_{g,n}(L)$'s through an example when $g = n = 1$. In this case, using Theorem 5.1 for a hyperbolic surface of genus one with one geodesic boundary component implies that for any $X \in \mathcal{T}(S_{1,1}, L)$, we have

$$\sum_{\gamma} \mathcal{D}(L, \ell_\gamma(X), \ell_\gamma(X)) = L,$$

where the sum is over all simple closed curves γ on $S_{1,1}$. Also, we have

$$\frac{\partial}{\partial L}\mathcal{D}(L,x,x) = \frac{1}{1+e^{x-\frac{L}{2}}} + \frac{1}{1+e^{x+\frac{L}{2}}}.$$

Using the method developed in [Mirz] for integrating the left hand side of the identity over $\mathcal{M}_{1,1}(L)$, we get

$$L \cdot V_{1,1}(L) = \int_0^\infty x\ \mathcal{D}(L,x,x)\ dx.$$

So we have

$$\frac{\partial}{\partial L}L\cdot V_{1,1}(L)=\int_0^\infty x\cdot(\frac{1}{1+e^{x+\frac{L}{2}}}+\frac{1}{1+e^{x-\frac{L}{2}}})\,dx.$$

By setting $y_1 = x + L/2$ and $y_2 = x - L/2$, we get

$$\int_0^\infty x\cdot(\frac{1}{1+e^{x+\frac{L}{2}}}+\frac{1}{1+e^{x-\frac{L}{2}}})\,dx=\int_{L/2}^\infty \frac{y_1-L/2}{1+e^{y_1}}dy_1+\int_{-L/2}^\infty \frac{y_2+L/2}{1+e^{y_2}}\,dy_2=$$

$$=2\int_0^\infty \frac{y}{1+e^y}\,dy+\int_0^{L/2}\frac{y-L/2}{1+e^y}\,dy+\int_0^{-L/2}\frac{y+L/2}{1+e^y}\,dy=$$

$$\frac{\pi^2}{6}+\int_0^{L/2}(y-L/2)(\frac{1}{1+e^y}+\frac{1}{1+e^{-y}})\,dy=\frac{\pi^2}{6}+\frac{L^2}{8},$$

since we have

$$\frac{1}{1+e^y}+\frac{1}{1+e^{-y}}=1.$$

Therefore, we have:

$$V_{1,1}(L)=\frac{L^2}{24}+\frac{\pi^2}{6}. \tag{5.6}$$

Remark. This result agrees with the result obtained in [NN].

Statement of the recursive formula. Now we state a recursive formula for $V_{g,n}(L)$, the Weil-Petersson volume of $\mathcal{M}_{g,n}(L)$ [Mirz].

The volume function $V_{g,n}(L_1,\ldots,L_n)$ is a symmetric function in $L_1,\ldots,L_n$. Hence for any set A of positive numbers with $|A| = n$, we can define $V_{g,n}(A)$ by

$$V_{g,n}(A)=V_{g,n}(a_1,\ldots,a_n),$$

where $\{a_1,\ldots,a_n\} = A$.

In the simplest case when $n = 3$ and $g = 0$, the moduli space $\mathcal{M}_{0,3}(L_1, L_2, L_3)$ consists of only one point, so we let

$$V_{0,3}(L_1,L_2,L_3)=1.$$

The function $V_{g,n}(L_1,\ldots,L_n)$ for any g and n $(2g-2+n>0)$ is determined recursively as follows :

- For any $L_1, L_2, L_3 \geq 0$, set

$$V_{0,3}(L_1, L_2, L_3) = 1,$$

and

$$V_{1,1}(L_1) = \frac{L_1^2}{24} + \frac{\pi^2}{6}.$$

- Let $\widehat{L} = (L_2, \ldots, L_n)$. When $(g,n) \neq (1,1), (0,3)$, the volume $V_{g,n}(L) = \mathrm{Vol}(\mathcal{M}_{g,n}(L))$ is inductively determined by :

$$\frac{\partial}{\partial L_1} L_1 V_{g,n}(L) = \mathcal{A}^{con}_{g,n}(L_1, \widehat{L}) + \mathcal{A}^{dcon}_{g,n}(L_1, \widehat{L}) + \mathcal{B}_{g,n}(L_1, \widehat{L}), \tag{5.7}$$

