INSTABILITY OF RENORMALIZATION

MARCO MARTENS AND BJÖRN WINCKLER

ABSTRACT. In the theory of renormalization for classical dynamical systems, e.g. unimodal maps and critical circle maps, topological conjugacy classes are stable manifolds of renormalization. Physically more realistic systems on the other hand may exhibit instability of renormalization within a topological class. This instability gives rise to new phenomena and opens up directions of inquiry that go beyond the classical theory. In phase space it leads to the coexistence phenomenon, i.e. there are systems whose attractor has bounded geometry but which are topologically conjugate to systems whose attractor has degenerate geometry; in parameter space it causes dimensional discrepancy, i.e. a topologically full family has too few dimensions to realize all possible geometric behavior.

1. INTRODUCTION

To understand the behavior of dynamical systems it is natural to consider: (a) the geometry of attractors, (b) bifurcation patterns of families, (c) topological and combinatorial aspects, as well as (d) measure theoretical aspects. Renormalization is a tool which was originally introduced into dynamics [4, 7] to analyze (a) and (b) but it turned out to connect all four of the above aspects. The renormalization of a system is a new system which describes the dynamics on a smaller scale; it is a microscope on attractors. Intrinsic to this scheme is a characterization of (c) which provides a natural setting for understanding (d) and culminating in a description of topological conjugacy classes as invariant manifolds of renormalization.

In classical systems, e.g. unimodal maps and critical circle maps,[1] instability of renormalization is exclusively associated with changes in topology: a topological conjugacy class is the stable manifold of a hyperbolic fixed point of renormalization and its unstable manifold is a topologically full family (see e.g. [2, 6, 23] and references therein). As a consequence there is rigidity and parameter universality. Rigidity is the phenomenon that two topologically conjugate systems are automatically smoothly conjugate on their attractors, i.e. topology determines geometry. Intuitively, repeated renormalization is like increasing the magnification factor whilst looking at an attractor under a microscope; successive renormalizations converge to the fixed point, so asymptotically the attractor looks like the attractor of the fixed point. Parameter universality is the phenomenon that the metric aspects of the bifurcation patterns of a family is determined by the bifurcation patterns of the unstable manifold. It is a consequence of the fact that a topologically full family has the same dimension as the unstable manifold; hence it meets the stable manifold and successive renormalizations of the family accumulate on the unstable manifold.

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[1]With some restriction on the topology; for simplicity, assume stationary combinatorics.

A question of great interest is what happens to these phenomena as more physically relevant systems are considered? Coullet and Tresser [4] conjectured that universality would occur in real world systems and this has since been confirmed in many contexts (e.g. [13, 14]). However, the main message here is that in more realistic systems there may be instability of renormalization inside a topological class, leading to new phenomena which are much more intricate than for the classical systems. The theory of renormalization is at the beginning of a new chapter which goes beyond the classical theory. The first indication of this was observed in Hénon dynamics where the rigidity paradigm turned out to only hold in a probabilistic sense: the conjugacy between two attractors of period-doubling type is smooth except on a set of measure zero where it is at most Hölder [3].

In this paper we show that within the context of Lorenz dynamics there is instability of renormalization inside most topological classes, even for stationary combinatorics. Lorenz maps are one-dimensional systems that are closely related to unimodal maps, but they are physically more relevant in the sense that they describe the dynamics of certain higher-dimensional flows (see e.g. [1, 9, 21] and references therein). Instability of renormalization has two distinct consequences for these systems: (i) degeneration of successive renormalizations within the topological class of a renormalization fixed point, (ii) unstable manifolds of renormalization fixed points have strictly larger dimension (≥ 3) than topologically full families (2).

Item (i) leads to what we call the *coexistence phenomenon*, which is when a topological class contains both systems whose attractor has bounded geometry and systems whose attractor has degenerate geometry. In particular, there is no rigidity in the traditional sense that the topological class is a rigidity class; instead the topological class is partitioned by rigidity classes. This has consequences on the bifurcation patterns of a family: in the classical setting they are given by intersections with topological classes, but here they are given by intersections with rigidity classes. Because of (ii), which we call *dimensional discrepancy*, a topologically full family (of dim 2) is too small to realize all geometric aspects of the possible bifurcation patterns. For example, such a family will not meet the stable manifold generically and hence the rigidity class of the renormalization fixed point will not be represented; neither will successive renormalizations of the family accumulate on the unstable manifold of the fixed point, as in the classical setting. In particular, there is no parameter universality in the traditional sense, but instead we see much more intricate behavior (see the conjecture in §1.1).

1.1. **Results.** Fix a topological class $\mathcal{T}$ of infinitely (a, b)–renormalizable Lorenz maps with a and b sufficiently large (see §2 for definitions). We would like to highlight two results, illustrated in Figure 1, which are the consequences of the instability of renormalization discussed in the introduction.

Coexistence Theorem. *There exist a nonempty open set $\mathcal{U} \subset \mathcal{T}$ and a renormalization fixed point $f^\star \in \mathcal{T} \setminus \mathcal{U}$ such that the successive renormalizations of $f \in \mathcal{U}$ degenerate and the successive renormalizations of $f \in \mathcal{T} \setminus \mathcal{U}$ have convergent subsequences.*

Dimensional Discrepancy Theorem. *The fixed point $f^\star$ has an unstable manifold $\mathcal{W}^u$. The dimension of $\mathcal{W}^u$ is at least three and every neighborhood of $f^\star$ in $\mathcal{W}^u$ intersects $\mathcal{T} \setminus f^\star$. Furthermore, $f^\star$ has a two-dimensional strong unstable manifold $\mathcal{W}^{uu} \subset \mathcal{W}^u$. It is a topologically full family and $\mathcal{W}^{uu} \cap \mathcal{T} = f^\star$.*

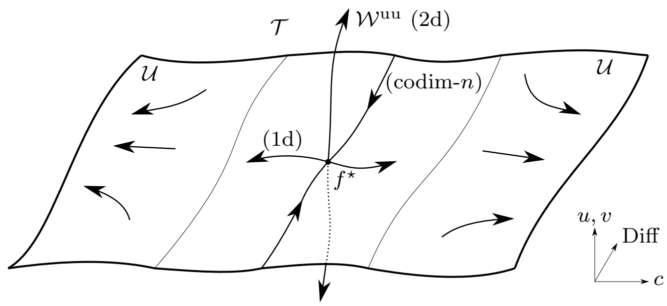


FIGURE 1. Illustration of the dynamics on $\mathcal{T}$.

The Coexistence Theorem follows from the Degeneration Theorem of §4 (which defines "degeneration") and the existence of fixed points, see §5. The Dimensional Discrepancy Theorem follows from the results of §8. That $\mathcal{W}^{\mathrm{uu}}$ meets $\mathcal{T}$ in a unique point is related to the question of monotonicity of entropy; in fact, we prove this intersection property for a class of families which contains the family $\mathcal{W}^{\mathrm{uu}}$. We do not prove anything related to the stable manifold, but we know that it has finite codimension (see the remark after Theorem 20).

We make the following conjecture regarding the structure of $\mathcal{T}$:

Conjecture. *$\mathcal{T}$ is a codim–2 manifold and $f^\star$ is hyperbolic. For a and b sufficiently large: (1) $\dim \mathcal{W}^u = 3$, (2) $\mathcal{T} \setminus \mathcal{U}$ is the stable manifold of $f^\star$, (3) $\mathcal{U}$ is foliated by codim–1 rigidity classes and $\mathcal{T} \setminus \mathcal{U}$ is the rigidity class of $f^\star$, (4) a generic 2–dim family $\mathcal{F}$ intersects $\mathcal{U}$ in a unique rigidity class and the domains of n times renormalizability inside $\mathcal{F}$ shrink super-exponentially in a universal way; if $\mathcal{F}$ intersects $\mathcal{U} \setminus \mathcal{T}$, then generically these domains shrink exponentially in a universal way.*

By the conjecture a generic 3–dim family intersects $\mathcal{T}$ in a curve and points on this curve corresponds to different rigidity classes. In other words, there is a kind of parameter universality for such families. The conjecture also shows that there are two kinds of parameter universality for 2–dim families, depending on whether they hit $\mathcal{U}$ (the generic case) or not. The above is in agreement with the Rigidity Conjecture [17] but it is slightly stronger as we do not expect to see probabilistic rigidity classes as in the Hénon case. It is important to note that the condition on a and b both being large is essential: e.g. conjecturally there is no instability within the topological classes for the combinatorics of [22] ($a = 1$, $b = 2$, critical exponent $\alpha = 2$) and [18] (a small, b large). Moreover, we conjecture that there are topological classes (e.g. $a = 2$, $b = 8$, critical exponent $\alpha = 2$) for which the fixed point has 2–dim unstable behavior, but where there still is instability inside the class due to the presence of a period–2 point of renormalization with 3–dim unstable behavior.

In the process of proving the above theorems we are able to get precise control over how the critical point moves under renormalization as well as to get bounds on the distortion, see §3. In §4 we use these results to derive many important dynamical properties for infinitely (a, b)–renormalizable Lorenz maps. In particular we show that there are no wandering intervals (implying that two such maps are topologically conjugate), see Theorem 12. Under what conditions there are no wandering intervals for Lorenz maps in general is an important question which is still wide open. Furthermore, we prove that such maps have a measure zero minimal Cantor attractor and describe the invariant measures on the attractor as well as its Hausdorff dimension.

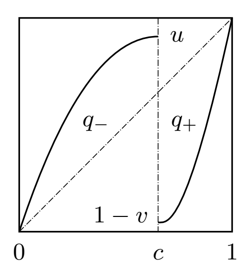
As a closing remark, note that our techniques are based on real analytical tools and work for arbitrary real exponents $\alpha > 1$. We generally work in the C^3 category which traditionally presents significant difficulties as the classical renormalization operator is not differentiable in this class [6]. This issue is avoided in §6 by defining renormalization over internal structures instead of over diffeomorphisms [15]. In particular, our renormalization operator is differentiable in this category, see Theorem 20. We construct unstable manifolds in the space of internal structures and by composing we obtain results for the actual system, see §8.3.

2. Preliminaries

In this section we define the space of Lorenz maps as well as the renormalization operator acting on this space. For more details, see [18].

2.1. **The space of Lorenz maps.** The *standard family* $(u, v, c) \mapsto q(x)$ is defined by $q|_{[0,c)} = q_-$ and $q|_{(c,1]} = q_+$, where $u, v \in [0, 1]$, $c \in (0, 1)$ and

$$\begin{cases} q_-(x) = u\Big(1 - \Big|\dfrac{c-x}{c}\Big|^\alpha\Big), \\ q_+(x) = 1 + v\Big(-1 + \Big|\dfrac{x-c}{1-c}\Big|^\alpha\Big). \end{cases} \tag{1}$$



Here $\alpha \in \mathbb{R}$ the *critical exponent*, and c the *critical point*; $\alpha > 1$ is fixed throughout.

Let Diff^2 denote the set of orientation-preserving C^2–diffeomorphisms on $[0, 1]$; it is a Banach space with norm $\|\phi\| = \sup|N\phi|$ and linear structure

$$t_1\phi_1 \oplus t_2\phi_2 = N^{-1}(t_1 N\phi_1 + t_2 N\phi_2), \qquad \phi_i \in \mathrm{Diff}^2,\ t_i \in \mathbb{R}.$$

The bijection $N : \mathrm{Diff}^2 \to C^0$; $\phi \mapsto D \log D\phi$, is called the *nonlinearity operator*. Let $\mathrm{Diff}^{\mathrm{S}} \subset \mathrm{Diff}^2$ denote the convex subset of C^3–diffeomorphisms with non-positive Schwarzian derivative, $S\phi = DN\phi - (N\phi)^2/2 \le 0$.

Let f be a map with two increasing branches $f_\pm$ of the form $f_- = \phi \circ q_-$ and $f_+ = \psi \circ q_+$, with $\phi, \psi \in \mathrm{Diff}^{\mathrm{S}}$. Note that $Sf < 0$ and that f is undefined at the critical point c, but the *critical values* $f_-(c)$ and $f_+(c)$ are well-defined. We call f a *Lorenz map* iff $x < f_-(x)$, $\forall x \in (0, c]$ and $f_+(x) < x$, $\forall x \in [c, 1)$; it is identified with the tuple (u, v, c, ϕ, ψ) using (1). The set of all Lorenz maps is denoted $\mathcal{L}$. We identify $\mathcal{L}$ with a subset of $\mathbb{R}^3 \times \mathrm{Diff}^{\mathrm{S}} \times \mathrm{Diff}^{\mathrm{S}}$ and use the product topology. Define

$$\mathcal{L}_\delta = \{(u, v, c, \phi, \psi) \in \mathcal{L} \mid \|\phi\|, \|\psi\| < \delta\}.$$

We say that a branch $f_\pm$ is *trivial* iff $f_\pm(c) = c$ and we say that $f_\pm$ is *full* iff $f_-(c) = 1$ resp. $f_+(c) = 0$. We say that f is *full* iff both branches are full.

2.2. **Renormalization.** We say that $f \in \mathcal{L}$ is *renormalizable* iff there exists a closed interval C such that $\operatorname{Int} C \ni c$, $C \neq [0,1]$, and such that the first-return map to C is affinely conjugate to some $g \in \mathcal{L}$. The *renormalization operator* $\mathcal{R}$ is defined by taking the largest such C and sending f to g. We call $\mathcal{R}f$ the *renormalization* of f and we call C the *return interval* of f. We say that f is *infinitely renormalizable* iff $\mathcal{R}^n f$ is renormalizable, $\forall n \geq 0$.

We say that f is (a,b)*–renormalizable* iff

$$f^1(C_-) > \cdots > f^a(C_-) > c, \quad C_- \subset f^{a+1}(C_-) \subset C, \quad C_- = C \cap \{x < c\},$$
$$f^1(C_+) < \cdots < f^b(C_+) < c, \quad C_+ \subset f^{b+1}(C_+) \subset C, \quad C_+ = C \cap \{x > c\}.$$

In this case $\partial_- C$ resp. $\partial_+ C$ are periodic points of periods $a+1$ resp. $b+1$, and identifying $\mathcal{R}f$ with $(u', v', c', \phi', \psi')$ we have

$$u' = \frac{|q(C_-)|}{|U|}, \quad v' = \frac{|q(C_+)|}{|V|}, \quad c' = \frac{|C_-|}{|C|}, \quad \phi' = [\Phi|U], \quad \psi' = [\Psi|V], \tag{2}$$

where $\Phi = f_+^a \circ \phi$, $\Psi = f_-^b \circ \psi$, $U = \Phi^{-1}(C)$, $V = \Psi^{-1}(C)$, and $[g|I]$ denotes the affine rescaling of the domain and range of $g|_I$ to $[0,1]$.[2] Note that $c' \neq c$ in general. This movement of the critical point is key in understanding the new renormalization phenomena we describe here.

We will almost exclusively consider (a,b)–renormalizable maps and will maintain the convention of using primes to denote variables associated with the renormalization (we consistently use D for derivatives).

3. Technical lemmas

In this section we state and prove technical lemmas which we then apply in the following section to prove things like ergodicity, non-existence of wandering intervals, etc. This section can be skipped on a first read-through and referenced back to later on. We use the letter K to denote an anonymous constant and follow the convention of reusing the same letter for different constants.

3.1. **A fundamental lemma.** The following lemma is the starting point for all other results. Its main content is that the critical values of a renormalizable map are "expanded away" from the critical point. It is fundamental in controlling the expansion along postcritical orbits.

Expansion Lemma. *For every $\delta < \infty$ and $\gamma \in (0,1)$ there exists $\rho > 0$ such that if $f \in \mathcal{L}_\delta$ is renormalizable and $c < 1-\gamma$, then $c - f_-^{-1}(c) \geq \rho c$ and $f_+^{-1}(c) - c \geq \rho c^{1/\alpha}$.*

Remark. We will repeatedly make use of this lemma in the following way: let f be (a,b)–renormalizable so that $f_-(c) > f_+^{-a}(c)$ and $f_+(c) < f_-^{-b}(c)$. Because of the lemma the backward orbits $f_+^{-a}(c)$ and $f_-^{-b}(c)$ approach 1 and 0 exponentially fast, respectively. Hence, all (a,b)–renormalizable maps are exponentially close (in $\min\{a,b\}$) to the set of full maps. This allows us to make perturbation arguments away from the full maps and this is the central idea behind all estimates.

Note that the convergence of the above backward orbits depends on γ and this leads to us having to treat the case where the critical point c is bounded away from 0 and 1 separately from the case where c is allowed to approach 0 or 1.

[2] That is, $[g|I] = h_{g(I)}^{-1} \circ g \circ h_I$, where $h_{[x,y]}(t) = (1-t)x + ty$.

Proof. We claim that $\forall\delta < \infty\ \exists\rho \in (0,1)$ such that

$$\frac{\phi^{-1}(c)}{u} \leq 1 - \rho^\alpha. \tag{3}$$

Let us show that (3) implies the lemma before proving the claim.

From (3), the fact that $f_-^{-1}(c) = q_-^{-1} \circ \phi^{-1}(c)$ and (1) we get $c - f_-^{-1}(c) \geq \rho c$. Since $f_+(c) < f_-^{-1}(c)$, $f_+(c) = \psi(1-v)$ and $v \leq 1$ we can apply this and (1) to get

$$\left(\frac{f_+^{-1}(c) - c}{1-c}\right)^\alpha = \frac{\psi^{-1}(c) - (1-v)}{v} = \frac{1}{v}\left|\psi^{-1}\left([f_+(c), c]\right)\right| \geq e^{-\delta}\left|c - f_-^{-1}(c)\right|.$$

Hence $f_+^{-1}(c) - c \geq \gamma(e^{-\delta}\rho c)^{1/\alpha}$ and the lemma follows.

We now prove (3). Let $I = [f_-^{-1}(c), c]$. Since f is renormalizable $f^2(I) \supset I$. Hence, $\exists t \in I$ such that $Df^2(t) \geq 1$ which together with (1) and (1) gives

$$1 \leq \frac{e^{2\delta}\alpha^2 u}{c(1-c)}\left(1 - \frac{f_-^{-1}(c)}{c}\right)^{\alpha-1} = \frac{e^{2\delta}\alpha^2 u}{c(1-c)}\left(1 - \frac{\phi^{-1}(c)}{u}\right)^{1-1/\alpha}.$$

Since $1 - c > \gamma$ and $\phi^{-1}(c) \leq e^\delta c$ this shows that

$$\frac{u}{\phi^{-1}(c)}\left(1 - \frac{\phi^{-1}(c)}{u}\right)^{1-1/\alpha} \geq \frac{1}{K}. \tag{4}$$

The left-hand side approaches 0 as $\phi^{-1}(c) \to u$, so (4) implies (3). $\square$

3.2. **Controlling Koebe space.** In this subsection we deal with the central task of controlling the distortion of first-entries to the return interval C of a renormalizable map. We employ the Koebe Lemma 33 for this purpose and to that end we spend most of this subsection controlling the "Koebe space" around C. Lemma 1 shows where the Koebe space around C is located and the remaining lemmas are concerned with estimating its size. The mechanisms which govern the size of the Koebe space are different when the critical point is bounded compared to when the critical point approaches the boundary. The former case is covered by Lemma 5 and the latter case is covered by Lemmas 2 and 4. Both cases are then summarized in Lemma 6.

Lemma 1 (Monotone extension)**.** *Let f be renormalizable with return times (m,n). If $f^k|_I : I \to C$ is a first-entry map to C with monotone extension $f^k|_J$, then $f^k(J)\backslash C$ contains $f^j(C_+)$ in the left component and $f^i(C_-)$ in the right component, for some $j \in \{1,\dots,n-1\}$ and $i \in \{1,\dots,m-1\}$.*

Remark. In particular, if f is (a,b)–renormalizable then the Koebe space around C extends at least to the preimages $f_-^{-1}(c)$ and $f_+^{-1}(c)$ since these points are contained in $f^b(C_+)$ and $f^a(C_-)$, respectively.

Proof. By definition $\exists i > 0$ such that $f^{k-i}(J) \setminus f^{k-i}(I)$ contains C_- in its right component. Hence $f^k(J)\backslash C$ contains $f^i(C_-)$ in its right component. If $i > m$, then $f^{k-i+m}(J)$ would contain c, which contradicts monotonicity of $f^k|_J$. If $i = m$, then $f^k(I) \cap C_- = \emptyset$ since $f^k|_J$ is monotone and $C_- \subset f^m(C_-) \subset C$ by renormalizability, which contradicts $f^k(I) = C$. Hence $i < m$. Repeat the argument on the left. $\square$

In the following lemma we consider the case where c is allowed to approach 0. In this case the right branch of f is called the "big branch" as it may take up most of the unit interval. We will only state and prove results for c close to 0. There is always a symmetrical statement and proof for c close to 1 which can be obtained by conjugating with the involution $x \mapsto 1 - x$.

Lemma 2 (Size of C in big branch)**.** *For every $\delta < \infty$ and $\varepsilon > 0$ there exists $\gamma > 0$ such that if $f \in \mathcal{L}_\delta$ is (a, b)–renormalizable, $c < \gamma$ and $b > \alpha + 1$, then $|C_+| \le \varepsilon \min\{|c - f_-^{-1}(c)|, |f_+^{-1}(c) - c|\}$.*

Proof. By the Expansion Lemma $f_+^{-1}(c) - c$ is larger than the whole domain of the small branch as $c \downarrow 0$, so we only need to prove that $|C_+| \le \varepsilon |c - f_-^{-1}(c)|$.