where the functions

$$\mathcal{A}^{con}_{g,n}(L_1, \widehat{L}) = \frac{1}{2} \int_0^\infty \int_0^\infty x\, y\, \widehat{\mathcal{A}}^{con}_{g,n}(x, y, L_1, \widehat{L})\, dx\, dy, \tag{5.8}$$

$$\mathcal{A}^{dcon}_{g,n}(L_1, \widehat{L}) = \frac{1}{2} \int_0^\infty \int_0^\infty x\, y\, \widehat{\mathcal{A}}^{dcon}_{g,n}(x, y, L_1, \widehat{L})\, dx\, dy, \tag{5.9}$$

and

$$\mathcal{B}_{g,n}(L_1, \widehat{L}) = \int_0^\infty x \cdot \widehat{\mathcal{B}}_{g,n}(x, L_1, \widehat{L})\, dx, \tag{5.10}$$

are defined in terms of the $V_{h,m}(L)$'s with $3h + m < 3g + n$ as follows.

First, we define the functions

$$\widehat{\mathcal{A}}^{con}_{g,n} : \mathbb{R}_+^{n+2} \to \mathbb{R}_+,$$

$$\widehat{\mathcal{A}}^{dcon}_{g,n} : \mathbb{R}_+^{n+2} \to \mathbb{R}_+,$$

and

$$\widehat{\mathcal{B}}_{g,n} : \mathbb{R}_+^{n+1} \to \mathbb{R}_+.$$

To do this, we need the function $H : \mathbb{R} \to \mathbb{R}_+$ defined by

$$H(x,y) = \frac{1}{1 + e^{\frac{x+y}{2}}} + \frac{1}{1 + e^{\frac{x-y}{2}}}.$$

Also, as in the Introduction, let

$$m(g,n) = \delta(g-1) \times \delta(n-1).$$

Namely, $m(g,n) = 0$ unless $g = 1$ and $n = 1$.

I) : Definition of $\widehat{\mathcal{A}}^{con}_{g,n}$. Define $\widehat{\mathcal{A}}^{con}_{g,n} : \mathbb{R}^{n+2}_{+} \to \mathbb{R}_{+}$ by

$$\widehat{\mathcal{A}}^{con}_{g,n}(x,y,L_1,\ldots,L_n) = \frac{V_{g-1,n+1}(x,y,\widehat{L})}{2^{m(g-1,n+1)}} \cdot H(x+y,L_1).$$

II) : Definition of $\widehat{\mathcal{A}^{dcon}}_{g,n}$. Let $\mathcal{I}_{g,n}$ be the set of ordered pairs

$$a = ((g_1,I_1),(g_2,I_2))$$

where $I_1, I_2 \subset \{2,\ldots,n\}$ and $0 \leq g_1, g_2 \leq g$ such that the following holds:

1. The two sets I_1 and I_2 are disjoint and $\{2,3,\ldots,n\} = I_1 \sqcup I_2$.
2. The numbers $g_1, g_2 \geq 0$ and $n_1 = |I_1|, n_2 = |I_2|$ satisfy

$$2 \leq 2g_1 + n_2,$$

$$2 \leq 2g_2 + n_2,$$

and

$$g_1 + g_2 = g.$$

For notational convenience, given $L = (L_1,\ldots,L_n)$ and $I \subset \{1,\ldots,n\}$ with $|I| = k$, define L_I by

$$L_I = (L_{j_1},\ldots,L_{j_k}),$$

where $I = \{j_1,\ldots,j_k\}$.

For

$$a = ((g_1,I_1),(g_2,I_2)) \in \mathcal{I}_{g,n},$$

let

$$V(a,x,y,\widehat{L}) = \frac{V_{g_1,n_1+1}(x,L_{I_1})}{2^{m(g_1,n_1+1)}} \times \frac{V_{g_2,n_2+1}(y,L_{I_2})}{2^{m(g_2,n_2+1)}}.$$

Finally, define $\widehat{\mathcal{A}}^{dcon}_{g,n} : \mathbb{R}^{n+2}_{+} \to \mathbb{R}_{+}$ by

$$\widehat{\mathcal{A}}^{dcon}_{g,n}(x,y,L_1,\widehat{L}) = \sum_{a \in \mathcal{I}_{g,n}} V(a,x,y,\widehat{L}) \cdot H(x+y,L_1).$$