We claim that $\forall \varepsilon > 0$, $\forall \delta < \infty$ $\exists \gamma > 0$ such that if $c < \gamma$ and $n > \alpha$, then

$$f_-^{-n}(c) \le (\varepsilon c)^\alpha. \tag{5}$$

Before proving the claim let us show how this implies the lemma. Since $f(C_+)$ first enters C after b steps $f(C_+) \subset [0, f_-^{-b+1}(c)]$, so $|f(C_+)| < (\varepsilon c)^\alpha$ for $b - 1 > \alpha$ by (5). Equation (1) can be used to estimate $|f(C_+)| \ge e^{-\delta} v(|C_+|/(1-c))^\alpha$. From $c \ge f_+(c) = \psi(1 - v)$ we get $v \ge 1 - e^\delta c$. Taken all together we get $|C_+| \le K\varepsilon c$. This finishes the proof, since $c - f_-^{-1}(c) \ge \rho c$ by the Expansion Lemma.

Let us prove (5). The Expansion Lemma gives $c - f_-^{-1}(c) \ge \rho c$ which means that 0 uniformly attracts $f_-^{-1}(c)$ under iteration of f_-^{-1}. Hence $f_-^{-n}(c)/c \le K Df(0)^{-n}$, $\forall n$. Equation (1) can be used to estimate $Df(0) \ge e^{-\delta}\alpha u/c$. Since f is renormalizable $f_+^{-1}(c) \le f_-(c) = \phi(u)$, so the Expansion Lemma implies that $u \ge c^{1/\alpha}/K$. Conclusively, $f_-^{-n}(c) \le Kc(Kc^{1-1/\alpha})^n \le (Kc^{1-1/\alpha})^{n-\alpha} c^\alpha$. Define γ by $(K\gamma^{1-1/\alpha})^{n-\alpha} = \varepsilon^\alpha$ and the claim follows. $\square$

The following lemma is a simple induction result on certain backward orbits of the critical point which will be needed in a few places.

Lemma 3. *Let $\kappa = e^{\delta/(\alpha-1)}\alpha/(\alpha - 1)$ and $\alpha_n = 1/\alpha^n$. Then $c - f_-^{-n}(c) \le \kappa c(1-c)^{\alpha_n}$ and $f_+^{-n}(c) - c \le \kappa(1-c)c^{\alpha_n}$, for every $n \ge 1$*

Proof. Let $s_n = 1 + \alpha_1 + \cdots + \alpha_n$. We claim that for all $n \ge 1$

$$f_+^{-n}(c) - c \le (1-c)e^{s_{n-1}\delta/\alpha} s_{n-1} c^{\alpha_n}. \tag{6}$$

From this the lemma follows since $s_n < \alpha/(\alpha - 1)$.

The proof of the claim is by induction. From (1) and $v \le 1$ we get

$$\frac{f_+^{-1}(x) - c}{1 - c} \le \psi^{-1}(x)^{1/\alpha} \le (e^\delta x)^{1/\alpha}.$$

Let $x = c$ to prove the base case. Make the induction assumption that (6) holds for some n. Then

$$\begin{aligned}\frac{f_+^{-(n+1)}(c) - c}{1 - c} &\le e^{\delta/\alpha} f_+^{-n}(c)^{1/\alpha} \le e^{\delta/\alpha}\left(c + e^{s_{n-1}\delta/\alpha} s_{n-1} c^{\alpha_n}\right)^{1/\alpha} \\ &\le e^{\delta/\alpha(1+s_{n-1}/\alpha)}\left(c^{1-\alpha_n} e^{-s_{n-1}\delta/\alpha} + s_{n-1}\right)^{1/\alpha} c^{\alpha_{n+1}}.\end{aligned}$$

The term in parenthesis is less than $1 + s_{n-1}$. Use the fact that $(1 + s_{n-1})^{1/\alpha} < 1 + s_{n-1}/\alpha = s_n$ to finish the proof. $\square$

The following lemma is the counterpart to Lemma 2 but for the “small branch,” i.e. the left branch when c is allowed to approach 0.

Lemma 4 (Size of C in small branch)**.** *For every $\delta < \infty$ and $\varepsilon > 0$ there exist $N < \infty$ and $\gamma > 0$ such that if $f \in \mathcal{L}_\delta$ is (a, b)–renormalizable, $c < \gamma$ and $a \geq N$, then $|C_-| \leq \varepsilon \min\{|c - f_-^{-1}(c)|, |f_+^{-1}(c) - c|\}$.*

Proof. We only need to consider the case $|C_-| > |C_+|$, else we could apply Lemma 2. As in the proof of that lemma it suffices to show that $|C_-| \leq \varepsilon |c - f_-^{-1}(c)|$.

Let $T = [c, f_-(c)]$, $R_k = f^{a+1-k}(C_-)$ and $\theta_0 = \varepsilon^\alpha e^{-\delta}$. If $|R_a| \leq \theta_0 |T|$, then (68) gives $|C_-|/|c - f_-^{-1}(c)| = (|\phi^{-1}(R_a)|/|\phi^{-1}(T)|)^{1/\alpha} \leq (e^\delta |R_a|/|T|)^{1/\alpha} \leq \varepsilon$. Hence we are done if there is enough space around R_a inside T. The following claim gives two sufficient conditions for this to happen.

Let $S_k = [f_+^{-k+1}(c), R_k]$ for $k \leq a$, $S_{a+1} = f_-^{-1}(S_a)$ and $L_k = S_k \setminus R_k$ for $k \leq a + 1$. We claim that $\forall \delta < \infty, \varepsilon > 0$ $\exists \theta_1, \theta_2 < \infty$ such that if

(i) $|R_k| \leq \theta_1 |L_k|$ for some $k \leq a$, or if
(ii) $[f_+(c), 1]$ contains a θ_2^{-1}–scaled neighborhood of R_k for some $k \leq a - 1$,

then $|R_a| \leq \theta_0 |T|$. Let use prove the claim.

Assume (i). From (70) we get

$$\frac{|L_{i+1}|}{|S_{i+1}|} = \frac{|q_+^{-1} \circ \psi^{-1}(L_i)|}{|q_+^{-1} \circ \psi^{-1}(S_i)|} \geq \frac{|\psi^{-1}(L_i)|}{|\psi^{-1}(S_i)|} \geq e^{-\delta |S_i|} \frac{|L_i|}{|S_i|} \tag{7}$$

and consequently $|L_a|/|S_a| \geq \exp\{-\delta \sum |S_i|\} |L_k|/|S_k| \geq e^{-2\delta} \theta_1^{-1}$, since $\sum |S_i| \leq 2$. Hence, $|R_a|/|S_a| \leq 1 - e^{-2\delta} \theta_1^{-1}$ and since $S_a \subset T$, it follows that $|R_a| \leq \theta_0 |T|$ for θ_1 sufficiently small.

Assume (ii). The branch $f^{a-k}|_{R_a} : R_a \to R_k$ has monotone extension whose image is $[f_+(c), 1]$ and its domain is in $[c, 1]$. Hence the claim follows from the Macroscopic Koebe Principle (see [20, p. 287]) by choosing θ_2 small enough. This concludes the proof of the claim.

Now assume that δ and ε have been chosen and that the constants θ_1 and θ_2 have been determined. If (i) was true then there would be nothing to prove so assume that it is false. Let $N \geq 3$ be given, assume $a \geq N$, and choose $\gamma > 0$ such that if $c < \gamma$, then $\operatorname{dist}(R_i, 1) \geq \theta_2^{-1} |R_i|$, $\forall i = 0, \ldots, N-1$. This is possible, since $R_i \subset [0, f_+^{-N}(c)]$ $\forall i = 0, \ldots, N-1$ and $f_+^{-N}(c) - c \leq K c^{1/\alpha^N}$ by Lemma 3. If one of the R_i also had space on the left, i.e. if $\operatorname{dist}(R_k, f_+(c)) \geq \theta_2^{-1} |R_k|$ for some $k < N$, then we could apply (ii) and be done. Consequently we assume that (ii) does not hold for any $k < N$. Let us prove that this leads to a contradiction.

Let $\hat{R}_i = [f_+(c), R_i]$ and $\hat{L}_i = [f_+(c), L_i]$. By assumption $|R_i| > \theta_1 |L_i|$ and $|\hat{R}_i| > (1 + \theta_2) |\hat{L}_i|$, $\forall i \in \{1, \ldots, N-1\}$. These are exactly the conditions needed for (71). Repeating the estimate in (7) but using (71) instead of (70) we get that $\exists \rho > 0$ (only depending on θ_1 and θ_2) such that

$$\frac{|L_i|}{|S_i|} \geq e^{-\delta |S_{i-1}|} \frac{1}{\rho} \frac{|L_{i-1}|}{|S_{i-1}|}, \qquad \forall i \in \{1, \ldots, N-1\}. \tag{8}$$

From (68) and (69) we get

$$\frac{|L_1|}{|S_1|} \geq e^{-\delta |S_0|/\alpha} \left(\frac{|L_0|}{|S_0|} \right)^{1/\alpha}, \qquad \frac{|L_{a+1}|}{|S_{a+1}|} \geq e^{-\delta |S_a|} \frac{1}{\alpha} \frac{|L_a|}{|S_a|}, \tag{9}$$

respectively. Since $L_{a+1} \subset L_0$, $R_{a+1} = C_- \subset R_0 \subset C$ we can estimate

$$\frac{|L_{a+1}|}{|S_{a+1}|} = \frac{|L_{a+1}|}{|L_{a+1}| + |R_0|} \left(1 + \frac{|R_0| - |R_{a+1}|}{|S_{a+1}|} \right) \leq \frac{|L_0|}{|S_0|} \left(1 + \frac{|C_+|}{|C_-|} \right). \tag{10}$$

By assumption $|C_-| > |C_+|$, which combined with (7)–(10) gives

$$\frac{|L_0|}{|S_0|} \geq \rho^{-N+2} \frac{1}{2\alpha} e^{-\delta \sum |S_i|} \left(\frac{|L_0|}{|S_0|} \right)^{1/\alpha}. \tag{11}$$

Since $\sum |S_i| \leq 2$ this implies that $(|L_0|/|S_0|)^{1-1/\alpha} \geq \rho^{-N}/K$ which is impossible for N large enough as the left-hand side is at most 1 but the right-hand side is unbounded in N. We conclude that either (i) holds for some $k \leq a$ or (ii) holds for some $k < N$. □

The following lemma controls the Koebe space when the critical point is bounded away from the boundary. Note that in this case we get explicit bounds on the size of the Koebe space. We need these explicit bounds later on.

Lemma 5 (Size of C)**.** *For every closed interval $\Delta \subset (0,1)$ and $\delta < \frac{1}{2}\log\alpha$ there exist $N < \infty$, $K < \infty$ and $\lambda > 1$ such that if $f \in \mathcal{L}_\delta$ is (a,b)–renormalizable,* $\min(a,b) \geq N$ *and $c \in \Delta$, then*

$$|C| \leq K \min\left\{ |c - f_-^{-1}(c)|, |f_+^{-1}(c) - c| \right\} \lambda^{-\min(a,b)/\alpha}.$$

Remark. The *only* reason for demanding $\delta < \frac{1}{2}\log\alpha$ is because it directly implies that $Df(x) > 1$ for $x = 0, 1$ as shown in the proof below. It is by no means necessary but it makes many arguments simpler. We will later see that the distortion of the renormalization is tiny when the return times are large, so this condition can be automatically satisfied by renormalizing once. Hence we allow ourselves the convenience to assume $\delta < \frac{1}{2}\log\alpha$ from now on.

Proof. Let λ be the infimum of $Df(0)$ over all f satisfying the assumptions of the lemma. Then $\lambda \geq e^{-\delta}\alpha u/c$. From $c \leq f_-(c) = \phi(u)$ and $\phi(u) \leq e^{\delta}u$, we get $u/c \geq e^{-\delta}$. Since $\delta < \frac{1}{2}\log\alpha$ it follows that $\lambda \geq \alpha e^{-2\delta} > 1$. By the Expansion Lemma $f_-^{-n}(c) \leq KDf(0)^{-n} \leq K\lambda^{-n}$. Now argue as in the proof of Lemma 2 to get $|C_+| \leq K\lambda^{-b/\alpha}$. Use the Expansion Lemma to see that the lemma holds with $|C_+|$ in place of $|C|$. Since c is bounded we can repeat this argument for $|C_-|$ and since it holds for both $|C_\pm|$, it must hold for $|C|$. □

Finally, the following lemma summarizes the previous results on the size of the Koebe space independently of the whether the critical point is bounded or not. It shows that we can make the Koebe space large by increasing the return times.

Lemma 6 (Koebe space)**.** *For every $\delta < \frac{1}{2}\log\alpha$ and $\tau > 0$ there exists $N < \infty$ such that if $f \in \mathcal{L}_\delta$ is (a,b)–renormalizable and* $\min\{a,b\} \geq N$*, then* $[f_-^{-1}(c), f_+^{-1}(c)]$ *contains a τ–scaled neighborhood of C.*

Proof. From Lemmas 2 and 4 we get an $N_0 < \infty$ and a closed interval $\Delta \subset (0,1)$ such that the statement is true if $c \notin \Delta$ and $\min\{a,b\} \geq N_0$. From Lemma 5 we get an $N_1 < \infty$ such that the statement holds for $c \in \Delta$ and $\min\{a,b\} \geq N_1$. Let $N = \max\{N_0, N_1\}$ to finish the proof. □

3.3. **Controlling the critical point.** Having established the necessary results to control distortion in the previous subsection we now turn to the main difficulty of Lorenz renormalization, namely to control how the critical point moves under renormalization. This is an essential problem — we will later see that the fact that the critical point may move under renormalization contributes to an "extra" unstable direction when the return times are large. The mechanism behind this phenomenon is given by the important Flipping Lemma below.

The following lemma provides a central relation between the critical point of f and the critical point of $\mathcal{R}f$.

Lemma 7 (Position of the critical point)**.** *Let $\tilde{c}(f) = c/(1-c)$ denote the relative critical point and let $\tilde{v}(f) = f_-(c)/(1-f_+(c))$ denote the relative critical value. If $f \in \mathcal{L}_\delta$ is (a,b)–renormalizable, then*

$$\frac{\tilde{c}(\mathcal{R}f)^\alpha}{\tilde{v}(\mathcal{R}f)} = \kappa \frac{\tilde{c}(f)^\alpha}{\tilde{v}(f)} \frac{Df^b(x)}{Df^a(y)},$$

for some $x \in f(C_+)$, $y \in f(C_-)$ and $e^{-2\delta} \le \kappa \le e^{2\delta}$.

Proof. By definition $\tilde{c}(\mathcal{R}f) = |C_-|/|C_+|$ and $\tilde{v}(\mathcal{R}f) = |f^{a+1}(C_-)|/|f^{b+1}(C_+)|$, so we are looking for an expression involving these quantities.

By the mean-value theorem there exist $x \in f(C_+)$, $x_0 \in q(C_+)$, and $x_1 \in [1-v, 1]$ such that: (i) $|f^{b+1}(C_+)| = Df^b(x)|f(C_+)|$, (ii) $|f(C_+)| = vD\psi(x_0)(|C_+|/(1-c))^\alpha$, by (1), and (iii) $D\psi(x_1)v = 1 - f_+(c)$, since $1 - f_+(c) = 1 - \psi(1-v) = |\psi([1-v,1])|$. Putting all of this together we get

$$|f^{b+1}(C_+)| = \frac{D\psi(x_0)}{D\psi(x_1)}(1 - f_+(c))Df^b(x)\left(\frac{|C_+|}{1-c}\right)^\alpha.$$

A similar argument gives an equation for $|f^{a+1}(C_-)|$. Divide the two equations and apply Lemma 32 to finish the proof. □

The rest of this subsection is dedicated to the proof of the Flipping Lemma. It shows that renormalization contracts the critical point very strongly toward the boundary. The name comes from the fact that if $c(f)$ is close to 0, then $c(\mathcal{R}f)$ is close to 1. That is, the position of the critical point "flips" under renormalization. Note that it is essential that the return times are not too small; otherwise this flipping does not occur (although we do not prove that statement here).

Flipping Lemma. *For every compact interval $P \subset \mathbb{R}^+$, $\delta < \infty$ and $\sigma \in (0,1)$ there exist $N < \infty$ and $\gamma > 0$ such that the following holds. Let $f \in \mathcal{L}_\delta$ be (a,b)–renormalizable with $\min\{a,b\} \ge N$ and $a/b \in P$. If $c(f) < \gamma$, then $1 - c(\mathcal{R}f) < \sigma c(f)$. If $1 - c(f) < \gamma$, then $c(\mathcal{R}f) < \sigma(1 - c(f))$.*

Remark. The condition on how large a and b have to be is explicitly given by (17). For $\alpha = 2$ it can be seen that this condition is satisfied if $\min\{a,b\} \ge 4$. Computer experiments indicate that this is the optimal lower bound on N in the sense that if $a \le 3$ then we can choose b such that the Flipping Lemma is false.

The following lemma is a simple induction needed in the lemma following it.

Lemma 8. *Let $\alpha_n = 1/\alpha^n$. For every $\delta < \infty$ and $\gamma \in (0,1)$ there exists $\rho \in (0,1)$ such that if $f \in \mathcal{L}_\delta$ is renormalizable and $c < 1 - \gamma$, then $f_+^{-n}(c) - c \ge \rho c^{\alpha_n}$, for all $n \ge 1$.*

Proof. Let $s_n = 1 + \alpha_1 + \cdots + \alpha_n$. We claim that $\exists \rho \in (0,1)$ such that

$$f_+^{-n}(c) - c \ge \left(\gamma e^{-\delta/\alpha}\right)^{s_{n-1}} (\rho c)^{\alpha_n}, \tag{12}$$

from which the lemma follows, since $\rho^{\alpha_n} \uparrow 1$ and $s_n < \alpha/(\alpha-1)$. The proof of (12) is by induction. The base case follows from the Expansion Lemma. Assume that (12) holds for some n. Then (1), $v \le 1$, $1-c > \gamma$ and $f_+(c) \le c$ imply that

$$\begin{aligned} f_+^{-(n+1)}(c) - c &= \frac{1-c}{v}\left|\psi^{-1}\left([f_+(c), f_+^{-n}(c)]\right)\right|^{1/\alpha} \\ &\ge \gamma e^{-\delta/\alpha}\left|c + \left(\gamma e^{-\delta/\alpha}\right)^{s_{n-1}}(\rho c)^{\alpha_n} - f_+(c)\right|^{1/\alpha} \ge \left(\gamma e^{-\delta/\alpha}\right)^{1+s_{n-1}/\alpha}(\rho c)^{\alpha_{n+1}}. \end{aligned}$$

This completes the proof, since $1 + s_{n-1}/\alpha = s_n$. $\square$

The following lemma gives bounds on the derivative of returns to C which appear in the formula of Lemma 7. We only need this in the proof of the Flipping Lemma.

Lemma 9. *Let $\alpha_n = 1/\alpha^n$. For every $\delta < \infty$ there exist $K < \infty$ and $\gamma > 0$ such that if $f \in \mathcal{L}_\delta$ is (a,b)–renormalizable, $b > \alpha + 1$ and $c < \gamma$, then $Df_-^b(x) \ge K^{-b}c^{-b(1-\alpha_a)}$, and $Df_+^a(y) \le K^a c^{1-\alpha_a}$ for every $x \in f_-^{-b}(C)$ and $y \in f_+^{-a}(C)$.*

Proof. We first prove the bound on Df_-^b. By (1)

$$Df_-^b(x) \ge \left(\frac{u}{Kc}\right)^b \prod_{i=0}^{b-1}\left(1 - \frac{f_-^i(x)}{c}\right)^{\alpha-1}. \tag{13}$$

Since f is (a,b)–renormalizable $\phi(u) = f_-(c) \ge f_+^{-a}(c)$, so $u/c \ge c^{-1+\alpha_a}/K$ by Lemma 8. Hence, it remains to show that the product on the right-hand side is bounded. To that end, let $L = [f_-^{-1}(c), c]$ and $R = [c, f_-(c)]$. By (69) $|f_-^{-1}(C_+)|/|L| \le e^\delta |C_+|/|R|$, which is small for $c < \gamma$ by Lemma 2. This and the Expansion Lemma show that $f^b(t)/c \le \rho$ for all $t \in f_-^{-1}(C)$, where $\rho \in (0,1)$ only depends on δ. Hence, any $t \in f_-^{-1}(C)$ is attracted to 0 under iteration of f_-^{-1} and the product in (13) has a uniform bound as claimed.

We now prove the bound on Df_+^a. Assume first that $y \in f_+^{-a}(C_-)$, so that $f_+^i(y) \le f_+^{-a+i}(c)$. Then we can use Lemma 3 and (1) to estimate

$$Df_+^a(y) \le \prod_{i=0}^{a-1} K\left(f_+^{-a+i}(c) - c\right)^{\alpha-1} \le \prod_{i=0}^{a-1} K\left(Kc^{\alpha_{a-i}}\right)^{\alpha-1} \le K^a c^{1-\alpha_a}. \tag{14}$$

To show that (14) also holds for $y \in f_+^{-a}(C_+)$ we argue as follows. The image of the monotone extension of $f_+^{-a}(C_+) \to C_+$ is $[f_+(c), 1]$ and this contains a 1–scaled neighborhood of C_+ for $c < \gamma$ by Lemma 2. Hence the Koebe Lemma 33 implies that $Df_+^a(y) \le KDf_+^a(c)$ for all $y \in f_+^{-a}(C_+)$. $\square$

Proof of the Flipping Lemma. Assume without loss of generality that c is close to 0 and let $N > \alpha + 1$ to begin with. We claim that $1 - c(\mathcal{R}f) \le \sigma c(f)$, if

$$\frac{(\sigma c^2)^\alpha}{e^{2\delta}}\frac{Df_-^b(x)}{Df_+^a(y)} \ge 1, \qquad \forall x \in f_-^{-b}(C), \forall y \in f_+^{-a}(C). \tag{15}$$

To prove this we will use Lemma 7 and the notation defined there. From $0 \le f_+(c) \le c \le f_-(c) \le 1$ we get $c \le \tilde{v}(f) \le (1-c)^{-1}$ and

$$\frac{c^\alpha}{(1-c)^{\alpha-1}} \le \frac{\tilde{c}(f)^\alpha}{\tilde{v}(f)} \le \frac{c^{\alpha-1}}{(1-c)^\alpha}. \tag{16}$$

Let $c' = c(\mathcal{R}f)$. Then (16) and Lemma 7 gives

$$\frac{1}{(1-c')^\alpha}\frac{1}{c^\alpha} \ge \frac{(c')^{\alpha-1}}{(1-c')^\alpha}\frac{(1-c)^{\alpha-1}}{c^\alpha} \ge \frac{\tilde{c}(\mathcal{R}f)^\alpha}{\tilde{v}(\mathcal{R}f)}\frac{\tilde{v}(f)}{\tilde{c}(f)^\alpha} \ge e^{-2\delta}\tau,$$

where $\tau = \inf Df_-^b(x)/Df_+^a(y)$ over all $x \in f_-^{-b}(C)$ and $y \in f_+^{-a}(C)$. This proves that $(1-c')^\alpha \le e^{2\delta}\tau^{-1}c^{-\alpha}$. Hence, if $e^{2\delta}\tau^{-1}c^{-\alpha} \le (\sigma c)^\alpha$, then $1 - c' \le \sigma c$. This concludes the proof of (15).

We now use (15) to prove the lemma. Let $\beta = a/b$, $t \in (0,1)$ and apply Lemma 9:

$$\tau \ge K^{-a}K^{-b}c^{-(b+1)(1-\alpha_a)} = \left(K^{-(\beta+1)}c^{-(1-t)(1-\alpha_a)}\right)^b c^{-(tb+1)(1-\alpha_a)},$$

where $\alpha_a = 1/\alpha^a$. Choose $\gamma \in (0,1)$ such that $K^{-(\beta+1)}\gamma^{-(1-t)(1-\alpha_a)} \ge 1$ for all $\beta \in P$. This is possible since P is compact and the exponent of γ is negative. Hence, if $c < \gamma$, then $\tau > c^{-(tb+1)(1-\alpha^{-a})}$ for all b. Insert this into (15) to get

$$\log\left\{\frac{(\sigma c^2)^\alpha \tau}{e^{2\delta}}\right\} \ge \log\left\{\frac{\sigma^\alpha}{e^{2\delta}}\right\} + \left(-2\alpha + (tb+1)(1-\alpha^{-a})\right)\log c^{-1}.$$

The term in front of $\log c^{-1}$ is positive if

$$b > \frac{1}{t}\left(\frac{2\alpha}{1-\alpha^{-a}} - 1\right). \tag{17}$$

This proves that (15) holds if (17) is satisfied and $c < \gamma$, for γ small enough. □

4. Applications

In this section we apply the lemmas of the previous section. The first result shows that the distortion of the renormalization may be chosen arbitrarily small by increasing the return times. Note that this holds irrespective of the position of the critical point. It was not previously known if the distortion might blow up as the critical point approached the boundary.

Proposition 10 (Distortion invariance)**.** *For every $\delta < \frac{1}{2}\log\alpha$ and $\varepsilon > 0$ there exists $N < \infty$ such that if $f \in \mathcal{L}_\delta$ is (a,b)–renormalizable and $\min\{a,b\} \ge N$, then $\mathcal{R}f \in \mathcal{L}_\varepsilon$. If in addition $c(f) \in \Delta$ then we may choose $\varepsilon \le \lambda^{\min\{a,b\}}$, where $\lambda < 1$ only depends on the closed interval $\Delta \subset (0,1)$ and δ.*

Proof. The renormalization is given by (2). By Lemmas 1 and 6 the monotone extensions of Φ and Ψ of (2) have images which contain arbitrary amounts of space around C. Hence the first statement follows from the Koebe Lemma 33. The second statement follows if we use Lemma 5 in place of Lemma 6. □

The rest of this section concerns infinitely renormalizable maps whose return times are not too small. Some results also require that the return time on the left is comparable to the return time on the right. To this end we say that f is of type $\Omega(N;P)$ if: (1) $\mathcal{R}^{k-1}f$ is (a_k,b_k)–renormalizable, (2) $\min\{a_k,b_k\} \ge N$, and (3) $a_k/b_k \in P$, for all $k \ge 1$. We write $\Omega(N)$ as shorthand for $\Omega(N;\mathbb{R}^+)$, i.e. maps of type $\Omega(N)$ need not have comparable return times on the left and on the right. Finally, we say that f is of *bounded type* if $\sup_k \max\{a_k,b_k\} < \infty$.

Given an infinitely renormalizable map f we are interested in the behavior of the *successive renormalizations*, f, $\mathcal{R}f$, $\mathcal{R}^2f$, etc. By historical precedence we strongly expect the successive renormalizations to have a convergent subsequence, at least for bounded type. However, for Lorenz maps this is not always the case and instead we see the new phenomenon of degeneration. We will say that the successive renormalizations of f *degenerate* if f satisfies the following theorem. Note that it is important here that the return times are comparable on the left and on the right. Without this condition it is more or less possible to choose the accumulation points of $\{c(\mathcal{R}^k f)\}_k$ (but we do not prove that statement here).

Degeneration Theorem. *For every compact interval $P \subset \mathbb{R}^+$ and $\delta < \frac{1}{2}\log\alpha$ there exist a closed interval $\Delta \subset (0,1)$ and $N < \infty$ such that if $\mathcal{R}^n f \in \mathcal{L}_\delta$ is infinitely renormalizable of type $\Omega(N;P)$ and $c(\mathcal{R}^n f) \notin \Delta$ for some $n \geq 0$, then $c(\mathcal{R}^{2k} f) \to 0$ and $c(\mathcal{R}^{2k+1} f) \to 1$ (or vice versa). The rate of convergence of $c(\mathcal{R}^k f)$ to $\{0,1\}$ is faster than λ^k for every $\lambda \in (0,1)$.*

Proof. Since distortion does not increase under renormalization (Proposition 10), we can apply the Flipping Lemma repeatedly. The rate of convergence is faster than exponential since the contraction constant σ in the Flipping Lemma is arbitrary. □

When the successive renormalizations of f have a convergent subsequence we say that f has *a priori bounds*. The exact conditions which give a priori bounds is the content of the following theorem.

A Priori Bounds. *For every $\delta < \frac{1}{2}\log\alpha$ there exists $N < \infty$ such that if $f \in \mathcal{L}_\delta$ is infinitely renormalizable of type $\Omega(N)$ and if there exists a closed interval $\Delta \subset (0,1)$ such that $c(\mathcal{R}^k f) \in \Delta$ for all $k \geq 0$, then $\{\mathcal{R}^k f\}_{k\geq 0}$ is a relatively compact family in the C^0–topology on $\mathcal{L}_\delta$.*

Proof. The successive renormalizations form an equicontinuous family since both the position of the critical point and the distortion is invariant under renormalization (by assumption and Proposition 10, respectively). Hence the result follows from the Arzelà–Ascoli Theorem. □

From the Degeneration Theorem we know that the topological classes of type $\Omega(N;P)$ exhibit degeneration. The following dichotomy shows that the only other alternative is a priori bounds. In this sense this can be seen as a precursor to the Coexistence Theorem. That each class contains at least one map with a priori bounds is the main content of the next section.

Theorem 11 (Dichotomy)**.** *For every compact interval $P \subset \mathbb{R}^+$ and $\delta < \frac{1}{2}\log\alpha$ there exists $N < \infty$ such that if $f \in \mathcal{L}_\delta$ is infinitely renormalizable of type $\Omega(N;P)$, then either the successive renormalizations of f degenerate, or f has a priori bounds.*

Proof. This is a direct consequence of the Degeneration Theorem and the conditions in the theorem on a priori bounds. □

We conclude this section with some dynamical properties of infinitely renormalizable maps. Note that the geometry of a degenerating map is very different from that of a map with a priori bounds as evidenced by the difference in Hausdorff dimension. Also, note that there is no quasi-symmetric rigidity for topological classes containing both degenerating maps and maps with a priori bounds. To our knowledge this is the first example without quasi-symmetric rigidity which is not contrived in some sense (e.g. such as considering conjugated maps with different critical exponents or differing number of critical points). This theorem is an improvement over previous results (see [18]) since it holds independently of the behavior of the critical points of the successive renormalizations.

Theorem 12 (Dynamical properties)**.** *For every $\delta < \frac{1}{2}\log\alpha$ there exists $N < \infty$ such that if $\mathcal{R}^n f \in \mathcal{L}_\delta$ is infinitely renormalizable of type $\Omega(N)$ for some $n \geq 0$, then*

(1) f has no wandering intervals and is ergodic,
(2) f has a minimal Cantor attractor Λ of measure zero,
(3) Λ supports one or two ergodic invariant probability measures.

Furthermore, if f is of bounded type then

(4) Λ is uniquely ergodic and the measure on Λ is physical,
(5) the Hausdorff dimension $\mathrm{HD}(\Lambda) = 0$ if the successive renormalizations of f degenerate, otherwise $\mathrm{HD}(\Lambda) \in (0, 1)$.

Remark. If f is of unbounded type, then it may not have a physical measure [19].

Proof. Assume without loss of generality that $f \in \mathcal{L}_\delta$ is infinitely renormalizable of type $\Omega(N)$. We will begin by discussing how to choose N.

By Proposition 10 we can choose N such that $\mathcal{R}^n f \in \mathcal{L}_\delta$, $\forall n \geq 0$. We will need this to apply Lemma 6 to $\mathcal{R}^n f$ for each n. Since f is infinitely renormalizable there exists a sequence of closed intervals $\{C_n\}$ such that $C_{n-1} \supset C_n$ and the first-return map of f to C_n is affinely conjugate to a map in $\mathcal{L}$, $\forall n$. In particular, the boundary points of C_n are periodic points whose orbits never enter $\operatorname{Int} C_n$. In other words, the intervals C_n are *nice* (see [18, §3]). Let $T_n : D_n \to C_n$ be the *first-entry* map of f to C_n. That is, for every $x \in D_n = \bigcup_{i \geq 0} f^{-i}(C_n)$ define $T(x) = f^{t(x)}(x)$, where $t(x)$ is the smallest non-negative integer such that $f^{t(x)}(x) \in C_n$. Now let $f^i : I \to C_n$ be any branch of T_n for some arbitrary n and let $f^i : \hat{I} \to \hat{C}_n$ be its monotone extension. By lemmas 6 and 1 we can choose N such that $\hat{C}_n \cap C_{n-1}$ contains a 1-scaled neighborhood of C_n. Since the branch and n was arbitrary this holds for all branches and for all n. This finishes our discussion on how to choose N.

To show that f has no wandering intervals it suffices to show that the branches of T_n shrink to points uniformly as $n \to \infty$ (cf. [18, Theorem 3.11]). Let $f^i : I \to C_{n-1}$ and $f^j : J \to C_n$ be branches of T_{n-1} and T_n, respectively, and assume that $I \supset J$. We claim that there exists a constant $\tau > 0$, not depending on f, such that I contains a τ-scaled neighborhood of J. From this claim it follows that $|J| \leq (1 + 2\tau)^{-n}$, i.e. all branches shrink to points uniformly in n. We will now prove the claim. Let $f^j : \hat{J} \to \hat{C}_n$ be the monotone extension of $f^j|_J$. By the above, $\hat{C}_n \cap C_{n-1}$ contains a 1-scaled neighborhood of C_n and by the Macroscopic Koebe Principle (see [20, p. 287]) it follows that $\hat{J} \cap I$ contains a τ-scaled neighborhood of J, for some $\tau > 0$ not depending on f. This proves the claim and hence f has no wandering intervals. That f is ergodic follows from [18, Theorem 3.12].

We claim that the ω-limit set of c, $\omega(c)$, is a Cantor attractor for f and hence that the attractor is minimal. Since f is not defined at c let us emphasize that $\omega(c)$ is defined as the union of the ω-limit sets of the two critical values of f. The claim follows from the fact that $|D_n| = 1$ (see [18, Proposition 3.7]) and $|C_n| \leq |C_{n-1}|/3$ (since C_{n-1} contains a 1-scaled neighborhood of C_n). That is, almost all points pass arbitrarily close to c after sufficiently many iterates.

To see that the Cantor attractor has zero measure note that we can cover it by branches of T_n. Let Λ_n be the smallest such cover. Then $|\Lambda_n| \leq (1 + 2\tau)^{-1}|\Lambda_{n-1}|$ since each branch J of T_n is contained in a τ-scaled neighborhood of a branch I of T_{n-1} by the above. Hence $|\omega(c)| \leq \lim|\Lambda_n| = 0$.

Unique ergodicity can be proved with techniques from [8], see also [19].

We now prove the last statement. If $\{\mathcal{R}^n f\}$ degenerate then the longest interval of Λ_n shrinks at a faster than exponential rate in n, by the Degeneration Theorem and Proposition 10. Since f is of bounded type, the number of intervals in Λ_n is at most exponential in n. It follows that $\mathrm{HD}(\Lambda) = 0$ (see e.g. [5, Proposition 4.1]).

If $\{\mathcal{R}^n f\}$ do not degenerate, then f has a priori bounds by Theorem 11. It follows from standard arguments that $\mathrm{HD}(\Lambda) \in (0, 1)$ (see e.g. [20, Theorem VI.2.1]). □

5. Existence of fixed points

This whole section is dedicated to the proof of the following theorem.

Theorem 13 (Existence of fixed points)**.** *For every $\beta \in \mathbb{Q}^+$ there exists $N < \infty$ such that $\mathcal{R}$ has an (a, b)–renormalizable fixed point, for all $b \geq N$ and $a/b = \beta$.*

Remark. The combinatorics here are different from those of [18, Theorem 6.1]. This causes drastic changes in the dynamics: here we see three unstable directions (see Theorem 29), whereas in [18] there appear only to be two. The extra unstable direction complicates the arguments, but on the other hand here we are able to prove the existence of fixed points for infinitely many different types of combinatorics.

Fix $\beta \in \mathbb{Q}^+$ and assume $a/b = \beta$. We are going to define a subset of the domain of (a, b)–renormalization and then prove that this subset contains a fixed point. The reason why we only consider a subset is that we cannot control the critical point and critical values of the renormalization on the whole domain. For any $b \geq 1$, closed interval $\Delta \subset (0, 1)$, $\gamma \in (0, \frac{1}{2})$ and $\delta \in (0, \frac{1}{2} \log \alpha)$, define $Q = [1 - \gamma, 1]^2$ and

$$\mathcal{D}(b, \Delta, \gamma, \delta) = \{(a, b)\text{–renormalizable } f \in \mathcal{L}_\delta \mid c \in \Delta,\ (u', v') \in Q,\ a/b = \beta\}.$$

Here f is identified with (c, u, v, ϕ, ψ) and $\mathcal{R}f$ is identified with $(c', u', v', \phi', \psi')$.

The next lemma gives very explicit control over how the relative critical point moves under renormalization. In particular, the relative critical point of the renormalization is a function of the relative critical point, up to a multiplicative factor close to 1 which we can control.

Lemma 14. *Let $\tilde{c} = c/(1 - c)$ and $\tilde{c}' = c'/(1 - c')$ denote the relative critical point of f and $\mathcal{R}f$, respectively, and let $h(\tilde{c})^\alpha = (\alpha(1 + \tilde{c}))^{1-\beta}/\tilde{c}$. There exists a continuously differentiable map $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that if $f \in \mathcal{D}(b, \Delta, \gamma, \delta)$, then*

$$\tilde{c}' = \tilde{c} g(\tilde{c}) h(\tilde{c})^b \kappa \cdot \left(1 + O(b(\delta + (1 - u) + (1 - v)) + \delta' + \alpha^{-b})\right),$$

where $1 - \gamma \leq \kappa^\alpha \leq (1 - \gamma)^{-1}$ and $\delta' = \max\{\|\phi'\|, \|\psi'\|\}$.

Proof. The proof is about controlling the factors appearing in the formula which relates $\tilde{c}'$ to $\tilde{c}$ in Lemma 7. First we control the factor $\tilde{v}(\mathcal{R}f)/\tilde{v}(f)$, then we control the factor $Df^b(x)/Df^a(y)$.

Note that by Lemma 32

$$\tilde{v}(f) = \frac{|\phi([0, u])|}{|\psi([1 - v, 1])|} = \frac{u}{v} \exp\{O(\delta)\} = 1 + O(\delta + (1 - u) + (1 - v)).$$

The same argument shows that $\tilde{v}(\mathcal{R}f) = u'/v'(1 + O(\delta'))$. Since $u', v' \in [1 - \gamma, 1]$ $\tilde{v}(\mathcal{R}f) = \kappa^\alpha \cdot (1 + O(\delta'))$, where $1 - \gamma \leq \kappa^\alpha \leq (1 - \gamma)^{-1}$. We have proved that

$$\frac{\tilde{v}(\mathcal{R}f)}{\tilde{v}(f)} = \kappa^\alpha \cdot (1 + O(\delta + (1 - u) + (1 - v) + \delta')) \,. \tag{18}$$

Next we show how to control $Df_-^b(x)/Df_+^a(y)$ by comparing with a full map of the standard family $\hat{q}$ (i.e. for which $u = 1 = v$):

$$\frac{Df^b(x)}{D\hat{q}^b(0)} = \frac{Df^b(x)}{Df^b(f_-^{-b}c)} \cdot \frac{Df^b(f_-^{-b}c)}{D\hat{q}^b(\hat{q}_-^{-b}c)} \cdot \frac{D\hat{q}^b(\hat{q}_-^{-b}c)}{D\hat{q}^b(0)}. \tag{19}$$

The first factor is $\exp\{O(\delta'+\delta)\}$, since the diffeomorphic part of the renormalization is given by $\psi' = [\Psi|\Psi^{-1}(C)]$, where $\Psi = f^b \circ \psi$, and $x \in f_-^{-b}(C)$. Consider the second factor. Let $c_k = f_-^{-k}(c)$, $\hat{c}_k = \hat{q}_-^{-k}(c)$ and use (1) to get

$$\frac{Df^b(f_-^{-b}c)}{D\hat{q}^b(\hat{q}_-^{-b}c)} = \prod_{k=1}^{b} D\phi(qc_k)\frac{Dq(c_k)}{D\hat{q}(\hat{c}_k)} = \prod_{k=1}^{b} D\phi(qc_k)u\left(1 + \frac{\hat{c}_k - c_k}{c - \hat{c}_k}\right)^{\alpha-1}.$$

Note that $|D\phi|^b = 1 + O(\delta b)$ and $u^b = 1 + O(b(1-u))$ so we only need to control how fast $l_k = \hat{c}_k - c_k$ shrinks. Use $l_{k+1} = \left|\hat{q}_-^{-1}(\hat{c}_k) - q_-^{-1}(\phi^{-1}c_k)\right|$ to estimate

$$\begin{aligned} l_{k+1} &\le \left|\hat{q}_-^{-1}(\hat{c}_k) - \hat{q}_-^{-1}(c_k)\right| + \left|\hat{q}_-^{-1}(c_k) - q_-^{-1}(c_k)\right| + \left|q_-^{-1}(c_k) - q_-^{-1}(\phi^{-1}c_k)\right| \\ &\le D\hat{q}_-^{-1}(t_k)l_k + Kc_k(1-u) + Dq_-^{-1}(s_k)\left|c_k - \phi^{-1}(c_k)\right|, \end{aligned}$$

for some $t_k \in [c_k, \hat{c}_k]$ and $s_k \in [\phi^{-1}(c_k), c_k]$. Because of the Expansion Lemma, $t_k, s_k \le K\lambda^k$ for some $\lambda \in (0,1)$. Let $D_k = D\hat{q}_-^{-1}(t_k)$, then $D_k \to D\hat{q}(0)^{-1} \le \alpha^{-1}$ exponentially fast and $Dq_-^{-1}(s_k)\left|c_k - \phi^{-1}(c_k)\right| \le K\delta c_k$. We have the relation

$$l_{k+1} \le D_k l_k + K\lambda^{k+1}(\delta + 1 - u).$$

Choose $\lambda \in (\alpha^{-1}, 1)$ so that $\limsup D_j/\lambda < 1$. An induction argument shows that $l_k \le K\lambda^k(1 - u + \delta)$. Thus

$$\sum \log\left(1 + \frac{l_k}{c - \hat{c}_k}\right) \le K(\delta + 1 - u).$$

We have shown that the second factor of (19) is $1 + O(b\delta + b(1-u))$.