III) : Definition of $\widehat{\mathcal{B}}_{g,n}$. Finally, define $\widehat{\mathcal{B}}_{g,n} : \mathbb{R}^{n+1}_{+} \to \mathbb{R}_{+}$ by

$$\widehat{\mathcal{B}}_{g,n}(x,L_1,\widehat{L}) = \frac{1}{2^{m(g,n-1)}} \times$$

$$\sum_{j=2}^{n} \frac{1}{2}(H(x, L_1+L_j)+H(x, L_1-L_j)) \cdot V_{g,n-1}(x, L_2, \ldots, \hat{L_j}, \ldots, L_n). \quad (5.11)$$

Remark. Note that in our recursive formula, we always have to divide by 2 when we are dealing with a simple closed geodesic γ separating off a one handle. The main reason is that in this case the stabilizer of γ contains a half twist. See [Mirz] for more details.

Connection with topology of the set of pairs of pants. Although the recursive formula 5.7 has been described in purely combinatorial terms, it is closely related to the topology of different types of pairs of pants in a surface. In fact, this formula gives us the volume of $\mathcal{M}_{g,n}(L)$ in terms of volumes of moduli spaces of Riemann surfaces that we get by removing a pair of pants containing at least one boundary component of $S_{g,n}$. Also, the second condition in the definition of $\mathcal{I}_{g,n}$ is equivalent to the condition that the universal covering spaces of the complementary regions of the pair of pants are both conformally equivalent to the upper half plane [Mirz].

Remark. The functions $\mathcal{A}^{con}_{g,n}$, $\mathcal{A}^{dcon}_{g,n}$ and $\mathcal{B}_{g,n}$ are determined by the functions $\{V_{i,j}\}$ where $3i+j < 3g+n$. Therefore equation (5.7) is a recursive formula for calculating $V_{g,n}(L)$. In fact, we can simplify this recursive formula and use it to prove that $V_{g,n}(L)$ is a polynomial in L (Theorem 1.1).

Calculating the coefficients of $V_{g,n}(b)$. The following elementary observations are our main tools for simplifying the recursive formula.

For $i \in \mathbb{N}$, define $F_{2i+1} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$F_{2k+1}(t) = \int_0^\infty x^{2k+1} \cdot H(x,t)\, dx.$$

These functions play a key role in the calculation of $\mathrm{Vol}_{g,n}(L)$. It is easy to express the other terms in $\mathcal{B}_{g,n}(b)$ in terms of the F's. By setting $z = x+y$, we get

$$\int_0^\infty \int_0^\infty x^{2i+1} \cdot y^{2j+1} \cdot H(x+y,t)\; dx\; dy = \int_0^\infty \int_0^z (z-y)^{2i+1} \cdot y^{2j+1} \cdot H(z,t)\; dy\; dz =$$

$$= \frac{(2i+1)! \cdot (2j+1)!}{(2i+2j+3)!} \int_0^\infty z^{2i+2j+3} H(z,t)\; dz.$$

Therefore, we have

$$\int_0^\infty \int_0^\infty x^{2i+1} \cdot y^{2j+1} \cdot H(x+y,t)\; dx\; dy = \frac{(2i+1)! \cdot (2j+1)!}{(2i+2j+3)!} F_{2i+2j+3}(t). \quad (5.12)$$

The following lemma helps us to proceed to the calculation of $\mathrm{Vol}_{g,n}(L)$.

Lemma 5.2. *The function $F_{2k+1}(t)$ is given by*

$$\frac{F_{2k+1}(t)}{(2k+1)!} = \sum_{i=0}^{k+1} \zeta(2i)\ (2^{2i+1} - 4) \cdot \frac{t^{2k+2-2i}}{(2k+2-2i)!}.$$

Therefore $F_{2k+1}(t)$ is a polynomial in t^2 of degree $k+1$ such that the coefficient of $m^{2k+2-2i}$ lies in $\pi^{2i} \cdot \mathbb{Q}_{>0}$.

Remark. Here $\zeta(0) = -1/2$, and therefore the leading coefficient of the polynomial $F_{2k+1}(t)$ is $t^{2k+2}/2k+2$.

Leading coefficients of $V_{g,n}(L)$. As we will see later, calculating the leading coefficients of $V_{g,n}(L)$ turns out to be easier than calculating other terms; the recursive formula simplifies when $\sum \alpha_i = 3g - 3 + n$.

Simplifying $\mathcal{A}^{con}_{g,n}$ and $\mathcal{A}^{dcon}_{g,n}$. we will use the following observation in order to simplify infinite integrals.