Consider the third factor of (19); call it $G_b^-(c)$. We claim that $G_n^-|_{[\varepsilon,1-\varepsilon]}$ converges in C^1 as $n \to \infty$, for every $\varepsilon > 0$. Let us prove the claim by showing that it holds for $\log G_n^-$. As in the above we can appeal to the Expansion Lemma to see that $\forall\varepsilon > 0$ there exists $m_\varepsilon < \infty$ and $K_\varepsilon < \infty$ such that

$$\hat{c}_n \le \min\left\{\varepsilon,\, K_\varepsilon\alpha^{-n}\right\}, \quad \forall n \ge m_\varepsilon,\, \forall c \in [\varepsilon, 1-\varepsilon]. \tag{20}$$

Here we used that the contraction rate of $\hat{c}_n \mapsto \hat{c}_{n+1}$ is $D\hat{q}_-^{-1}(0) = c/\alpha \le 1/\alpha$. We will need the relation

$$\left(1 - \frac{\hat{c}_{n+1}}{c}\right)^{\alpha} = 1 - \hat{c}_n, \tag{21}$$

which is a consequence of (1). By definition, $G_n^-(c) = D\hat{q}^n(\hat{c}_n)/D\hat{q}^n(0)$, which combined with (21) and (1) gives

$$\log G_n^-(c) = \frac{\alpha - 1}{\alpha}\sum_{k=0}^{n-1}\log\left(1 - \hat{c}_k\right). \tag{22}$$

This sequence is uniformly convergent and hence $G^- = \lim G_n^- \in C^0$, since

$$\sum_{k \ge m_\varepsilon}\left|\log\left(1 - \hat{c}_k\right)\right| \le \sum_{k \ge m_\varepsilon}\frac{\hat{c}_k}{1 - \hat{c}_k} \le \frac{K_\varepsilon}{(1-\varepsilon)}\sum_{k \ge m_\varepsilon}\alpha^{-k} = \frac{K_\varepsilon\alpha^{-(m_\varepsilon - 1)}}{(\alpha - 1)(1 - \varepsilon)}.$$

The tail can be written as $\log\{G^-(c)/G_n^-(c)\}$, so this also shows that

$$\frac{G_b^-(c)}{G^-(c)} = 1 + O(\alpha^{-b}), \quad \forall c \in \Delta.$$

Before proving that G^- is C^1 let us get back to (19), which by the above is now

$$\frac{Df^b(x)}{D\hat{q}^b(0)} = (1 + O(\delta + \delta'))(1 + O(b\delta + b(1-u)))G^-(c)(1 + O(\alpha^{-b})).$$

Now do an analogous estimate for $Df^a(y)$, using that $a/b = \beta$, to get

$$\frac{Df^b(x)}{Df^a(y)} = \frac{D\hat{q}^b(0)}{D\hat{q}^a(1)} \frac{G^-(c)}{G^+(c)} \left(1 + O(b(\delta + (1-u) + (1-v)) + \delta' + \alpha^{-b})\right), \tag{23}$$

where G^+ is defined analogously to G^-. Define $G(c) = G^-(c)/G^+(c)$ and

$$H(c) = \left(\frac{D\hat{q}^b(0)}{D\hat{q}^a(1)}\right)^{1/b} = \alpha^{1-\beta}\frac{(1-c)^\beta}{c}.$$

Let $\tau(c) = c/(1-c)$ be the diffeomorphism which sends an absolute critical point to its relative critical point. Define $g(\tilde{c})^\alpha = G \circ \tau^{-1}(\tilde{c})$ and $h(\tilde{c})^\alpha = H \circ \tau^{-1}(\tilde{c})$. To finish the proof of the main statement apply Lemma 7, using equations (18) and (23) together with the definitions of h and g.

It remains to show that $g : \mathbb{R}^+ \to \mathbb{R}^+$ is C^1, which we prove by showing that $D\log G_n^-$ converges uniformly to $D\log G^-$ on $[\varepsilon, 1-\varepsilon]$, $\forall \varepsilon > 0$. Differentiating (22) gives (nb. were write ∂_c instead of $\frac{\partial}{\partial c}$)

$$D\log G_n^-(c) = -\frac{\alpha - 1}{\alpha}\sum_{k=0}^{n-1}(1-\hat{c}_k)^{-1}\partial_c\hat{c}_k. \tag{24}$$

Solving (21) for $\hat{c}_{n+1}$ and differentiating gives

$$\partial_c\hat{c}_{n+1} = \partial_c\hat{c}_n\frac{c}{\alpha}(1-\hat{c}_n)^{-1+1/\alpha} + 1 - (1-\hat{c}_n)^{1/\alpha}.$$

By (20), for $n+1 \geq m_\varepsilon$ and $\varepsilon \leq c \leq 1-\varepsilon$ we get

$$|\partial_c\hat{c}_{n+1}| \leq |\partial_c\hat{c}_n|\frac{1}{\alpha}(1-\varepsilon)^{1/\alpha} + \frac{1}{\alpha}(1-\varepsilon)^{-1+1/\alpha}K_\varepsilon\alpha^{-n}.$$

Here we used that $\varphi(t) = 1 - (1-t)^{1/\alpha}$ is convex on $(0,1)$ and $\varphi(0) = 0$, so $\varphi(t) \leq D\varphi(t)t$. An induction argument gives

$$|\partial_c\hat{c}_{m_\varepsilon+k}| \leq |\partial_c\hat{c}_{m_\varepsilon}|\alpha^{-k} + K'_\varepsilon\alpha^{-(m_\varepsilon+k)},$$

so the tail of (24) converges absolutely. Hence $D\log G_n^-$ converges uniformly. □

From the previous lemma we will be able to deduce that each $\beta = a/b$ uniquely defines the asymptotic critical point of an (a,b)–renormalization fixed point (as $b \to \infty$ with $a/b = \beta$). Call this asymptotic critical point $c_\infty(\beta)$; it is defined as the unique solution to

$$\frac{x}{\alpha} = \left(\frac{1-x}{\alpha}\right)^\beta.$$

We will write c_∞ instead of $c_\infty(\beta)$ when there is no need to emphasize the dependence on β. Let $\tilde{c}_\infty = c_\infty/(1-c_\infty)$ be the relative asymptotic critical point. Note that c_∞ is defined by the relative asymptotic critical point satisfying $h(\tilde{c}_\infty) = 1$, where h is defined in the previous lemma.

The next lemma serves to define the subset of the domain of renormalization where we can control the critical point and the critical values of the renormalization. The main point is that within this subset the critical point of the renormalization expands away from c_∞ while staying bounded, and that the critical values are close to 1. It also shows that any renormalization fixed point must have critical point close to c_∞. Some of the conclusions of the lemma will only be needed later when we prove the existence of unstable manifolds.

Lemma 15. *For every $\varepsilon > 0$, l and r satisfying $0 < l \pm \varepsilon < c_\infty < r \pm \varepsilon < 1$, there exist $N < \infty$, $0 < \gamma < \min\{l-\varepsilon, 1-(r+\varepsilon)\}$, $K < \infty$ and closed intervals $\{\Delta_n \subset (l+\varepsilon, r-\varepsilon)\}_{n \geq N}$ such that if $f \in \mathcal{D}(b, \Delta_b, \gamma, \delta_b = 1/b^2)$ and $b \geq N$, then the following holds:*

1. *$c(\mathcal{R}f) \in (l-\varepsilon, r+\varepsilon)$,*
2. *if $c(f) \in \partial\Delta_b$ then $c(\mathcal{R}f) \notin [l+\varepsilon, r-\varepsilon]$,*
3. *if $c(f) = c(\mathcal{R}f)$ then $\operatorname{dist}(c(\mathcal{R}f), \partial\Delta_b) \geq K^{-1}|\Delta_b|$,*
4. *$K^{-1} \leq b|\Delta_b| \leq K$,*
5. *$u, v > 1-\gamma$ and $\mathcal{R}f \in \mathcal{L}_{\delta_b'}$ where $\delta_b' < \delta_b$,*
6. *$\mathcal{R}f$ does not have a trivial branch.*

Proof. Let $\tau : (0,1) \to \mathbb{R}^+$ be the diffeomorphism taking a critical point to its relative critical point, $\tau(x) = x/(1-x)$. We will work in $\mathbb{R}^+$ and then transfer back to $(0,1)$ via τ^{-1}. Let $\rho(x) = xg(x)h(x)^b$, where g and h are defined in Lemma 14. Note that ρ describes how the relative critical point moves under renormalization, up to a multiplicative factor which we need to control.

We will now construct Δ_b. Let $J = \tau([l,r])$ and let $J_\varepsilon = \tau((l+\varepsilon, r-\varepsilon))$. Define $\tilde{\Delta}_b = \rho^{-1}(J) \cap J_\varepsilon$ and $\Delta_b = \tau^{-1}(\tilde{\Delta}_b)$ so that $\Delta_b \subset (l+\varepsilon, r-\varepsilon)$. Note that ρ maps the open interval J_ε over J for b sufficiently large, since h is strictly decreasing, $h(\tilde{c}_\infty) = 1$ and $\tilde{c}_\infty \in J_\varepsilon$. Hence we may choose $N < \infty$ such that ρ maps $\tilde{\Delta}_b$ onto J for $b \geq N$. By Lemma 14 ρ is differentiable, $g > 0$ and $h > 0$, so

$$D \log \rho(x) = b \cdot D \log h(x) + D \log g(x) + 1/x.$$

Since $Dh < 0$, J is compact, and $g, h \in C^1$, this shows that $-Kb \leq D\log\rho|_J \leq -b/K$ for b sufficiently large. Since $\rho > 0$ is bounded on $\rho^{-1}(J)$, this in turn shows that $\exists N < \infty$ such that

$$D\rho|_J < 0 \quad \text{and} \quad b/K \leq \left|D\rho|_{\rho^{-1}(J)\cap J}\right| \leq Kb, \quad \forall b \geq N. \tag{25}$$

In particular, ρ is injective on J so $\rho : \tilde{\Delta}_b \to J$ is a diffeomorphism and consequently $\tilde{\Delta}_b$ and Δ_b are compact intervals. Property 4 is a direct consequence of (25), since $D\tau^{-1}$ has bounded distortion on compact intervals.

We will now prove the remaining properties. Consider the multiplicative factor of Lemma 14. Note that $1-u$, $1-v$ and δ' are exponentially small in b since c is bounded (and $\delta_b < \frac{1}{2}\log\alpha$); the Expansion Lemma gives the statement about u and v (see the remark after the lemma), whereas Proposition 10 gives the statement about δ'. This proves property 5, and together with Lemma 14 shows that we may choose $N < \infty$ large and $\gamma > 0$ small enough so that for $b \geq N$:

$$1 - \varepsilon \leq \frac{\tilde{c}'}{\rho(\tilde{c})} \leq 1 + \varepsilon. \tag{26}$$

It is clear that we can ensure $\gamma < \min\{l - \varepsilon, 1 - (r + \varepsilon)\}$. A calculation shows that

$$1 - \varepsilon < \frac{\tau(x)}{\tau(y)} < 1 + \varepsilon \implies |x - y| < \varepsilon. \tag{27}$$

Since $\rho : \tilde{\Delta}_b \to J$ is a diffeomorphism (26) and (27) imply properties 1 and 2.

To prove property 3 note first that $\rho|_J$ has a unique fixed point $\tau(p_b) \in \tilde{\Delta}_b$. Since $\tau(p_b) \in J_\varepsilon$, it sits at a bounded distance away from ∂J, so $\operatorname{dist}(\tau(p_b), \partial J) \geq K^{-1}|J|$. By (25) $\rho|_{\tilde{\Delta}_b}$ has bounded distortion (and so does $\tau|_{\Delta_b}$), so it follows that $\operatorname{dist}(p_b, \partial\Delta_b) \geq K^{-1}|\Delta_b|$. It remains to prove that any critical point which is fixed under renormalization must be close to p_b (relative to $|\Delta_b|$). By equations (26) and (27) $|c' - c| < \varepsilon$ for all $c \in \Delta_b$. In particular, any fixed point $c = c'$ must satisfy $|c - p_b| \leq K\varepsilon / D(\tau^{-1} \circ \rho \circ \tau)(s)$, for some $s \in \Delta_b$. By the mean-value theorem $D(\tau^{-1} \circ \rho \circ \tau)(t) = (r - l)/|\Delta_b|$ for some $t \in \Delta_b$. Since $\tau^{-1} \circ \rho \circ \tau$ has bounded distortion on Δ_b, $|c - p_b| \leq K\varepsilon|\Delta_b|/(r - l)$. From this and the discussion before (26) it follows that we may choose N and γ so that $|c - p_b|/|\Delta_b|$ is as small as we wish (independently of b) so property 3 follows.

Finally, property 6 holds since $\gamma < c' < 1 - \gamma$ whereas $u', v' \geq 1 - \gamma$. Explicitly, if the left branch of $\mathcal{R}f$ is trivial, then $u' = \phi'(c')$, but this is impossible since $\phi'(c') \to c'$ as $b \to \infty$. A similar argument holds for the right branch of $\mathcal{R}f$. $\square$

For the remainder of this section we assume that $\Delta = [l, r]$, ε, N and γ have been chosen such that the previous lemma holds and assume that $b \geq N$. Define

$$\mathcal{D}_b = \mathcal{D}(b, \Delta_b, \gamma, \delta_b).$$

The next lemma describes the topology of $\mathcal{D}_b$. It relies on Lemma 19, which is at the end of this section as it is independent of the presentation of the set $\mathcal{D}_b$.

Let π_{uv} be the projection onto the (u, v)–coordinates and let $\mathcal{F}_\nu$ denote the two-dimensional family $(u, v) \mapsto (u, v, c, \phi, \psi)$, where $\nu = (c, \phi, \psi)$.

Lemma 16. *$\mathcal{D}_b$ is connected and $\pi_{uv} \circ \mathcal{R}$ maps $\mathcal{D}_b \cap \mathcal{F}_{\nu_0}$ diffeomorphically onto Q, for every family $\mathcal{F}_{\nu_0}$ intersecting $\mathcal{D}_b$. In particular, the intersection of $\mathcal{D}_b$ with the standard family is homeomorphic to $Q \times \Delta_b$.*

Proof. Assume that $\mathcal{F}_{\nu_0}$ intersects $\mathcal{D}_b$ and let $\xi_{\nu_0} = \pi_{uv} \circ \mathcal{R}|_{\mathcal{F}_{\nu_0}}$. Note that ξ_{ν_0} is a local diffeomorphism since $\det D\xi_{\nu_0}(u, v) > 0$ by Lemma 19. We will show that $\xi_{\nu_0} : \xi_{\nu_0}^{-1}(Q) \to Q$ is a diffeomorphism.

We begin by proving that $\xi_{\nu_0} : J \to Q$ is a diffeomorphism for any connected component $J \subset \xi_{\nu_0}^{-1}(Q)$. The boundary of J is either a preimage of ∂Q, or a boundary point of $\mathcal{D}_b$. A point in $\partial\mathcal{D}_b$ which is not in $\xi_{\nu_0}^{-1}(\partial Q)$ must have a renormalization with a trivial branch. Hence Lemma 15(6) implies that $\partial J \subset \xi_{\nu_0}^{-1}(\partial Q)$ and from this we get that if $y \in \operatorname{Int} Q$ has a preimage $x \in J$, then $x \in \operatorname{Int} J$. But ξ_{ν_0} is a local diffeomorphism, so a neighborhood of x is mapped to a neighborhood of y. Thus $\xi_{\nu_0}(J)$ is open (in the subspace topology on Q). But $\xi_{\nu_0}(J)$ is also closed, since it is compact by the fact that $\xi_{\nu_0}^{-1}(Q)$ is closed. Thus $\xi_{\nu_0}(J) = Q$. Since $\xi_{\nu_0} : J \to Q$ is proper, J is path-connected and Q is simply connected, it follows that ξ_{ν_0} is also injective (see [10]).

It remains to show that $\xi_{\nu_0}^{-1}(Q)$ is connected. This will be done in two steps: (1) showing that if $\xi_{\nu_0}^{-1}(Q)$ is not connected, then the two-dimensional standard family $\mathcal{F}_{(c,\mathrm{id},\mathrm{id})}$ has more than one full vertex (i.e. a map whose renormalization has both branches full), and (2) showing that any two-dimensional standard family has a unique full vertex.

If $\xi_{\nu_0}^{-1}(Q)$ is not connected, then there exist full vertices $p_1(\nu_0) \neq p_2(\nu_0)$ in $\mathcal{F}_{\nu_0}$, since every connected component of $\xi_{\nu_0}^{-1}$ maps onto Q. That is, $\xi_{\nu_0}(p_i(\nu_0)) = (1,1)$, for $i = 1,2$. Both of these full vertex lies at the intersection of two smooth curves $\hat{\chi}_i^{\pm} \subset \mathcal{F}_{\nu_0}$ defined by the left (resp. right) branch of the renormalization having a full branch. The intersection is transversal, since ξ_{ν_0} maps $\hat{\chi}_i^{\pm}$ diffeomorphically into $\{u = 1\}$ and $\{v = 1\}$, respectively. Hence we can perturb ν away from ν_0 and get that $p_i(\nu_0)$ persists in a neighborhood V of ν_0; i.e. $\xi_\nu(p_i(\nu)) = (1,1)$, $\forall \nu \in V$. Since ξ_ν is a local diffeomorphism this transversality argument shows that $p_i(\nu)$ is well-defined for all ν such that $\mathcal{F}_\nu \cap \mathcal{D}_b \neq \emptyset$. Furthermore, the facts that $p_1(\nu_0) \neq p_2(\nu_0)$ and that ξ_ν is a local diffeomorphism implies that $p_1(\nu) \neq p_2(\nu)$. In particular, the two-dimensional standard family $\mathcal{F}_{\nu_1}$, $\nu_1 = (c, \mathrm{id}, \mathrm{id})$, has two full vertices (for every c). This completes the first step.

Note that if there was a unique full vertex then the above argument implies that $\mathcal{D}_b$ is connected. Hence, connectedness follows from step two which we now prove.

In light of [16, Prop. 6.1] it suffices to show that there is a unique trivial vertex (i.e. whose renormalization has trivial branches) and that any connected component of the domain of (a,b)–renormalization in $\mathcal{F}_{\nu_1}$ has at most one full vertex.

We will now show that there is a unique trivial vertex in $\mathcal{F}_{\nu_1}$. The maps q with branches $q_\pm$ in $\mathcal{F}_{\nu_1}$ are given by (1). Note that q_- depends on u and q_+ depends on v. A trivial vertex lies at every intersection of the two curves

$$\hat{\sigma}_l = \{(u,v) \mid q_+^{-a}(c) = u\} \quad \text{and} \quad \hat{\sigma}_r = \{(u,v) \mid q_-^{-b}(c) = 1 - v\}, \tag{28}$$

inside $(c,1) \times (1-c,1)$. Note, if $(u,v) \in \hat{\sigma}_l$ (resp. $(u,v) \in \hat{\sigma}_r$), then $\mathcal{R}q$ has a trivial left (resp. right) branch. For every $v \in [1-c,1]$ there is a unique $\sigma_l(v) \in [c,1]$ such that $(\sigma_l(v), v) \in \hat{\sigma}_l$. Hence $\hat{\sigma}_l$ is the graph of a function $\sigma_l : [1-c,1] \to [c,1]$ of v. Similarly, $\hat{\sigma}_r$ is the graph of some function $\sigma_r : [c,1] \to [1-c,1]$ of u. We claim that σ_l and σ_r are concave (the proof is below). From (28) it follows that $\sigma_l(1-c) = c$ and $\sigma_r(c) = 1-c$, and from (1) it follows that that the derivative of σ_l (resp. σ_r) is unbounded at $1-c$ (resp. c). Hence $(\sigma_l(v), v) = (u, \sigma_r(u))$ has a unique solution in $(c,1) \times (1-c,1)$.

In order to reach a contradiction, assume that some connected component I of the domain of (a,b)–renormalization has two full vertices p_i. As was noted above each p_i lies at the transversal intersection of two curves $\hat{\chi}_i^{\pm}$. By [16, Prop. 6.1] $\hat{\chi}_i^{\pm}$ are graphs over the diagonal $\{u = v\}$ locally around p_i and we may assume that $\hat{\chi}_i^{+}$ lies over $\hat{\chi}_i^{-}$ (as graphs over the diagonal) at both p_1 and p_2. By assigning an orientation to ∂I we see that this implies that $\hat{\chi}_1^{+}$ (resp. $\hat{\chi}_1^{-}$) goes toward (resp. away from) p_1 and the opposite at p_2 (or vice versa). Hence p_1 and p_2 have neighborhoods which are mapped with different orientation to neighborhoods of $(1,1)$ by ξ_{ν_1}. This contradicts the fact that $D\xi_{\nu_1}$ preserves orientation.

We will now prove that σ_r is concave (the proof for σ_l is similar). By (28) $\sigma_r(u) = 1 - q_-^{-b}(c)$. Let $c_n = q_-^{-n}(c)$ and $g_n(u) = -\partial_u c_n = D\sigma_r(u)$. We will show that $Dg_n < 0$, $\forall n > 1$. Write $g_n(u) = -\partial_u(q_-^{-1}(c_{n-1}))$ and differentiate to get

$$Dq_-(c_n) g_n(u) = g_{n-1}(u) + \hat{q}(c_n),$$

where $\hat{q}_-$ is the full branch (i.e. q_- with $u = 1$). Apply ∂_u to both sides to get

$$q_-(c_n) Dg_n(u) = Dg_{n-1}(u) - g_n(u)\left(2D\hat{q}_-(c_n) - D^2 q_-(c_n) g_n(u)\right).$$

From $g_0 = 0$ we get $g_1 > 0$ and $Dg_1 < 0$, so $g_n > 0$ and $Dg_n < 0$ by induction, $\forall n \geq 1$ (use $q_- > 0$, $Dq_- > 0$, and $D^2 q_- < 0$).