Let $P(x, y)$ be a polynomial of degree d in x^2 and y^2 of the form

$$P(x, y) = \sum_{1 \leq i+j \leq d} C(i, j)\ x^{2i}\ y^{2j}.$$

Then equation 5.12 and Lemma 5.2 imply that the function

$$\widehat{P}(x) = \int_0^\infty \int_0^\infty y_1\ y_2\ H(y_1 + y_2, x)\ P(y_1, y_2)\ dy_1\ dy_2$$

is a polynomial in x^2 whose leading term is

$$\sum_{i+j=d} \frac{(2i+1)!(2j+1)!}{(2d+4)!} C(i, j) x^{2d+4}.$$

Simplifying $\mathcal{B}_{g,n}$. Let $Q(x)$ be a polynomial of degree d in x^2 of the form

$$Q(x) = \sum_{i=0}^{d} C(i)\ x^{2i}.$$

Then the function

$$\widehat{Q}(x, y) = \frac{1}{2} \int_0^\infty t\ Q(t)\ (H(t, x+y) + H(t, x-y))\ dt$$

is a polynomial of degree $d+2$ in x^2. Using Lemma 5.2 we can calculate this polynomial explicitly, and prove that the term corresponding to $x^{2i}y^{2j}$ when $i+j=d+2$ is equal to

$$(2d+1)!\,C(d)\,\frac{x^{2i}y^{2j}}{(2i)!\,(2j)!}.$$

Notation. As we mentioned in the Introduction, the case of $g=n=1$ is exceptional. We will see later that it would be easier to work with

$$\widehat{C}_g(\alpha)=\frac{C_g(\alpha)}{2^{m(g,n)}},$$

where $m(g,n)=\delta(g-1)\times\delta(n-1)$ which is zero except when $g=n=1$.

For

$$a=((g_1,I_1),(g_2,I_2))\in\mathcal{I}_{g,n},$$

let $i(a),j(a)\in\mathbb{Z}$ be such that

$$i(a)+\sum_{k=1}^{n_1}\alpha_{i_k}=3g_1-3+n_1+1$$

$$j(a)+\sum_{k=1}^{n_2}\alpha_{j_k}=3g_2-3+n_2+1.$$

Infact, we have

$$i(a)+j(a)=\alpha_1-2$$

Finally, let $F[\alpha]$ denote the coefficient of $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ in the polynomial $F(x_1,\ldots,x_n)$. Now, we can use the recursive formula for the volume polynomials to prove the following result:

Lemma 5.3. *In terms of the above notation, the coefficients of the polynomials $\mathcal{A}^{dcon}_{g,n}(L)$, $\mathcal{A}^{con}_{g,n}(L)$ and $\mathcal{B}_{g,n}(L)$ are given by*

$\mathcal{A}^{dcon}_{g,n}(L)[\alpha]=$

$$\frac{2\alpha_1+1}{2}\sum_{a\in\mathcal{I}_{g,n}}\frac{(2i(a)+1)!\,(2j(a)+1)!}{(2\alpha_1+1)!}\,\widehat{C}_{g_1}(i(a),\alpha_{i_1},\ldots,\alpha_{i_{n_1}})\cdot\widehat{C}_{g_2}(j(a),\alpha_{j_1},\ldots,\alpha_{j_{n_2}}),$$

$\mathcal{A}^{con}_{g,n}(L)[\alpha]=$

$$\frac{2\alpha_1+1}{2}\sum_{i+j=\alpha_1-2}\frac{(2i+1)!\,(2j+1)!}{(2\alpha_1+1)!}\,\widehat{C}_{g-1}(i,j,\alpha_2,\ldots,\alpha_n),$$

and

$\mathcal{B}_{g,n}(L)[\alpha] =$

$$(2\alpha_1+1)\ \sum_{j=2}^{n} \frac{(2(\alpha_1+\alpha_j-1)+1)!}{(2\alpha_1+1)!\,(2\alpha_j)!}\ \widehat{C}_g(\alpha_1+\alpha_j-1,\alpha_2,\ldots,\widehat{\alpha_j},\ldots,\alpha_n).$$

Proof. Here we sketch the proof of the first part. Fix $a \in \mathcal{I}_{g,n}$. It is enough to find the coefficient of L^{α} in

$$\int_0^{\infty}\int_0^{\infty} x\ y\ V_{g_1,n_1}(x,L_{I_1}) \times V_{g_2,n_2}(x,L_{I_2})\ H(x+y,L_1)dx\,dy.$$

Now using Theorem 1.1 , $V_{g_1,n_1}(L_{I_1}) \times V_{g_2,n_2}(L_{I_2})$ is a polynomial in L. So can use the preceding lemma to calculate the double integral; it is enough to calculate

$$\int_0^{\infty}\int_0^{\infty} x^{2i+1}\ y^{2j+1}\ C_{g_1}(i(a),\alpha_{I_1}) \times C_{g_2}(j(a),\alpha_{I_2})\ H(x+y,L_1)dx\,dy =$$

$$= C_{g_1}(i(a),\alpha_{I_1}) \times C_{g_2}(j(a),\alpha_{I_2})\ \int_0^{\infty}\int_0^{\infty} x^{2i(a)+1}\ y^{2j(a)+1} H(x+y,L_1)dx\,dy.$$

Now equation 5.12 allows us to use Lemma 5.2 to prove the result. $\square$

6 Virasoro equations

In this section we use the recursive formula for the volume polynomials and the relationship between these polynomials and the intersection numbers of tautological classes to derive the Virasoro equations. Let

$$(\alpha_1,\ldots\alpha_n)_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}\cdots\psi_n^{\alpha_n}.$$

String and dilaton equation. If one of the α_i's is 0 or 1, the coefficient of $L^{2\,\alpha}$ in $\mathcal{A}^{dcon}_{g,n}$ and $\mathcal{A}^{con}_{g,n}$ equals zero. Then by using Lemma 5.3 and Theorem 4.4, when $\sum_{i=1}^{n}\alpha_i = 3g-3+n$ we have :

- String equation: $(1,\alpha_1,\ldots,\alpha_n)_g = (2g+n-2)\,(\alpha_1,\ldots,\alpha_n)_g$,

- Dilaton equation: $(0,\alpha_1,\dots,\alpha_n)_g = \sum_{\alpha_i\neq 0}(\alpha_1,\dots,\alpha_i-1,\dots)_g$.

For a simple algebro-geometric proof of the preceding result see [Har].
Virasoro constraints. Let

$$F_g(t_0,t_1,\dots) = \sum_{\{d_i\}}\langle\prod\tau_{d_i}\rangle_g \prod_{r>0} t_r^{n_r}/n_r!$$

where the sum is over all sequences of nonnegative integers with finitely many nonzero terms and $n_r = \mathrm{Card}(i : d_i = r)$. Let

$$F = \sum_{g=0}^{\infty}\lambda^{2g-2}F_g.$$

Define the sequence of differential operators $L_{-1}, L_0, \dots L_n, \dots$ by

$$L_{-1} = \frac{\partial}{\partial t_0} + \frac{\lambda^{-2}}{2}t_0^2 + \sum_{i=1}^{\infty} t_{i+1}\frac{\partial}{\partial t_i},$$

$$L_0 = \frac{3}{2}\frac{\partial}{\partial t_1} + \sum_{i=1}^{\infty}\frac{2i+1}{2}\frac{\partial}{\partial t_i} + \frac{1}{16},$$

and for $n \geq 1$

$$L_n = -\left(\frac{(2n+3)!!}{2^{n+1}}\right)\frac{\partial}{\partial t_{n+1}} + \sum_{i=0}^{\infty}\left(\frac{(2i+2n+1)!!}{(2i-1)!!2^{n+1}}\right)t_i\frac{\partial}{\partial t_{i+n}}$$

$$+\frac{\lambda^2}{2}\sum_{i=0}^{n-1}\left(\frac{(2i+1)!!\,(2n-2i-1)!!}{2^{n+1}}\right)\frac{\partial^2}{\partial t_i\partial t_{n-1-i}},$$

where $(2i+1)!! = 1\cdot 3\dots\cdot(2i+1)$.

Then we obtain a new proof of Witten's conjecture:

Theorem 6.1. *For $k \geq -1$, we have*

$$L_k(\exp(F)) = 0.$$

Remark. In fact, although we show that for any $k \geq -1$, $L_k(e^F) = 0$, since the sequence $\{L_i\}$ satisfies

$$[L_m, L_n] = (m-n)L_{m+n},$$

it is enough to show that $L_2(e^F)=0$.

Proof. It is easy to see that L_{-1} and L_0 are associated to the dilaton and string equation.