Finally, the map $(u, v, c) \mapsto (\pi_{uv} \circ \mathcal{R}(u, v, c, \mathrm{id}, \mathrm{id}), c)$ is injective and continuous, hence it is a homeomorphism. □

We claim that $\mathcal{D}_b$ contains a fixed point of $\mathcal{R}$. To prove this we will reduce the fixed point problem from an infinite-dimensional space to a three-dimensional space using the following result:

Homotopy Lemma ([18, Appendix A])**.** *Let Y be a normal topological space, let $X \subset Y$ be a closed subset and let $h : X \times [0, 1] \to Y$ be a homotopy between f and g. If every extension of $g|_{\partial X}$ to X has a fixed point and if $h(x, t) \neq x$ for all $x \in \partial X$ and $t \in [0, 1]$, then f has a fixed point.*

The homotopy we will use is

$$\pi_t(u, v, c, \phi, \psi) = (u, v, c, (1 - t)\phi, (1 - t)\psi).$$

Recall that $(1 - t)\phi$ means scaling the nonlinearity of ϕ and note that π_1 is the projection to the standard family. From the Homotopy Lemma and the next two lemmas it immediately follows that $\mathcal{R}$ has a fixed point in $\mathcal{D}_b$.

Lemma 17. *$\pi_t \circ \mathcal{R}f \neq f$, $\forall f \in \partial\mathcal{D}_b$, $\forall t \in [0, 1]$.*

Proof. We will consider each boundary piece of $\mathcal{D}_b$ one at a time:

(1) If $\mathcal{R}f$ has a full branch, then $u' = 1$ or $v' = 1$, but both $u < 1$ and $v < 1$ for a renormalizable map.
(2) If $\|\phi\| = \delta_b$, then $\|\phi'\| < \delta_b$ by Lemma 15(5), hence $\|(1 - t)\phi'\| < \delta_b$ (and similarly for $\|\psi\|$).
(3) If $c \in \partial\Delta_b$, then $c' \notin \Delta_b$ by Lemma 15(2).
(4) If $\mathcal{R}f$ has a trivial branch, then $f \notin \mathcal{D}_b$ by Lemma 15(6). Hence this boundary piece is not part of $\partial\mathcal{D}_b$.

This shows that $\pi_t \circ \mathcal{R}$ has no fixed point in $\partial\mathcal{D}_b$. □

Lemma 18. *Let D be the intersection of $\mathcal{D}_b$ with the standard family and let $R = \pi_1 \circ \mathcal{R}|_D$. Every extension of $R|_{\partial D}$ to D has a fixed point.*

Proof. Let $\rho : \mathbb{R}^3 \setminus \{0\} \to \mathbb{S}^2$ be the radial projection $\rho(x) = x/\|x\|$ and define the *displacement map* of R, $d : \partial D \to \mathbb{S}^2$, by $d(f) = \rho(R(f) - f)$. Let $H_u \subset \mathbb{S}^2$ denote the open hemisphere around the positive u-axis and let $-H_u$ denote its antipodal set. Define H_v, H_c and their antipodal sets similarly. By Lemma 15, the d–image of each boundary piece of ∂D lies in a hemisphere: if $u' = 1$, then $u < u' = 1$, so $d(f) \in H_u$; if $u' = 1 - \gamma$, then $u' < u$, so $d(f) \in -H_u$, if $c = \partial^-\Delta_b$, then $c' > c$, so $d(f) \in H_c$; if $c = \partial^+\Delta_b$, then $c' < c$, so $d(f) \in -H_c$; and similarly, $v' = 1 \implies d(f) \in H_v$, $v' = 1 - \gamma \implies d(f) \in -H_v$. We call what we have just described the *hemisphere property* of the displacement map.

By Lemma 16 ∂D is a topological sphere so there is a well defined notion of the degree of d. We claim that $\deg d \neq 0$. Note that this claim would finish the proof, since if an extension of $R|_{\partial D}$ to D did not have a fixed point then its displacement map would extend to all of D and consequently have degree zero. We now prove the claim by constructing a homotopy from d to a homeomorphism. By Lemma 16 there is a homeomorphism $h : K \to \partial D$, where $K = \partial[-1, 1]^3$. We choose h so that it takes faces of K onto boundary pieces of ∂D in the following manner: the bottom face of K is mapped onto $\{v' = 1-\gamma\} \cap D$, the top face is mapped onto $\{v' = 1\} \cap D$, the left face is mapped to $\{c = \partial^-\Delta_b\} \cap D$, and so on. To be clear, here we choose the right-handed coordinate system for K so that "right" corresponds to increasing c and "up" corresponds to increasing v. Define $i_K : K \to K$ by $i_K(x,y,z) = (-x,y,z)$. This is an orientation reversing homeomorphism, so $\deg i_K = -1$. Define $i : \partial D \to \mathbb{S}^2$ by $i = \rho \circ i_K \circ h^{-1}$. For each (closed) boundary piece F of ∂D define a homotopy from d to i by interpolating along geodesics. This is a well-defined continuous operation since $d(F)$ and $i(F)$ lie in the same open hemisphere because of the hemisphere property and because of how we defined i. In fact, this defines a homotopy between d and i all of ∂D since each piece F is closed and their union is all of ∂D. Thus $\deg d = \deg i \neq 0$. $\square$

This concludes the proof of Theorem 13. We end with the fundamental lemma which was used earlier. Here $\partial_x = \frac{\partial}{\partial x}$ and the remaining notation is defined in (2).

Lemma 19. *For every $\delta < \frac{1}{2}\log\alpha$, $\varepsilon > 0$ and compact intervals $\Delta \subset (0,1)$ and $P \subset \mathbb{R}^+$ there exists $N < \infty$ such that if $f \in \mathcal{L}_\delta$ is (a,b)–renormalizable, $\min\{a,b\} \geq N$ and $a/b \in P$, then*

$$|U|\partial_u u' = 1 + \frac{1-u'}{\lambda_0'-1} + o(b^{-n}), \quad |V|\partial_v u' = -\frac{u'}{\lambda_1'-1} + o(b^{-n}),$$
$$|U|\partial_u v' = -\frac{v'}{\lambda_0'-1} + o(b^{-n}), \qquad |V|\partial_v v' = 1 + \frac{1-v'}{\lambda_1'-1} + o(b^{-n}), \tag{29}$$

for every $n > 0$. Furthermore

$$|U||V|\det\begin{pmatrix}\partial_u u' & \partial_v u' \\ \partial_u v' & \partial_v v'\end{pmatrix} > \frac{\alpha(\alpha-1)}{(\alpha-c')(\alpha-1+c')} - \varepsilon. \tag{30}$$

Proof. We will begin by proving (29) for $\partial_u u'$ and $\partial_v u'$. The proofs for $\partial_u v'$ and $\partial_v v'$ are identical so we will not include them here.

By (2) $u' = |q_-(C_-)|/|U|$, hence $\partial|U|u' + |U|\partial u' = \partial|q_-(C_-)|$. Let $C = [l,r]$. From $|U| = \Phi^{-1}(r) - \Phi^{-1}(l)$, $|q_-(C_-)| = q_-(c) - q_-(l) = u - \Phi^{-1}(l)$ we get

$$|U|\partial u' = \partial u - (1-u')\partial(\Phi^{-1}l) - u'\partial(\Phi^{-1}r). \tag{31}$$

In order to use (31) we need to evaluate terms like $\partial(T^{-1}x)$, where T is a first-entry map to C and $y = T(x)$ is a fixed point of f^k for some k. By the chain rule, $\partial y = \partial T(x) + DT(x)\partial x$, we get

$$\partial(T^{-1}y) = \frac{\partial y - \partial T(T^{-1}y)}{DT(T^{-1}y)}. \tag{32}$$

The chain rule applied to $y = f^k(y)$ gives $\partial y = \partial f^k(y) + Df^k(y)\partial y$. In particular, $l = f^{a+1}(l) = \Phi \circ q_-(l)$ and $r = f^{b+1}(r) = \Psi \circ q_+(r)$, so

$$\partial l = -\frac{\partial(\Phi\circ q_-)(l)}{\lambda_0'-1} \quad \text{and} \quad \partial r = -\frac{\partial(\Psi\circ q_+)(r)}{\lambda_1'-1}. \tag{33}$$

Affine conjugation preserves derivatives, so $Df^{a+1}(l) = \lambda_0'$ and $Df^{b+1}(r) = \lambda_1'$.

We will now calculate $\partial_u u'$. Equations (32), (33), (1) and $\partial_u\Phi = 0$ gives

$$\partial_u(\Phi^{-1}l) = \frac{\partial_u l}{D\Phi(\Phi^{-1}l)} = -\frac{\partial_u q_-(l)}{\lambda_0'-1} = -\frac{1}{\lambda_0'-1} + \frac{(|C_-|/c)^\alpha}{\lambda_0'-1} \tag{34}$$

and, using $\partial_u q_+ = 0$ as well,

$$\partial_u(\Phi^{-1}r) = \frac{\partial_u r}{D\Phi(\Phi^{-1}r)} = -\frac{\partial_u\Psi(\Psi^{-1}r)}{D\Phi(\Phi^{-1}r)(\lambda_1' - 1)}. \tag{35}$$

By Proposition 10 we can choose N such that $\mathcal{R}f \in \mathcal{L}_{\delta'}$ with $\delta' < \frac{1}{2}\log\alpha$ and hence there exists a $\lambda > 1$ (not depending on f) such that $\lambda_0', \lambda_1' \geq \lambda$ for N large enough (see the remark after Lemma 5). This, together with Lemma 5, $c \in \Delta$ and $a/b \in P$ shows that the second term in the right-hand side of (34) is of the order $o(b^{-n})$, $\forall n > 0$. We claim that $|\partial_u\Psi|$ is of the order $O(b)$, the proof of which is provided below. On the other hand, $|D\Phi|$ grows exponentially in a by the Expansion Lemma and Proposition 10. Since $a/b \in P$, this implies that (35) is of the order $o(b^{-n})$, $\forall n > 0$. This finishes the proof for $\partial_u u'$.

We will now calculate $\partial_v u'$. Equations (32), (33), (1) and $\partial_v q_- = 0$ gives

$$\partial_v(\Phi^{-1}l) = \frac{\partial_v l - \partial_v\Phi(\Phi^{-1}l)}{D\Phi(\Phi^{-1}l)} = -\frac{\partial_v\Phi(\Phi^{-1}l)(1 + (\lambda_0' - 1)^{-1})}{D\Phi(\Phi^{-1}l)}, \tag{36}$$

$$\partial_v(\Phi^{-1}r) = \frac{\partial_v r - \partial_v\Phi(\Phi^{-1}r)}{D\Phi(\Phi^{-1}r)} = -\frac{D\Psi(q_+r)\partial_v q_+(r)}{D\Phi(\Phi^{-1}r)(\lambda_1' - 1)} - \frac{\partial_v\Phi(\Phi^{-1}r)}{D\Phi(\Phi^{-1}r)}. \tag{37}$$

We claim that $|\partial_v\Phi|$ is of the order $O(a)$, the proof of which is postponed. As in the above, $|D\Phi|$ grows exponentially with b. Hence (36) and the second term of (37) are both of the order $o(b^{-n})$. The first term of (37) can be estimated as follows. There exists $x \in V$, $y \in U$ such that $D\Psi(x) = |C|/|V|$ and $D\Phi(y) = |C|/|U|$. Let $\rho_x = D\Psi(q_+r)/D\Psi(x)$ and $\rho_y = D\Phi(\Phi^{-1}r)/D\Phi(y)$. Then, using (1), we get

$$-\frac{D\Psi(q_+r)\partial_v q_+(r)}{D\Phi(\Phi^{-1}r)(\lambda_1' - 1)} = \frac{|U|}{|V|}\frac{\rho_x}{\rho_y}\frac{1 - (|C_+|/(1-c))^\alpha}{\lambda_1' - 1}.$$

But ρ_x/ρ_y is of the order $O(e^{\delta'})$ by Lemma 32 and δ' is exponentially small in b. So is $|C_+|$ by Lemma 5 and hence $\partial_v(\Phi^{-1}r) = ((\lambda_1' - 1)^{-1} + o(b^{-n}))|U|/|V|$. Now put all the above together in (31) to finish the proof for $\partial_v u'$.

We will now prove the claim that $|\partial_u\Psi| \leq O(b)$; that $|\partial_v\Phi| \leq O(a)$ follows from an identical argument. An induction argument using the chain rule shows that

$$\partial_u\Psi(x) = \sum_{i=0}^{b-1} Df^{b-1-i}(f^{i+1}\circ\psi(x))\partial_u f_-(f^i\circ\psi(x)).$$

By (1), $\partial_u f_-(t) = D\phi(q_-t)q_-(t)/u$. By the mean-value theorem there exists $x_i \leq f^{b-1-i}(x)$ such that $Df^{b-1-i}(x_i)f^{i+1}\circ\psi(x) = \Psi(x)$. Since $|D\phi| \leq e^\delta$ and $|\Psi| \leq 1$,

$$\partial_u\Psi(x) \leq e^\delta\sum_{i=0}^{b-1}\frac{Df^{b-1-i}(f^{i+1}\circ\psi(x))}{Df^{b-1-i}(x_i)}\frac{q_-(f^i\circ\psi(x))}{f^{i+1}\circ\psi(x)}.$$

By the Expansion Lemma 0 is an attracting fixed point of f_-^{-1} with multiplier bounded by λ^{-1}. Hence each term of the sum is bounded for every b, so $|\partial_u\Psi| \leq Kb$.

Let us finish by proving (30). The right-hand sides of (29) all have bounded modulus since $\lambda_0', \lambda_1' \geq \lambda > 1$ (see above). A calculation using this fact and the expressions (29) for the partial derivatives gives

$$|U||V|\det\begin{pmatrix}\partial_u u' & \partial_v u' \\ \partial_u v' & \partial_v v'\end{pmatrix} = \frac{\lambda_0'\lambda_1' - v'\lambda_0' - u'\lambda_1'}{(\lambda_0' - 1)(\lambda_1' - 1)} + o(b^{-n}).$$

By Proposition 10 the diffeomorphic parts of the renormalization tend to identity maps as $N \to \infty$. Hence, (1) implies that $\lambda_0' \to \alpha u'/c'$ and $\lambda_1' \to \alpha v'/(1-c')$ as $N \to \infty$. This, together with $u', v' \leq 1$, gives

$$\frac{\lambda_0'\lambda_1' - v'\lambda_0' - u'\lambda_1'}{(\lambda_0'-1)(\lambda_1'-1)} \to \frac{\alpha(\alpha-1)}{(\alpha - \frac{c'}{u'})(\alpha - \frac{1-c'}{v'})} \geq \frac{\alpha(\alpha-1)}{(\alpha-c')(\alpha-1+c')}. \qquad \square$$

6. Internal structures and renormalization

A major problem with the classical renormalization operator in §2 is that it is *not* differentiable [6]. The solution to this problem is to avoid composition [15], which leads to the space of internal structures and a new renormalization operator. This new renormalization operator is always differentiable (see Theorem 20).

In this section we define internal structures and the renormalization operator acting on internal structures. An internal structure of a diffeomorphism ϕ is a sequence of diffeomorphisms which when composed yields ϕ (§6.1). In the definition of the renormalization operator, composition of diffeomorphisms is replaced by juxtaposition of internal structures; this defines the new renormalization operator (§6.2). Limits of renormalization have internal structures whose diffeomorphisms are pure. Properties of pure internal structures are discussed in §6.3. See [18, §7–8] for more details (nb. internal structures are also called "decompositions").

6.1. **Internal structure of diffeomorphisms.** A set T is called a *time set* iff it is countable, has a total order and an associated function $\operatorname{depth}: T \to \mathbb{N}$ with finite level sets. When emphasis is required, we write $(T, \operatorname{depth})$ and call it a time set. Elements of T are called *times*; those at depth 0 are called *top level* times.

Let $\operatorname{Diff}^3 \subset \operatorname{Diff}^2$ be the subspace of C^3–diffeomorphisms with norm $\|\phi\| = \max\{|N\phi|, |DN\phi|\}$. Let $\ell^1(\operatorname{Diff}^3; T)$ be the space of all $\bar\phi : T \to \operatorname{Diff}^3$ with the ℓ^1–norm $\|\bar\phi\| = \sum\|\bar\phi(\tau)\|$ and the linear structure induced by Diff^3. If $\bar\phi \in \ell^1(\operatorname{Diff}^3; T)$, then the diffeomorphisms of $\bar\phi$ can be composed in the order of T to obtain a $\phi \in \operatorname{Diff}^2$. Explicitly, let $T_k = \{\tau \in T \mid \operatorname{depth}\tau \leq k\}$ so that $T_k = \{\tau_1, \dots, \tau_{n(k)}\}$ and $\tau_i < \tau_j$ if $i < j$, and let $\phi_k = \bar\phi(\tau_{n(k)}) \circ \cdots \circ \bar\phi(\tau_1)$. Then $\phi = \lim_k \phi_k$ and we write $\phi = \bigcirc\bar\phi$. It is essential here that the derivative of the nonlinearity is bounded. We say that $\bar\phi$ is the *internal structure* of ϕ and we call

$$\bigcirc : \ell^1(\operatorname{Diff}^3; T) \to \operatorname{Diff}^2$$

the *composition operator*. The composition operator is Lipschitz on bounded subsets of $\ell^1(\operatorname{Diff}^3; T)$. See [15, Proposition 4.1] and [18, Proposition 7.5] for proofs of the above statements. Any subset $S \subset T$ is also a time set, so the restriction of $\bar\phi \in \ell^1(\operatorname{Diff}^3; T)$ to S can be composed. Such *partial composition* is denoted $\bigcirc\bar\phi|_S$.

6.2. **Renormalization.** Fix two time sets $(T_\pm, \operatorname{depth}_\pm)$. Let $u, v, c \in \mathbb{R}$, $\bar\phi_\pm \in \ell^1(\operatorname{Diff}^3; T_\pm)$, $\phi_\pm = \bigcirc\bar\phi_\pm$, and assume that $f = (u, v, c, \phi_\pm)$ is (a, b)–renormalizable. Denote $\mathcal{R}f = (u', v', c', \phi_\pm')$. We will define time sets $(T_\pm', \operatorname{depth}_\pm')$, and internal structures $\bar\phi_\pm' \in \ell^1(\operatorname{Diff}^3; T_\pm')$ such that $\phi_\pm' = \bigcirc\bar\phi_\pm'$. The operator sending $(u, v, c, \bar\phi_\pm)$ to $(u', v', c', \bar\phi_\pm')$ defines the *renormalization operator acting on internal structures*. Below $\square = \pm$ when $k = 0$, else $\square = \mp$, $n_- = a$ and $n_+ = b$.

(1) Define $T_\pm'$ to be the disjoint unions $(T_\mp \sqcup \{n_\pm\}) \sqcup \cdots \sqcup (T_\mp \sqcup \{1\}) \sqcup T_\pm$, i.e.

$$T_\pm' = \big\{(\tau, k) \mid \tau \in T_\square \text{ if } k \text{ even}, \ \tau = (k+1)/2 \text{ if } k \text{ odd}\big\}_{k=0}^{2n_\pm},$$

with order $(\tau_1, k_1) < (\tau_2, k_2)$ iff $k_1 < k_2$, or $k_1 = k_2$ and $\tau_1 < \tau_2$. Define

$$\mathrm{depth}'_\pm(k, 2k-1) = 0, \quad \mathrm{depth}'_\pm(\tau, 2k) = 1 + \mathrm{depth}_\square(\tau).$$

(2) Define the intervals $\{\bar{C}_\pm(\tau')\}_{\tau'}$ as follows: pull back C under f

$$\bar{C}_\pm(k, 2k-1) = f_\mp^{-(n_\pm+1-k)}(C),$$

then pull back these top level intervals through the internal structures

$$\bar{C}_\pm(\tau, 2k) = \big(\bigcirc\bar{\phi}_\square|_{\{t\geq\tau\}}\big)^{-1}\big(\bar{C}_\pm(k+1, 2k+1)\big).$$

For $k = n_\pm$ we use the convention $\bar{C}_\pm(n_\pm + 1, 2n_\pm + 1) = C$.

(3) Define $\bar{\phi}'_\pm$ by restricting to $\bar{C}_\pm$ and rescaling (see §2 for notation):

$$\bar{\phi}'_\pm(k, 2k-1) = \big[q_\mp|\bar{C}_\pm(k, 2k-1)\big],$$
$$\bar{\phi}'_\pm(\tau, 2k) = \big[\bar{\phi}_\square(\tau)|\bar{C}_\pm(\tau, 2k)\big].$$

Note that the diffeomorphisms at the top level of $\bar{\phi}'_\pm$ are rescaled restrictions of the standard family; we call them pure maps. An internal structure with pure maps at all times is called a *pure internal structure*. Renormalization fixed points and their unstable manifolds have pure internal structures.

6.3. **Pure maps.** A map of the form $[q|I]$ is called a *pure map* iff $c \notin I$. The map $s \mapsto \zeta_s$ is a bijection between $\mathbb{R}$ and the set of pure maps, where

$$\zeta_s(x) = \frac{\big(1 + (e^{s/(\alpha-1)} - 1)x\big)^\alpha - 1}{e^{\alpha s/(\alpha-1)} - 1}, \quad x \in [0,1]. \tag{38}$$

Note that $\zeta_0(x) = x$, $\zeta_s(x) \to x^\alpha$ as $s \to \infty$, and $\zeta_s(x) \to 1 - (1-x)^\alpha$ as $s \to -\infty$. The parameter s is called the *signed distortion* of ζ_s, since the distortion of ζ_s is $|s|$. A calculation using (38) shows that distortion is related to nonlinearity by

$$|N\zeta_s| = (\alpha-1)\big(e^{|s|/(\alpha-1)} - 1\big) \quad \text{and} \quad (\alpha-1)|DN\zeta_s| = |N\zeta_s|^2. \tag{39}$$

Pure maps are invariant under rescaling; i.e. if ϕ is a pure map, then so is $[\phi|I]$.

Given a time set T, define $\ell^1(\mathbb{R}; T)$ to be the space of sequences $\bar{s} : T \to \mathbb{R}$ with the ℓ^1–norm. Let $\bar{s} \mapsto \bar{\phi}$ be defined by $\bar{\phi}(\tau) = \zeta_{\bar{s}(\tau)}$. Then $\bar{\phi} \in \ell^1(\mathrm{Diff}^3; T)$ and $\forall S < \infty\ \exists K < \infty$ such that if $\|\bar{s}\| \leq S$ then $\|\bar{s}\| \leq \|\bar{\phi}\| \leq K\|\bar{s}\|$, by (39). We call $\bar{\phi}$ the *pure internal structure associated* with $\bar{s}$. Since $\bar{\phi} \in \ell^1(\mathrm{Diff}^3; T)$, $\phi = \bigcirc\bar{\phi} \in \mathrm{Diff}^2$, but in fact $\phi \in \mathrm{Diff}^{\mathrm{S}}$ (it is even analytic), see [18, §7]. We will write $\bigcirc\bar{s}$ to mean the same thing as $\bigcirc\bar{\phi}$.

Fix two time sets $T_\pm$. Given $u, v, c \in \mathbb{R}$, $\bar{s}_\pm \in \ell^1(\mathbb{R}; T_\pm)$, let $\bar{\phi}_\pm$ be the internal structures associated with $\bar{s}_\pm$ and let $\phi_\pm = \bigcirc\bar{\phi}_\pm$. We say that $(u, v, c, \bar{s}_\pm)$ is a *pure Lorenz map* iff $(u, v, c, \phi_\pm) \in \mathcal{L}$. The space of pure Lorenz maps is a subset of the Banach space $\mathbb{R}^3 \times \ell^1(\mathbb{R}; T_-) \times \ell^1(\mathbb{R}; T_+)$. The renormalization operator on pure Lorenz maps is defined in the obvious way: let $(u', v', c', \bar{\phi}'_\pm)$ be the renormalization of $(u, v, c, \bar{\phi}_\pm)$ and define $\mathcal{R}(u, v, c, \bar{s}_\pm) = (u', v', c', \bar{s}'_\pm)$ in such a way that $\bar{\phi}'_\pm$ are the internal structures associated with $\bar{s}'_\pm$. This makes sense because pure maps are invariant under rescaling, so $\bar{\phi}'_\pm$ are pure internal structures since $\bar{\phi}_\pm$ are.

6.4. **Results.** Here we collect results related to internal structures that will be needed in later sections. The main results are that the renormalization operator is differentiable and that the norm of the internal structure of the renormalization is small for a, b large. This subsection can be skipped on a first read-through.

Theorem 20. *$\mathcal{R}$ acting on internal structures is differentiable.*

Remark. $D\mathcal{R}_{f^\star}$ has finitely many expanding eigenvalues at any fixed point $f^\star$. The idea of the proof is that the internal structures of $f^\star$ are pure and lie inside the space whose weighted ℓ^1–norm is bounded, $\|\bar{s}\|_\mu = \sum \mu^{-\operatorname{depth}\tau}|\bar{s}(\tau)| < \infty$, for some $\mu \in (0,1)$. From this and the expressions for the partial derivatives it can be shown that $D\mathcal{R}_{f^\star}$ is compact as an operator from ℓ^1 with the $\|\cdot\|_\mu$–norm to ℓ^1 with the usual norm. Compactness can then be used to prove the statement. However, we do not need this result here so we leave it as a remark.

Differentiability is essentially a consequence of the following lemma:

Lemma 21. $V_x : \ell^1(\mathrm{Diff}^3;T) \to \mathbb{R}$; $\bar{\phi} \mapsto \bigcirc\bar{\phi}(x)$, *is differentiable,* $\forall x \in [0,1]$.

Proof. Given $\tau \in T$, let $\phi_\tau = \bar{\phi}(\tau)$ and define the partial compositions $\phi_{<\tau} = \bigcirc\bar{\phi}|_{\{t<\tau\}}$, $\phi_{>\tau} = \bigcirc\bar{\phi}|_{\{t>\tau\}}$ and $\phi_{\leq\tau} = \phi_\tau \circ \phi_{<\tau}$. Then $\bigcirc\bar{\phi} = \phi_{>\tau} \circ \phi_\tau \circ \phi_{<\tau}$. Let us first show that V_x is differentiable when we use the regular linear structure of C^3. Make a C^3–perturbation $\bar{h} : T \to C^3$ in the τ–direction, i.e. $h_t = \bar{h}(t) = 0$, $\forall t \neq \tau$ (and $h_\tau \neq 0$). A Taylor expansion gives

$$\begin{aligned} V_x(\bar{\phi} + \bar{h}) &= \phi_{>\tau} \circ (\phi_\tau + h_\tau) \circ \phi_{<\tau}(x) \\ &= V_x(\bar{\phi}) + D\phi_{>\tau}(\phi_{\leq\tau}(x)) \cdot h_\tau(\phi_{<\tau}(x)) + o(|h_\tau|). \end{aligned} \tag{40}$$

The linear operator which takes h_τ to the second term is the partial derivative of $V_x(\bar{\phi})$ in the direction of τ. The partial derivative depends continuously on $\bar{\phi}$, since $\bigcirc\bar{\phi}$ is C^1. Now $DV_x(\bar{\phi})$ is the sum of all partial derivatives over τ, proving that V_x is differentiable.

To prove that V_x is differentiable on $\ell^1(\mathrm{Diff}^3;T)$, make a perturbation $\bar{h} \in \ell^1(\mathrm{Diff}^3;T)$ in the τ–direction, i.e. $h_t = \bar{h}(t) = \mathrm{id}$, $\forall t \neq \tau$ (and $h_\tau \neq \mathrm{id}$). The inverse of the nonlinearity operator is differentiable (N^{-1} has an explicit formula, see [18, Lemma B.7], from which DN^{-1} can be calculated), so

$$\phi_\tau \oplus h_\tau = N^{-1}(N\phi_\tau + Nh_\tau) = \phi_\tau + DN^{-1}(N\phi_\tau) \cdot Nh_\tau + o(|Nh_\tau|).$$

Plug this into (40) to prove that the partial derivative in the τ–direction is well-defined and then argue as above to get differentiability for V_x on $\ell^1(\mathrm{Diff}^3;T)$. □

Proof of Theorem 20. Let $\sigma = (u, v, c, \bar{\phi}_\pm)$ be renormalizable, let f_σ be its associated Lorenz map, and let C be the return interval. We claim that C depends differentiably on σ. If $p \in \partial C$, then p is a repelling n–periodic point of f_σ, for some n. Define $F(\sigma, x) = f_\sigma^n(x) - x$. By Lemma 21 $\sigma \mapsto f_\sigma^n(x)$, and hence F, is differentiable. By the Implicit Function Theorem (using the fact that p is repelling), if σ is perturbed slightly then the periodic point persists and its new position depends differentiably on σ.

The renormalization is defined by pulling back C through internal structures, restricting and rescaling. By Lemma 21 this operation is differentiable, so the internal structures of the renormalization depend differentiably on σ. By (2) and the above the same is true for u', v' and c'. Hence $\mathcal{R}$ is differentiable. □

The next result is an analog of Proposition 10 but for pure Lorenz maps. Since the composition operator is Lipschitz, given $\delta > 0$ there exists a maximal $S_\delta > 0$ such that $\|\bar{s}_\pm\| \le S_\delta$ implies $\|\bigcirc \bar{s}_\pm\| \le \delta$. This defines S_δ below.

Proposition 22 (Bound on internal structures)**.** *For every closed interval $\Delta \subset (0,1)$ and $\delta < \frac{1}{2}\log\alpha$ there exist $N < \infty$ and $K < \infty$ such that if $(u, v, c, \bar{s}_\pm)$ is a pure (a,b)–renormalizable Lorenz map, $\min\{a,b\} \ge N$, $c \in \Delta$, and $\|\bar{s}_\pm\| \le S_\delta$, then $\|\bar{s}'_\pm\| \le K|C|$.*

Nonlinearities are contracted under rescaling, whereas for the distortion we have:

Lemma 23. *Let $\zeta_{s_j} = [\zeta_s|I_j]$. For all $S < \infty$ there exists $K < \infty$ such that if $|s| < S$, then $\sum |s_j| \le K|s|\sum |I_j|$.*

Proof. From the chain-rule, $N(g \circ h) = Ng \circ h \cdot Dh + Nh$, we get $|N\zeta_{s_j}| \le |I_j||N\zeta_s|$. Combined with (39) this implies $|s_j| \le |N\zeta_{s_j}| \le |I_j||N\zeta_s| \le K|s||I_j|$. □

Proof of Proposition 22. Let f be the Lorenz map associated with $(u, v, c, \bar{s}_\pm)$, and let $\hat{T}'_\pm \subset T'_\pm$ be the top level times. By definition

$$\|\bar{s}'_\pm\| = \sum_{\tau' \in \hat{T}'_\pm} |\bar{s}'_\pm(\tau')| + \sum_{\tau' \notin \hat{T}'_\pm} |\bar{s}'_\pm(\tau')|. \tag{41}$$

The idea of the proof is that the top level term contributes a definite amount of distortion whereas the lower levels are contracted (we only prove a Lipschitz bound).

Consider the top level term of (41). A calculation shows that if $\zeta_s = [q|I]$, then $|s| \le (\alpha - 1)|I|/\operatorname{dist}(I, c)$. Hence $\bar{s}'_\pm(k, 2k-1) \le K\big|\bar{C}_\pm(k, 2k-1)\big|/d_k^\pm$, where $d_k^\pm$ is the distance from $\bar{C}_\pm(k, 2k-1)$ to the critical point. The notation $\bar{C}_\pm$ is given by step 2 of the definition of $\mathcal{R}$. By the Expansion Lemma and Lemma 5 we can choose N such that $d_k^\pm \ge \rho$ for some $\rho > 0$ not depending on f. The assumption $\delta < \frac{1}{2}\log\alpha$ and the Expansion Lemma imply that 0 and 1 are attracting fixed points of $f_\pm^{-1}$ with uniformly bounded multipliers. Hence $\sum_k |\bar{C}_\pm(k, 2k-1)| \le K|C|$ and

$$\sum_{\tau' \in \hat{T}'_\pm} |\bar{s}'_\pm(\tau')| \le K|C|/\rho. \tag{42}$$

Consider the lower level term of (41). Every partial composition $\bigcirc \bar{s}_\pm|_{\{t \ge \tau\}}$ has norm bounded by S_δ, so by Proposition 32 $|\bar{C}_\pm(\tau, 2k)| \le K|\bar{C}_\pm(k+1, 2k+1)|$. By the above $\sum_k |\bar{C}_\pm(k, 2k-1)| \le K|C|$, so Lemma 23 gives

$$\sum_{\tau' \notin \hat{T}'_\pm} |\bar{s}'_\pm(\tau')| \le K \sum |\bar{C}_\pm(k, 2k-1)| \|\bar{s}_\pm\| \le K|C|\|\bar{s}_\pm\|. \tag{43}$$

Combine (41), (42) and (43) to see that $\|\bar{s}'_\pm\| \le K(\rho^{-1} + \|\bar{s}_\pm\|)|C|$. □

7. Derivative estimates

In this section we compute the partial derivatives of the renormalization operator acting on pure internal structures. These results will be applied in the next section. This section can be skipped on a first read-through and referenced back to later on.

We use the same notation in this section as in §2, with the following additions: $\lambda_x = Df(x)$ and $\lambda'_x = D(\mathcal{R}f)(x)$. We write $g = o(b^{-n})$ to mean that $b^n g \to 0$ as $b \to \infty$, for every $n > 0$. Note that we write ∂_x instead of $\frac{\partial}{\partial x}$.

Lemma 24. *For every $\delta < \frac{1}{2}\log\alpha$ and compact intervals $\Delta \subset (0,1)$ and $P \subset \mathbb{R}^+$ there exists $N < \infty$ such that if $f \in \mathcal{L}_\delta$ is (a,b)–renormalizable, $c \in \Delta$, $\min\{a,b\} \geq N$ and $a/b \in P$, then*

$$\partial_u \begin{pmatrix} u' \\ v' \\ c' \end{pmatrix} = \frac{w_0 + \varepsilon_0}{|U|}, \quad \partial_v \begin{pmatrix} u' \\ v' \\ c' \end{pmatrix} = \frac{w_1 + \varepsilon_1}{|V|}, \quad \partial_c \begin{pmatrix} u' \\ v' \\ c' \end{pmatrix} = \frac{w_2 - Aw_0 + Bw_1 + \varepsilon_2}{|C|},$$

where $A = |O(a)|$, $B = |O(b)|$, $\|\varepsilon_i\| = o(b^{-n})$, and $\{w_i\}$ are the columns of

$$W = \begin{pmatrix} 1 + \frac{1-u'}{\lambda_0'-1} & -\frac{u'}{\lambda_1'-1} & \frac{u'\lambda_1'}{\lambda_1'-1} - \frac{(1-u')\lambda_0'}{\lambda_0'-1} \\ -\frac{v'}{\lambda_0'-1} & 1 + \frac{1-v'}{\lambda_1'-1} & -\frac{v'\lambda_0'}{\lambda_0'-1} + \frac{(1-v')\lambda_1'}{\lambda_1'-1} \\ \frac{1-c'}{\lambda_0'-1} & -\frac{c'}{\lambda_1'-1} & 1 - \frac{c'\lambda_1'}{\lambda_1'-1} - \frac{(1-c')\lambda_0'}{\lambda_0'-1} \end{pmatrix}. \tag{44}$$

Proof. The statement about the upper-left 2×2 matrix was proved in Lemma 19. Here we will calculate $\partial_c u'$, $\partial_c c'$ and $\partial_u c'$. The calculations for $\partial_c v'$ and $\partial_v c'$ are identical so we will not include them here.

First of all, let us discuss how to choose N. Let δ' denote the bound on the norm of the diffeomorphic parts of the renormalization, i.e. $\mathcal{R}f \in \mathcal{L}_{\delta'}$. By Lemma 10 we may choose N such that $\delta' < \frac{1}{2}\log\alpha$. For this choice of N there exists $\lambda > 1$ (not depending on f) such that both $\lambda_x' \geq \lambda$ and $\lambda_x > \lambda$ for $x \in \{0,1\}$. This follows from (1) and Lemma 32. Assume that N has been chosen in this way.

We will derive expressions for $\partial_c l$ and $\partial_c r$, where $C = [l,r]$. Define

$$A = -\partial_c\Phi(\Phi^{-1}c), \qquad B = -\partial_c\Psi(\Psi^{-1}c),$$

where the notation is from (2). We claim that $A = |O(a)|$, $B = |O(b)|$ and that

$$A - \left|\partial_c\Phi(\Phi^{-1}x)\right| = o(b^{-n}), \quad B - \left|\partial_c\Psi(\Psi^{-1}y)\right| = o(b^{-n}), \quad \forall x \in U, y \in V. \tag{45}$$

We postpone the proof of this claim. From (33) and (1) we get

$$\partial_c l = \frac{\frac{l}{c}\lambda_0' - \partial_c\Phi(\Phi^{-1}l)}{\lambda_0' - 1} = \frac{\lambda_0' + A}{\lambda_0' - 1} + \frac{\frac{|C_-|}{c}\lambda_0' - \partial_c\Phi(\Phi^{-1}l) - A}{\lambda_0' - 1}.$$

We claim that the second fraction in the right-hand side is $o(b^{-n})$. This follows from (45) and the facts that: $|C| = o(b^{-n})$ by Lemma 5, $c \in \Delta$, and that $\lambda_0' \geq \lambda > 1$. An identical argument for $\partial_c r$ gives

$$\partial_c l = \frac{\lambda_0' + A}{\lambda_0' - 1} + o(b^{-n}), \qquad \partial_c r = \frac{\lambda_1' + B}{\lambda_1' - 1} + o(b^{-n}). \tag{46}$$

We now calculate $\partial_c u'$. From (32), (45) and (46) we get

$$\partial_c(\Phi^{-1}l) = \frac{1}{D\Phi(\Phi^{-1}l)}\left(\frac{\lambda_0' + A}{\lambda_0' - 1} + A + o(b^{-n})\right).$$

By the mean-value theorem there exists $x \in U$ such that $D\Phi(x) = |C|/|U|$. Furthermore $D\Phi(x)/D\Phi(\Phi^{-1}l) = 1 + O(\delta')$ by Lemma 32. Thus $D\Phi(\Phi^{-1}l) = (1 + o(b^{-n}))|C|/|U|$, since δ' is exponentially small in b by Proposition 10. An identical argument for $\partial_c(\Phi^{-1}r)$ and using the bounds on A, B and λ_x' gives

$$\frac{|C|}{|U|}\partial_c(\Phi^{-1}l) = \frac{\lambda_0'(1+A)}{\lambda_0' - 1} + o(b^{-n}), \quad \frac{|C|}{|U|}\partial_c(\Phi^{-1}r) = \frac{\lambda_0' + B}{\lambda_0' - 1} + A + o(b^{-n}).$$

Insert these expressions into (31) to get

$$|C|\partial_c u' = -\frac{(1-u')\lambda_0'}{\lambda_0'-1} - \frac{u'\lambda_1'}{\lambda_1'-1} - A\left(1+\frac{1-u'}{\lambda_0'-1}\right) - B\frac{u'}{\lambda_1'-1} + o(b^{-n}). \tag{47}$$

An identical argument gives

$$|C|\partial_c v' = \frac{(1-v')\lambda_1'}{\lambda_1'-1} + \frac{v'\lambda_0'}{\lambda_0'-1} + B\left(1+\frac{1-v'}{\lambda_1'-1}\right) + A\frac{v'}{\lambda_0'-1} + o(b^{-n}). \tag{48}$$

We will now calculate the partial derivatives of c'. By definition $|C|c' = c - l$ and taking partial derivatives gives

$$|C|\partial c' = \partial c - (1-c')\partial l - c'\partial r. \tag{49}$$

Inserting (46) into 49 gives

$$|C|\partial_c c' = 1 - \frac{(1-c')\lambda_0'}{\lambda_0'-1} - \frac{c'\lambda_1'}{\lambda_1'-1} - A\frac{1-c'}{\lambda_0'-1} - B\frac{c'}{\lambda_1'-1} + o(b^{-n}). \tag{50}$$

From (34) and (35) we get expressions for $\partial_u l$ and $\partial_u r$, respectively. Together with (49) and the above trick we see that $D\Phi(\Phi^{-1}l)|U|/|C| = 1 + o(b^{-n})$ and hence

$$|U|\partial_u c' = \frac{1-c'}{\lambda_0'-1} + o(b^{-n}). \tag{51}$$

An identical argument gives

$$|V|\partial_v c' = -\frac{c'}{\lambda_1'-1} + o(b^{-n}). \tag{52}$$

Equations (47), (48), (50), (51) and (52) imply the lemma. It only remains to prove (45) and the bounds on A and B.

Let $x \in V$ and note that $\Psi(x) = f^{b-i}(f^i \circ \psi(x))$. By the mean-value theorem there exists $x_i \in [0, f^i \circ \psi(x)]$ such that $Df^{b-i}(x_i) f^i \circ \psi(x) = \Psi(x)$. Using the chain-rule to compute $\partial_c \Psi$ and then applying (1) and the above we get

$$-\partial_c \Psi(x) = \frac{1}{c}\sum_{i=0}^{b-1} Df^{b-i}(f^i \circ \psi(x)) f^i \circ \psi(x) = \frac{\Psi(x)}{c}\sum_{i=0}^{b-1}\frac{Df^{b-i}(f^i\circ\psi(x))}{Df^{b-i}(x_i)}. \tag{53}$$

Since $\lambda_0 \geq \lambda > 1$, 0 is a uniformly attracting fixed point for f_-^{-1} so the summands are bounded. Hence $-\partial_c\Psi(x) \leq Kb\Psi(x)/c$ and since $c \in \Delta$ this proves the claim. The proof that $A = |O(a)|$ is identical.

We now prove (45). Let $x, y \in V$, $x_i = f^i \circ \psi(x)$, and $y_i = f^i \circ \psi(y)$. By (53)

$$\begin{aligned}
|\partial_c\Psi(y) - \partial_c\Psi(x)| &= \frac{1}{c}\left|\sum_{i=0}^{b-1} Df^{b-i}(x_i)\left(\frac{Df^{b-i}(y_i)}{Df^{b-i}(x_i)}y_i - x_i\right)\right| \\
&\leq \frac{1}{c}\sum_{i=0}^{b-1} Df^{b-i}(x_i)\left(\left|\frac{Df^{b-i}(y_i)}{Df^{b-i}(x_i)} - 1\right| y_i + |y_i - x_i|\right).
\end{aligned}$$

The distortion of f^{b-i} on $f^i\circ\psi(V)$ is exponentially small in b by Proposition 10. In particular, the distortion is $o(b^{-n})$. Hence, using a mean-value theorem argument as in the above, we get

$$|\partial_c\Psi(y) - \partial_c\Psi(x)| \leq \frac{1}{c}Kb\left(o(b^{-n}) + |\Psi(y)-\Psi(x)|\right) = o(b^{-n}),$$

since $|\Psi(y) - \Psi(x)| \leq |C| = o(b^{-n})$ by Lemma 5. $\square$

In the remainder of this section we use the notation of §6. Let $T = (T_-, T_+)$ be a pair of time sets. Since the composition operator is Lipschitz there exists a maximal $S_\delta > 0$ such that if $\bar{s}_\pm \in \ell^1(\mathbb{R}; T_\pm)$ and $\|\bar{s}_\pm\| \le S_\delta$, then $\|\bigcirc \bar{s}_\pm\| \le \delta$.