Define $A^{con}(F)$, $A^{dcon}(F)$ and $B(F)$ by

$$B_n(F)=\sum_{i=0}^{\infty} a_{n,i} t_i \frac{\partial}{\partial t_{i+n}} F,$$

$$A_n^{con}(F)=\frac{\lambda^2}{2}\sum_{i=0}^{n-1} b_{n-i-1,i}\frac{\partial^2}{\partial t_i \partial t_{n-1-i}} F,$$

$$A_n^{dcon}(F)=\frac{\lambda^2}{2}\sum_{i=0}^{n-1} b_{n-i-1,i}\frac{\partial}{\partial t_i} F\cdot \frac{\partial}{\partial t_{n-1-i}} F,$$

where

$$a_{n,i}=\frac{(2i+2n+1)!!}{(2i-1)!!(2n+3)!!},$$

and

$$b_{i,j}=\frac{(2i+1)!!\,(2j+1)!!}{(2i+2j+3)!!}.$$

We have to show that

$$\frac{\partial}{\partial t_{k+1}} F = A_k^{con}(F)+A_k^{dcon}(F)+B_k(F). \tag{6.1}$$

When $k\geq 1$, we use the recursive formula for the coefficient of $L_1^{2\alpha_1}\cdots L_n^{2\alpha_n}$ in $V_{g,n}(L)$ to prove 6.1.

More precisely, from the recursive formula for the volume polynomials in Section 5, we have

$$(2\alpha_1+1)\cdot V_{g,n}(b)[\alpha]=\mathcal{A}_{g,n}^{con}(b)[\alpha]+\mathcal{A}_{g,n}^{dcon}(b)[\alpha]+\mathcal{B}_{g,n}(b)[\alpha].$$

Using Lemma 5.3, we can write $\mathcal{A}_{g,n}^{con}(b)[\alpha]$, $\mathcal{A}_{g,n}^{dcon}(b)[\alpha]$ and $\mathcal{B}_{g,n}(b)[\alpha]$ in terms of $\widehat{C}_{h,m}(\alpha)$ where $3h+m<3g+n$. On the other hand, by Theorem 4.4, we have

$$\widehat{C}_{g,n}(\alpha)=\frac{C_g(\alpha)}{2^{m(g,n)}}=\frac{1}{2^{|\alpha|}\,\alpha!}\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}\cdots\psi_n^{\alpha_n}.$$

Now we use Lemma 5.3 to show that $\mathcal{A}_{g,n}^{con}$, $\mathcal{A}_{g,n}^{dcon}$ and $\mathcal{B}_{g,n}$ correspond to the terms $A_{\alpha_1-1}^{con}$, $A_{\alpha_1-1}^{dcon}$ and B_{α_1-1} in equation (6.1).

For notational brevity, let

$$[\alpha_1, \dots, \alpha_n]_g = \lambda^{2g-2}(\int\limits_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}) \cdot \prod_i \frac{t_i^{n_i}}{n_i!}$$

where $n_r = \mathrm{Card}(i \in (\alpha_1, \dots, \alpha_n) : \alpha_i = r)$.

Since

$$\frac{(2n)!}{2^n n!} = (2n-1)!!,$$

by using Lemma 5.3 we have:

$$\frac{\partial}{\partial t_{\alpha_1}}[\alpha_1, \dots, \alpha_n]_g = \tag{6.2}$$

$$\frac{\lambda^2}{2} \sum_{a \in \mathcal{I}_{g,n}} b_{j(a),i(a)} \, \frac{\partial}{\partial t_{i(a)}}[i(a), \dots, \alpha_{i_{n_1}}]_{g_1} \times \frac{\partial}{\partial t_{j(a)}}[j(a), \dots, j_{n_2}]_{g_2} +$$

$$+ \frac{\lambda^2}{2} \sum_{i+j=\alpha_1-1} b_{i,j} \, \frac{\partial}{\partial t_i \, \partial t_j}[i, j, \alpha_2, \dots, \alpha_n]_{g-1}$$

$$+ \sum_{j=2}^{n} a_{n,i} \, t_{\alpha_1+\alpha_j-1} \, \frac{\partial}{\partial t_{\alpha_j}}[\alpha_1 + \alpha_j - 1, \dots, \widehat{\alpha_j}, \dots, \alpha_n]_g.$$

Note that for $a \in \mathcal{I}_{g,n}$, $i(a) + j(a) = \alpha_1 + 2$. So for obtaining equation (6.1), we just have to add up the corresponding equations containing the term t_{α_1}.
$\square$

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