Lemma 25. *For every $\delta < \frac{1}{2}\log\alpha$ and compact intervals $\Delta \subset (0,1)$ and $P \subset \mathbb{R}^+$ there exists $N < \infty$ and $K < \infty$ such that if $(u, v, c, \bar{s}_\pm)$ is a pure (a,b)–renormalizable Lorenz map, $c \in \Delta$, $\min\{a,b\} \ge N$, $a/b \in P$ and $\|\bar{s}_\pm\| \le S_\delta$, then*

$$|C||\partial_s u'| \le Kb, \qquad |C||\partial_s v'| \le Kb, \qquad |C||\partial_s c'| \le Kb, \tag{54}$$

$$|U| \sum_{\tau' \in T'_\pm} |\partial_u \bar{s}'_\pm(\tau')| = o(b^{-n}), \qquad |V| \sum_{\tau' \in T'_\pm} |\partial_v \bar{s}'_\pm(\tau')| = o(b^{-n}), \tag{55}$$

$$|C| \sum_{\tau' \in T'_\pm} |\partial_c \bar{s}'_\pm(\tau')| = o(b^{-n}), \qquad |C| \sum_{\tau' \in T'_\pm} |\partial_s \bar{s}'_\pm(\tau')| = o(b^{-n}). \tag{56}$$

Here ∂_s denotes partial derivative with respect to an arbitrary variable of $\ell^1(\mathbb{R}; T_\pm)$.

Proof. Let f be the map associated with $(u, v, c, \bar{s}_\pm)$ and let $C = [l, r]$. Assume that N has been chosen as in the beginning of the proof of Lemma 24, so that $\delta' < \frac{1}{2}\log\alpha$ and $\lambda'_x \ge \lambda > 1$ for $x \in \{0, 1\}$. Throughout the proof assume without loss of generality that ∂_s is the partial derivative with respect to the variable at a time $\tau \in T_-$ and write $s = \bar{s}_-(\tau)$. That is, we will perturb $s = \bar{s}_-(\tau)$ slightly and see how the renormalization changes.

We will need the following bounds (see (2) for notation):

$$|\partial_s l| \le Kb, \quad |\partial_s r| \le Kb, \quad |\partial_s \Phi| \le Kb, \quad |\partial_s \Psi| \le Kb. \tag{57}$$

The first two bounds follow from the last two bounds, $\lambda'_x \ge \lambda$, and (33) so let us prove the last two bounds. Note that

$$|\partial_t \zeta_t(x)| \le Kx(1-x). \tag{58}$$

This can be seen by taking partial derivatives of (38) to get

$$\partial_t \zeta_t(x) = \frac{D\zeta_t(x)x - D\zeta_t(1)\zeta_t(x)}{N\zeta_t(1)}. \tag{59}$$

From this (58) follows by Taylor expanding $\zeta_t(x)$ and $D\zeta_t(x)$ around 0 and 1.

Let $\bar{\phi}$ be the internal structure associated with $\bigcirc \bar{s}_-$. We write $\phi = \phi_{>\tau} \circ \phi_\tau \circ \phi_{<\tau}$, where $\phi = \bigcirc \bar{\phi}$, $\phi_{>\tau} = \bigcirc \bar{\phi}|_{\{t > \tau\}}$, $\phi_{<\tau} = \bigcirc \bar{\phi}|_{\{t<\tau\}}$ and $\phi_\tau = \bar{\phi}(\tau) = \zeta_s$. We get

$$\partial_s f_-(x) = D\phi_{>\tau}(\zeta_s \circ \phi_{<\tau} \circ q(x)) \partial_s \zeta_s(\phi_{<\tau} \circ q(x)).$$

A Taylor expansion shows that (here $\phi_{\ge\tau} = \phi_{>\tau} \circ \zeta_s$)

$$\partial_s \zeta_s(\phi_{<\tau} \circ q(x)) = \partial_s \zeta_s(fx) + D\partial_s \zeta_s(t)(\phi_{\ge\tau}^{-1} \circ f(x) - f(x)),$$

for some t. By differentiating (59) we see that $|D\partial_s \zeta_s| \le K$, since $|D^2 \zeta_s(x)| \le K$ and $D\zeta_s(x) \le K$. By Lemma 32 $K^{-1} \le |D\phi_{\ge\tau}^{-1}| \le K$, so

$$|\phi_{\ge\tau}^{-1} \circ f(x) - f(x)| \le Kf(x)(1 - f(x)).$$

Combine these two facts with (58) to see that

$$|\partial_s f(x)| \le Kf(x)(1 - f(x)). \tag{60}$$

Note that this holds for both branches of f, since s was assumed to be at a time $\tau \in T_-$, so $\partial_s f_+ = 0$. The chain-rule and (60) gives

$$|\partial_s f^{n+1}(x)| \leq K \sum_{i=0}^{n} Df^{n-i}(f^{i+1}x) f^{i+1}(x)(1 - f^{i+1}(x)). \tag{61}$$

The same argument as was used to prove $|\partial_c \Psi| \leq Kb$ in the proof of Lemma 24 combined with (61) shows that $|\partial_s f^n| \leq Kn$. Since $\Phi = f_+^a \circ \phi$ and $\Psi = f_-^b \circ \psi$, where $\psi = \bigcirc \bar{s}_+$, this implies that $|\partial_s \Phi| \leq Kb$ (nb. $a/b \in P$) and $|\partial_s \Psi| \leq Kb$.

From (49) and (57) we immediately get the bound on $\partial_s c'$ claimed in (54). By the mean value theorem $D\Phi(x) = |C|/|U|$ for some $x \in U$. Since the distortion of $\Phi|_U$ is bounded by δ' we get that $|D\Phi| \leq K|C|/|U|$. From (32), (57) and $|D\Phi| \leq K|C|/|U|$ we get $|\partial_s(\Phi^{-1}l)| \leq Kb|U|/|C|$ and similarly for $|\partial_s(\Phi^{-1}r)|$. Inserting this into (31) proves the bound on $\partial_s u'$ in (54). The bound on $\partial_s v'$ follows from an identical argument.

We will now estimate the partial derivatives $\partial s'_\pm(\tau')$. Assume without loss of generality that $\tau' \in T'_+$ and let $s' = \bar{s}'_+(\tau')$. By definition $\zeta_{s'} = [h|I]$ for some map h and interval $I = [x, y]$. By step 3 of §6.2 either: (i) τ' is at the top level and $h = q_-$, or (ii) $\mathrm{depth}'_+(\tau') > 0$ and $h = \zeta_{\hat{s}}$, where we may assume without loss of generality that $\hat{s} = \bar{s}_-(\hat{\tau})$, for some $\hat{\tau} \in T_-$. Since distortion is invariant under rescaling $s' = \log\{Dh(y)/Dh(x)\}$. The chain-rule gives

$$\partial s' = Nh(y)\partial y - Nh(x)\partial x + \frac{\partial(Dh)(y)}{Dh(y)} - \frac{\partial(Dh)(x)}{Dh(x)} = \partial s'_0 + \partial s'_1, \tag{62}$$

where $\partial s'_0$ and $\partial s'_1$ denote the two difference terms.

Consider the difference term $\partial s'_0$ of (62). By definition I is mapped diffeomorphically to $C = [l, r]$ by some first-entry map $F : I \to C$. In case (i) F is of the form $F = f_-^k$, else $F = f_-^k \circ \phi_{\geq \hat{\tau}}$, for some k. Either way, we have that

$$\partial x = \frac{\partial l - \partial F(x)}{DF(x)}, \qquad \partial y = \frac{\partial r - \partial F(y)}{DF(y)}.$$

We can bound $|\partial F| \leq Kb$ using the same estimates we did for $\partial \Phi$ and $\partial \Psi$ in the above and in the proofs of Lemmas 19 and 24. The distortion of F tends to zero as $N \to \infty$ by Proposition 10. Hence $|I||DF| \geq |C|/K$. Using the above, $N\zeta_{s'}(1) = |I|Nh(y)$ and $N\zeta_{s'}(0) = |I|Nh(x)$ we get

$$|\partial s'_0| = |Nh(y)\partial y - Nh(x)\partial x| \leq \frac{K|N\zeta_{s'}| \max\{|\partial l|, |\partial r|, b\}}{|C|}.$$

We know from the proof of Lemma 19 that $\partial_u l$ is of the order $|C|/|U|$ and $\partial_v r$ is of the order $|C|/|V|$. All other partial derivatives ∂l and ∂r are at most of the order of b, by the above and the proofs of Lemmas 19 and 24. Apply (39) to see that

$$|\partial_u s'_0| \leq K|s'|/|U|, \quad |\partial_v s'_0| \leq K|s'|/|V|, \quad |\partial_\star s'_0| \leq Kb|s'|/|C|,$$

for $\star = c, s$. By Lemma 5 and Proposition 22, $b\|\bar{s}'_+\| = o(b^{-n})$. Since $\sum |s'| = \|\bar{s}'_+\|$ it only remains to bound the $\partial s'_1$ term.

Consider the difference term $\partial s'_1$ of (62). A calculation using (1) shows that $\partial_u s'_1 = \partial_v s'_1 = 0$ and hence there is nothing more to prove for (55). It is similarly straightforward from (1) to calculate $\partial_c s'_1 = -|F^{-1}(C)|/c$ in case (i) and $\partial_c s'_1 = 0$ otherwise. From the Expansion Lemma and Lemma 5 we get $\sum |f_-^{-i}(C)| \leq K|C| = o(b^{-n})$, and the first equation of (56) follows. The argument for the second equation of (56) follows similarly from $\partial_s s_1' = 0$ in case (i), and the claim that in case (ii) $|\partial_s s_1'| \le K|F^{-1}(C)|$ if $s = \hat{s}$ and $\partial_s s_1' = 0$ if $s \ne \hat{s}$. To prove the claim, use (59) to get $\partial_s s_1' = (N\zeta_s(y)y - N\zeta_s(x)x)/N\zeta_s(1)$ and then Taylor expand $N\zeta_s(y)y$ around x and use that $D(N\zeta_s(x)x)$ and $N\zeta_s(1)$ are bounded. □

8. Unstable manifolds

In this section we prove the existence of unstable manifolds at the fixed points of §5. We begin by deriving conditions on $D\mathcal{R}$ which are sufficient for the existence of local unstable manifolds and use the results of §7 to show that they are satisfied. After this we describe the global unstable manifolds. Topologically full families only need two parameters so the unstable manifolds were expected to be two-dimensional, but we prove that they are at least three-dimensional. The "extra" unstable dimension is related to the movement of the critical point and it causes infinitely renormalizable maps to appear inside the unstable manifold. In particular, the infinitely renormalizable maps cannot form a stable manifold as was expected.

8.1. **Cone fields and local unstable manifolds.** In this subsection we will give conditions for the existence of a local unstable manifold that are suitable for our setup. We could not find a reference applicable to our situation, because: (1) we have to use unconventional bounds like the third inequality of (63), (2) we do not have good enough bounds on the derivative to get hyperbolicity, and (3) our map lives on an infinite-dimensional space and it is not a diffeomorphism. The first issue is the most significant difference to textbook examples. Our proof is an adaptation of [12, Theorem 6.2.8].

The setup is as follows. We have a smooth map $F : D \subset X \times Y \to X \times Y$ on an open neighborhood D of 0, which is a fixed point of F. Here $X = \mathbb{R}^n$ for some n and Y is a Banach space. We will use the notation

$$F(x, y) = (\xi(x, y), \eta(x, y)),$$

and write $z = (x, y)$. The derivative of F is assumed to satisfy the bounds

$$\frac{\|D_x\xi(z)u\|}{\|u\|} \ge \mu,\ \frac{\|D_y\xi(z)\|}{\|D_x\xi(z)u\|} \le \frac{\nu}{\|u\|},\ \frac{\|D_x\eta(z)u\|}{\|D_x\xi(z)u\|} \le \lambda,\ \frac{\|D_y\eta(z)\|}{\|D_x\xi(z)u\|} \le \frac{1-\tau}{\|u\|}, \tag{63}$$

$\forall z \in D$ and $\forall u \in X \setminus \{0\}$. These bounds state that the maximum expansion of $D_y\xi$, $D_x\eta$ and $D_y\eta$ are comparable to the minimum expansion of $D_x\xi$.

We will begin by giving conditions for the existence of an invariant cone field. To this end, let H_θ denote the standard horizontal cone

$$H_\theta = \{(u, v) \in X \times Y \mid \|v\| \le \theta\|u\|\},$$

and write $w = (u, v)$.

Lemma 26 (Invariant cone field)**.** *Define*

$$\theta' = \frac{\lambda + (1-\tau)\theta}{1 - \nu\theta}, \quad \theta_0 = \frac{2\lambda}{\tau}, \quad \theta_1 = \frac{\tau}{\nu} - \frac{2\lambda}{\tau}, \quad \theta_2 = \frac{\mu - 1}{\mu\nu + 1}.$$

If $2\sqrt{\nu\lambda} < \tau$*, then* $\theta_0 < \theta_1$*,* $\theta' < \theta$*, and*

$$DF(z)(H_\theta) \subset H_{\theta'}, \quad \forall z \in D,\ \forall \theta \in (\theta_0, \theta_1).$$

If furthermore $\mu > (\tau + 2\lambda)/(\tau - 2\lambda\nu)$, then $\theta_2 > \theta_0$ and

$$\frac{\|DF(z)w\|}{\|w\|} \geq \mu\frac{1-\nu\theta}{1+\theta} > 1, \quad \forall z \in D,\ \forall w \in H_\theta \setminus \{0\},\ \forall \theta < \theta_2.$$

Proof. Let $w = (u, v) \in H_\theta$ and let $w' = (u', v') = DF(z)w$. Then

$$\frac{\|v'\|}{\|u'\|} = \frac{\|D_x\eta(z)u + D_y\eta(z)v\|}{\|D_x\xi(z)u + D_y\xi(z)v\|} \leq \frac{\lambda + (1-\tau)\theta}{1-\nu\theta} = g(\theta).$$

Assume without loss of generality that $\nu > 0$. Solving $g(\theta) = \theta$ and using $4\nu\lambda < \tau^2$ gives two fixed points $\theta_\pm > 0$ and $g(\theta) < \theta$ for $\theta_- < \theta < \theta_+$. Use $\sqrt{1-t} \geq 1-t$ for $t = 4\lambda\nu\tau^{-2}$ to see that $\theta_- \leq \theta_0 < \theta_1 \leq \theta_+$. This proves invariance.

Expansion follows from the assumption $\theta < (\mu - 1)/(\mu\nu + 1)$ and

$$\frac{\|w'\|}{\|w\|} \geq \frac{\|u'\|}{\|u\|(1+\theta)} \geq \frac{\mu - \mu\nu\theta}{1+\theta} > 1.$$

Insert $\mu > (\tau + 2\lambda)/(\tau - 2\lambda\nu)$ into the definition of θ_2 to see that $\theta_2 > \theta_0$. □

Lemma 27 (Local unstable manifold)**.** *Under the assumptions of Lemma 26, F has a local unstable manifold at 0 which:*

(1) is the graph of a θ_0–Lipschitz map $\gamma^* : U \to Y$, where $U \subset X$ is a ball around 0,
(2) is unique in the sense that if $F(W) \supset W$, where $W \ni 0$ is the graph of a θ–Lipschitz map on U and $\theta < \min\{\theta_1, \theta_2, \theta_3\}$, where $\theta_3 = \tau(2\nu)^{-1}$, then $W = \operatorname{graph}\gamma^*$.

Proof. Let $U \subset X$ be a closed ball around 0 and define G to be the set of θ–Lipschitz maps $\gamma : U \to Y$ fixing 0, where $\theta_0 < \theta < \min\{\theta_1, \theta_2\}$. Assume that U is small enough so that $\operatorname{graph}\gamma \subset D$, $\forall\gamma \in G$. The graph transform $\Gamma : G \to G$ is defined by sending γ to γ' where

$$\operatorname{graph}\gamma' = F(\operatorname{graph}\gamma) \cap (U \times Y).$$

We claim that Γ is well-defined for U small enough. To see that the right-hand side is the graph of some $\gamma' : U \to Y$ we need only show that $x \mapsto \xi(x, \gamma(x))$ maps some U injectively over itself, since $F(x, \gamma(x)) = (\xi(x, \gamma(x)), \dots)$. In order to so, note that by the smoothness of ξ, $\forall\varepsilon > 0$ we may choose U so that $\|\xi(x, 0)\| \geq \|D_x\xi(0)x\| - \varepsilon\|x\|$. This and the cone expansion constant of Lemma 26 shows that

$$\begin{aligned}\|\xi(x,\gamma(x))\| &\geq \|\xi(x,0)\| - \|\xi(x,\gamma(x)) - \xi(x,0)\| \geq (\mu-\varepsilon)\|x\| - \mu\nu\|\gamma(x)\| \\ &\geq (\mu(1-\nu\theta)-\varepsilon)\|x\| > (1+\theta-\varepsilon)\|x\|.\end{aligned} \tag{64}$$

But $1 + \theta - \varepsilon > 1$ for $\varepsilon < \theta$, proving that $x \mapsto \xi(x, \gamma(x))$ maps some U injectively over itself. The cone invariance of Lemma 26 implies that γ' is θ–Lipschitz.

The set G is turned into a complete metric space by endowing it with the metric

$$d(\gamma_1, \gamma_2) = \sup_{x \in U} \frac{\|\gamma_1(x) - \gamma_2(x)\|}{\|x\|}.$$

The completeness of Y and compactness of U ensures that G is complete. As a remark, the interpretation of this metric is that if $\rho = d(\gamma_1, \gamma_2)$, then ρ is the smallest number such that the graph of $\gamma_1 - \gamma_2$ is contained in H_ρ. Choose $\theta < \theta_3$. This is possible, since $\theta_0/\theta_3 = 4\lambda\nu\tau^{-2} < 1$ by assumption. We claim that Γ is a contraction in this metric with this choice of θ and U small enough. To prove the claim let $\gamma_i \in G$ and let $\gamma_i' = \Gamma(\gamma_i)$, for $i = 1, 2$. By definition $(x_i', \gamma_i'(x_i')) = (\xi(x, \gamma_i(x)), \eta(x, \gamma_i(x)))$. By (64) and the bounds on $\|D_y\xi(z)\|$ and $\|D_y\eta(z)\|$:

$$\begin{aligned} d(\gamma_1', \gamma_2') &= \sup_{x'} \frac{\|\gamma_1'(x') - \gamma_2'(x')\|}{\|x'\|} \\ &= \sup_{\xi(x,\gamma_1(x))} \frac{\|\eta(x, \gamma_1(x)) - \eta(x, \gamma_2(x)) + \gamma_2'(\xi(x, \gamma_2(x))) - \gamma_2'(\xi(x, \gamma_1(x)))\|}{\|\xi(x, \gamma_1(x))\|} \\ &\le \sup_{x} \frac{\|\eta(x, \gamma_1(x)) - \eta(x, \gamma_2(x))\| + \theta\|\xi(x, \gamma_2(x)) - \xi(x, \gamma_1(x))\|}{(\mu(1 - \nu\theta) - \varepsilon)\|x\|} \\ &\le \sup_{x} \frac{(\mu(1 - \tau) + \mu\nu\theta)\|\gamma_1(x) - \gamma_2(x)\|}{(\mu(1 - \nu\theta) - \varepsilon)\|x\|} \le \frac{1 - \tau + \nu\theta}{1 - \nu\theta - \varepsilon/\mu} d(\gamma_1, \gamma_2). \end{aligned}$$

Using $\nu\theta < \tau/2$ we see that the factor in front of $d(\gamma_1, \gamma_2)$ is smaller than 1 for ε small enough. Hence we may choose U small so that Γ is a contraction.

Since Γ is a contraction it has a unique fixed point $\gamma^* \in G$ by the Contraction Mapping Theorem. By construction the graph of γ^* is an invariant manifold of F and it is unstable by the cone expansion of Lemma 26. We may choose θ arbitrarily close to θ_0, so γ^* is θ_0–Lipschitz. This proves property 1. Property 2 is a consequence of Γ having a unique fixed point. □

8.2. **Global unstable manifolds.** We now apply §8.1 to get local unstable manifolds of $\mathcal{R}$; these are then iterated to get global unstable manifolds. As the results below show these global manifolds are not complicated; in fact, they are still graphs if we cut off maps whose renormalization has a branch which is almost trivial. They also show that the unstable manifold is at least three-dimensional and that it contains a two-dimensional strong unstable manifold which is a full family.

We need some notation before stating the theorems. Let $\delta_b = 1/b^2$, let $f^\star_{a,b}$ denote an (a, b)–fixed point (which exists for a and b large, by Theorem 13) and let $\mathrm{Diff}^{\mathrm{S}}_{\delta_b} \subset \mathrm{Diff}^{\mathrm{S}}$ denote the ball of radius δ_b. Note that we may assume $f^\star_{a,b} \in \mathcal{L}_{\delta_b}$ by Proposition 10. Let Q and $\mathcal{D}_b$ be defined as in Lemma 16. Choose $\Delta \subset (0, 1)$ so that $c(\mathcal{R}f) \notin \Delta$ if $c(f) \in \partial\Delta_b$ (this is possible by Lemma 15(2)).

Theorem 28. *For every $\beta \in \mathbb{Q}^+$ there exist $N < \infty$, $K < \infty$, $\lambda < 1$ and $\theta > 0$ such that for every $b \ge N$ and $a/b = \beta$, $f^\star_{a,b}$ has a two-dimensional strong unstable manifold $\mathcal{W}^{uu}_b$ with the following properties:*

(1) $\mathcal{W}^{uu}_b$ is the graph of a θ–Lipschitz map $\gamma^{uu}_b : Q \to \Delta_b \times \mathrm{Diff}^{\mathrm{S}}_{\delta_b} \times \mathrm{Diff}^{\mathrm{S}}_{\delta_b}$,

(2) $\mathcal{W}^{uu}_b$ is unique in the sense that if $\gamma : Q \to \Delta_b \times \mathrm{Diff}^{\mathrm{S}}_{\delta_b} \times \mathrm{Diff}^{\mathrm{S}}_{\delta_b}$ is θ–Lipschitz, $\mathcal{R}(\operatorname{graph}\gamma \cap \mathcal{D}_b) \supset \operatorname{graph}\gamma$ and $f^\star_{a,b} \in \operatorname{graph}\gamma$, then $\gamma = \gamma^{uu}_b$,

(3) $\mathcal{R}^{-1} : \mathcal{W}^{uu}_b \to \mathcal{W}^{uu}_b$ is well-defined and $\|\mathcal{R}^{-n}f - f^\star_{a,b}\| \le K\lambda^n$, $\forall f \in \mathcal{W}^{uu}_b$, $\forall n \ge 0$,

(4) $f^\star_{a,b}$ is the unique representative of its topological class in families of the above type; that is, if $\gamma : Q \to \Delta_b \times \mathrm{Diff}^{\mathrm{S}}_{\delta_b} \times \mathrm{Diff}^{\mathrm{S}}_{\delta_b}$ is any θ–Lipschitz map whose graph contains $f^\star_{a,b}$, then the graph of γ does not contain any other infinitely (a, b)–renormalizable maps,

(5) $\mathcal{W}^{uu}_b$ extends to a 2–dim unstable manifold which is a full family.

Theorem 29. *For every $\beta \in \mathbb{Q}^+$ there exist $N < \infty$, $K < \infty$ and $\lambda < 1$ such that the following holds. For every $b \ge N$ and $a/b = \beta$, there exists $\theta_b > 0$ such that $f^\star_{a,b}$ has a three-dimensional unstable manifold $\mathcal{W}^u_b$ with the following properties:*

(1) $\mathcal{W}_b^u$ *is the graph of a* θ_b*–Lipschitz map* $\gamma_b^u : Q \times \Delta \to \mathrm{Diff}^{\mathrm{S}}_{\delta_b} \times \mathrm{Diff}^{\mathrm{S}}_{\delta_b}$*,*

(2) $\mathcal{W}_b^u$ *is unique in the sense that if* $\gamma : Q \times \Delta \to \mathrm{Diff}^{\mathrm{S}}_{\delta_b} \times \mathrm{Diff}^{\mathrm{S}}_{\delta_b}$ *is* θ_b*–Lipschitz and* $\mathcal{R}(\operatorname{graph}\gamma \cap \mathcal{D}_b) \supset \operatorname{graph}\gamma$*, then* $\gamma = \gamma_b^u$*,*

(3) $\mathcal{R}^{-1} : \mathcal{W}_b^u \to \mathcal{W}_b^u$ *is well-defined and* $\|\mathcal{R}^{-n} f - f^\star_{a,b}\| \le K\lambda^n$*,* $\forall f \in \mathcal{W}_b^u$*,* $\forall n \ge 0$*,*

(4) *any neighborhood of* $f^\star_{a,b}$ *in* $\mathcal{W}_b^u$ *contains an infinitely* (a,b)*–renormalizable map* $f \ne f^\star_{a,b}$*.*

Remark. (1) We do not prove that there are exactly three unstable eigenvalues at $f^\star_{a,b}$; there may be more, but we do not think so. (2) It is possible to prove that the unstable manifolds are C^1 with the techniques used here. The key is proving that the composition operator is differentiable on pure internal structures. In fact, the unstable manifolds are analytic. This can be proved by extending the pure maps of the internal structures to neighborhoods in $\mathbb{C}$ and using a holomorphic graph transform. We omit proofs of these statements for the sake of brevity.

8.3. **Proofs of the unstable manifold theorems.** The proofs rely on the fact that the fixed point is a pure Lorenz map; i.e. it has pure internal structures indexed by time sets $T_b^\pm$ which are uniquely determined by b and β (since the combinatorics is stationary). The composition operator is Lipschitz so there exists a maximal $S(\delta_b) < \infty$ such that if we define

$$E_b = \left\{ (\bar{s}_-, \bar{s}_+) \in \ell^1(\mathbb{R}; T_b^-) \times \ell^1(\mathbb{R}; T_b^+) \mid \|\bar{s}_\pm\| \le S(\delta_b) \right\},$$

then $\bigcirc \bar{s}_\pm \in \mathrm{Diff}^{\mathrm{S}}_{\delta_b}$, $\forall (\bar{s}_-, \bar{s}_+) \in E_b$. Define

$$\bar{\mathcal{D}}_b = \{(u,v,c,\bar{s}) \in \mathbb{R}^3 \times E_b \mid (u,v,c,\bigcirc \bar{s}_\pm) \in \mathcal{D}_b\}.$$

We will write $\bar{f} = (u,v,c,\bar{s})$ to denote pure Lorenz maps in $\bar{\mathcal{D}}_b$; $\bar{f}_b^\star = (u_b^\star, v_b^\star, c_b^\star, \bar{s}_b^\star)$ denotes the renormalization fixed point.

The idea of the proofs is to construct unstable Lipschitz manifolds inside the pure space $\mathbb{R}^3 \times E_b$. By composing the internal structures this gives us manifolds inside $\mathcal{L}$ which are Lipschitz since the composition operator is Lipschitz. Explicitly, if $\gamma : \mathbb{R}^3 \to E_b$ is Lipschitz, then $O \circ \gamma : \mathbb{R}^3 \to \mathrm{Diff}^{\mathrm{S}} \times \mathrm{Diff}^{\mathrm{S}}$ is also Lipschitz, where $O(\bar{s}_-, \bar{s}_+) = (\bigcirc \bar{s}_-, \bigcirc \bar{s}_+)$. So, once we prove that the graph of some γ is an unstable manifold in the pure space, then the graph of $O \circ \gamma$ is an unstable manifold in the space of Lorenz maps.

The proofs are divided into steps, starting with: (1) proving existence of a local unstable manifold, (2) showing that the graph transform can be extended, (3) growing the local unstable manifold to a global manifold.

Proof of Theorem 28. Step 1. We will prove the existence of an expanding cone field and a local unstable manifold.

Write $\mathcal{R}(\bar{f}) = (\xi(\bar{f}), \eta(\bar{f}))$ and $z = (x,y)$, where $\xi(\bar{f}), x \in \mathbb{R}^2$ $\eta(\bar{f}), y \in \mathbb{R} \times E_b$. Let $\hat{x} = (x_1/|U|, x_2/|V|)$ where U and V are as in §7. We claim that $\forall \bar{f} \in \bar{\mathcal{D}}_b$

$$\begin{aligned} &\|D_{u,v}\xi(\bar{f})x\| \ge \frac{\|\hat{x}\|}{K}, \qquad \|D_{c,\bar{s}}\xi(\bar{f})\| \le \frac{Kb}{|C|}, \\ &\|D_{u,v}\eta(\bar{f})x\| \le K\|\hat{x}\|, \quad \|D_{c,\bar{s}}\eta(\bar{f})\| \le \frac{Kb}{|C|}. \end{aligned} \tag{65}$$

Note first that each of these operators can be thought of as a matrix whose entries are the partial derivatives we estimated in §7 and that we are using the ℓ^1–norm, hence the operator norm can be bounded by the supremum over all column norms. Let us prove the claim.

The first column of $D_{c,\bar{s}}\xi(\bar{f})$ is bounded by $Kb/|C|$ by Lemma 24 and the remaining columns have the same bound by Lemma 25(54).

Each partial derivative in the first row of $D_{c,\bar{s}}\eta(\bar{f})$ is bounded by $Kb/|C|$ by Lemma 24 (for the first entry) and Lemma 25(54) (for the remaining entries). The norm of each column, disregarding the first row, is much smaller than the entries of the first row by Lemma 25(56).

The first row of $D_{u,v}\eta(\bar{f})x$ has two entries which are bounded by $K/|U|$ and $K/|V|$, respectively, by Lemma 24. Using Lemma 25(55) we get that these are bounds for the respective norms of the two columns as well. Use the triangle inequality to get the desired bound on $\|D_{u,v}\eta(\bar{f})x\|$.

By Lemma 19, $D_{u,v}\xi(\bar{f})x = \tilde{W}\hat{x}$ where $\det \tilde{W} \geq K^{-1} - o(b^{-n})$ and the columns of $\tilde{W}$ all have bounded norm. Lemma 31 gives the desired bound on $\|D_{u,v}\xi(\bar{f})x\|$.

This concludes the proof of (65). We will now show how these bounds give us an invariant expanding cone field and a local unstable manifold.

We will use (65) to determine constants μ, ν, λ and τ of (63). Choose $\mu = \inf_{\bar{f}}\|\hat{x}\|/(K\|x\|)$. Note that $\|\hat{x}\|/\|x\|$ is a linear combination of $|U|^{-1}$ and $|V|^{-1}$, which is at least as large as the smaller of the two. From this it follows that we may choose ν, and hence $1-\tau$, of the order $b\max\{|U|,|V|\}/|C|$. Finally $\lambda \leq K^2$. From the Expansion Lemma it follows that $\mu \geq \rho^b$, for some $\rho > 1$ not depending on $\bar{f}$. The fractions $|C|/|U|$ and $|C|/|V|$ are proportional to the respective derivative of the first-entry map to C (since the first-entries have bounded distortion by Proposition 10) which are exponentially large in b, again by the Expansion Lemma. Hence $\nu \leq \sigma^b$ for some $\sigma < 1$ not depending on $\bar{f}$.

From these bounds on the constants of (63) it follows that the conditions of Lemma 26 are satisfied by choosing b large. Furthermore, $\theta_0 \leq K$ so we may choose θ constant (independent of b) and get an invariant expanding cone field over the standard cone H_θ. Since $\nu \ll 1$, the expansion constant is of the order $\mu \gg 1$.

By Lemma 27 there is a local unstable manifold W_{loc} which is the graph of a θ–Lipschitz map $\gamma_{\mathrm{loc}} : B \to (0,1)\times E_b$ for some open set $B \subset Q$ containing $(u_b^\star, v_b^\star)$.

Step 2. We will now define the graph transform on graphs over Q.

Let G be the set of θ–Lipschitz maps $\gamma : Q \to \Delta_b \times E_b$ such that $\bar{f}_b^\star \in \operatorname{graph}\gamma$. We claim that $\operatorname{graph}\gamma \cap \bar{\mathcal{D}}_b$ is mapped diffeomorphically onto Q by ξ, for every $\gamma \in G$. This implies that $\mathcal{R}(\operatorname{graph}\gamma \cap \bar{\mathcal{D}}_b)$ is a graph and from the invariant cone field it follows that it is θ–Lipschitz. Hence the graph transform on G is well-defined. We will now prove the claim.

Let $\ell(u,v) = \xi^{-1}(u,v)$. By Lemma 19 every $(u,v) \in Q$ is a regular value of ξ (since $\det D_{u,v}\xi \neq 0$), so $\ell(u,v)$ is a manifold of codimension two. Now pick some arbitrary $\gamma_1 \in G$ and $(u,v) \in Q$. We will prove the claim by showing that $\ell(u,v)$ meets the graph of γ_1 in a unique point. Let $\gamma_0 \in G$ be the standard two-dimensional family through $\bar{f}_b^\star = (u_b^\star, v_b^\star, c_b^\star, \bar{s}_b^\star)$, i.e. $\gamma_0(u,v) = (u, v, c_b^\star, \bar{s}_b^\star)$. Because of Lemma 16, the graph of γ_0 meets $\ell(u,v)$ exactly once. Homotope γ_0 to γ_1 via $\gamma_t = t\gamma_0 + (1-t)\gamma_1$ and note that $\gamma_t \in G$, for all $t \in [0,1]$. Now let us see for how long the intersection persists under this homotopy. First of all note that any intersection must be transversal; all tangent vectors of $\ell(u,v)$ lie in the complementary cone, since $\xi(\ell(u,v))$ is the graph of a constant map $(0,1)\times E_b \to Q$, and since the complementary cone field is invariant under $D\mathcal{R}^{-1}$.

Due to transversality there are only two possibilities: either the graph of γ_t contains a boundary point of $\ell(u,v)$ for some $t \in (0,1)$, or $\ell(u,v)$ meets γ_1 in a unique point. We will show that the first option cannot occur.

Let $\gamma \in G$ and consider the projection of the graph of γ to $\Delta_b \times E_b$. The diameter of this projection is exponentially small in b (since θ is fixed and $1-u$ and $1-v$ are exponentially small in b as a consequence of the Expansion Lemma; see its accompanying remark) but the diameter of $\Delta_b \times E_b$ is at least of the order $1/b^2$ (by Lemma 15(4) and since $\delta_b = 1/b^2$). Furthermore, $(c^\star_b, \bar{s}^\star_b)$ is bounded away from the boundary of $\Delta_b \times E_b$ by Lemma 15(3) and since $\|\bar{s}^\star_b\|$ is exponentially small in b (by Lemma 5 and Proposition 22). Hence the graph of γ is far away from the boundary of $\ell(u,v)$ for all $\gamma \in G$. In particular, it holds for all γ_t so γ_1 must meet $\ell(u,v)$ in a unique point.

Step 3. We will now grow the local unstable manifold and prove properties 1 to 3.

Let $\gamma_0 \in G$ be some map whose graph coincides with the local unstable manifold on some neighborhood of $\bar{f}^\star_b$. That is, $\gamma_0|_B = \gamma_{\mathrm{loc}}|_B$ for some open neighborhood $B \subset Q$ of $(u^\star_b, v^\star_b)$. Let $\gamma_k = \Gamma^k(\gamma_0)$ and $W_k = \operatorname{graph}\gamma_k$, where $\Gamma : G \to G$ is the graph transform from step 2. We claim that there exists $n < \infty$ such that $W_n \subset \mathcal{R}^n(W_{\mathrm{loc}})$. Since the global unstable manifold at $\bar{f}^\star_b$ is given by $\bigcup_{k\ge 0} \mathcal{R}^k(W_{\mathrm{loc}})$, this implies that $\mathcal{W}^{\mathrm{uu}}_b = W_n$ and hence property 1 follows.

To see why the claim holds, pick an arbitrary $\bar{f}_n \in W_n$ and an arbitrary curve $\sigma_n : [0,1] \to W_n$ such that $\sigma_n(0) = \bar{f}^\star_b$ and $\sigma_n(1) = \bar{f}_n$. Assume without loss of generality that σ_n is differentiable. Let $\sigma_k : [0,1] \to W_k$ be curves in the backward orbit of σ_n and let $|\sigma_k|$ denote the length of σ_k. By step 1 the standard cone H_θ is expanded by $D\mathcal{R}(\bar{f})$ for all $\bar{f} \in \bar{\mathcal{D}}_b$, so the tangent vectors along σ_k are also expanded since they lie inside H_θ. Hence there exists $\lambda < 1$ such that $|\sigma_k| \le \lambda|\sigma_{k+1}|$, for $k = 0,\dots,n-1$. If we let $\bar{f}_0 = \sigma_0(1)$, then it follows that $\|\bar{f}_0 - \bar{f}^\star_b\| \le |\sigma_0| \le \lambda^n|\sigma_n|$. Since W_n is the graph of a Lipschitz function over a bounded domain, there exists $K < \infty$ (not depending on $\bar{f}_n$) such that $|\sigma_n| \le K$ for some choice of σ_n. But $\bar{f}_n \in W_n$ was arbitrary, so this shows that $\|\bar{f} - \bar{f}^\star_b\| \le K\lambda^n$ for every $\bar{f} \in \mathcal{R}^{-n}(W_n)$. In particular, there exists $n < \infty$ (depending on B) such that $\mathcal{R}^{-n}(W_n) \subset W_{\mathrm{loc}}$, since $\gamma_0|_B = \gamma_{\mathrm{loc}}|_B$. This concludes the proof of the claim and also proves property 3 (injectivity of $\mathcal{R}^{-1}$ follows from the claim of step 2).

If the graph of γ is invariant in the sense of property 2, then it must contain W_{loc} by Lemma 27. By the above claim it follows that $\gamma = \gamma^{\mathrm{uu}}_b$, which proves property 2.

Step 4. We will now prove property 4.

Let $\gamma \in G$, $W = \operatorname{graph}\gamma$, and assume $\bar{f} \in W$ is infinitely (a,b)–renormalizable. Note that the only (a,b)–renormalizable maps which are not in $\bar{\mathcal{D}}_b$ can be at most once renormalizable. Explicitly, a twice renormalizable map must have $(u',v') \in Q$ by the Expansion Lemma (for b large enough) which together with Lemma 16 implies $\bar{f} \in \bar{\mathcal{D}}_b$. By deforming γ (without leaving G)[3] we may additionally assume that $W|_V = W_{\mathrm{loc}}|_V$ on some neighborhood V of $\bar{f}^\star_b$. By step 3, $\mathcal{R}^n(W_{\mathrm{loc}} \cap V) \supset \mathcal{W}^{\mathrm{uu}}_b$ for some $n < \infty$. Since $\bar{f}$ is infinitely renormalizable this means that $\bar{f}$ must have been in W_{loc} to begin with. For the same reason, $\mathcal{R}^n\bar{f} \in \mathcal{W}^{\mathrm{uu}}_b \cap \bar{\mathcal{D}}_b$, $\forall n \ge 0$. Hence property 3 implies that $\|\bar{f} - \bar{f}^\star_b\| \le K\lambda^n$, $\forall n \ge 0$. That is, $\bar{f} = \bar{f}^\star_b$.

[3]If $\bar{f}$ was in the boundary of the cone $H_\theta + \bar{f}^\star_b$ then this deformation would take us outside G, but then we could simply make θ slightly larger and Γ would still be well-defined.

Step 5. We will now prove property 5.

We claim that there is a full island $I \subset Q$. Hence it is inside the domain of γ_b^{u} so $\mathcal{R}(\operatorname{graph} \gamma_b^{\mathrm{uu}}|_I)$ is a full family by [16, Prop. 2.1]; it is also a 2–dim unstable manifold extending $\mathcal{W}_b^{\mathrm{uu}}$. To prove the claim, note that γ_b^{u} can be extended to a family satisfying [16, Prop. 2.1] (see step 4 of the proof of Theorem 29) and hence it contains a full island I; $I \subset Q$ for b large, by the Expansion Lemma. □

Proof of Theorem 29. Step 1. We will prove the existence of an expanding cone field and a local unstable manifold. The arguments used here are identical to step 1 of the proof of Theorem 28 but we write them out in detail since the resulting invariant cone fields are quite different.

Write $\mathcal{R}(\bar{f}) = (\xi(\bar{f}), \eta(\bar{f}))$ and $z = (x, y)$, where $\xi(\bar{f}), x \in \mathbb{R}^3$ and $\eta(\bar{f}), y \in E_b$. Let $\hat{x} = (x_1/|U|, x_2/|V|, x_3/|C|)$ where U, V and C are as in §7. We claim that $\forall \bar{f} \in \bar{\mathcal{D}}_b$

$$\begin{aligned} \|D_{u,v,c}\xi(\bar{f})x\| &\geq \frac{\|\hat{x}\|}{Kb}, & \|D_{\bar{s}}\xi(\bar{f})\| &\leq \frac{Kb}{|C|}, \\ \|D_{u,v,c}\eta(\bar{f})x\| &\leq \|\hat{x}\| o(b^{-n}), & \|D_{\bar{s}}\eta(\bar{f})\| &\leq \frac{o(b^{-n})}{|C|}. \end{aligned} \tag{66}$$

Note first that each of these operators can be thought of as a matrix whose entries are the partial derivatives we estimated in §7 and that we are using the ℓ^1–norm, hence the operator norm can be bounded by the supremum over all column norms. Let us prove the claim.

By adding the bounds of Lemma 25(54) we see that all columns of $D_{\bar{s}}\xi(\bar{f})$ have norm bounded by $Kb/|C|$, proving the second inequality of (66).

All columns of $D_{\bar{s}}\eta(\bar{f})$ have norm bounded by $o(b^{-n})/|C|$ by the second equation of Lemma 25(56), proving the fourth inequality of (66).

The three columns of $D_{u,v,c}\eta(\bar{f})$ have norms bounded by $o(b^{-n})/|U|$, $o(b^{-n})/|V|$ and $o(b^{-n})/|C|$, respectively, by Lemma 25 (55) and the first equation of (56). Apply the triangle inequality to finish the proof of the third inequality of (66).

Let us finally prove the first inequality of (66). By Lemma 24 $\|D_{u,v,c}\xi(\bar{f})x\| = \|\tilde{W}\xi\|$ where $\tilde{W} = W + (\varepsilon_0, \varepsilon_1, \varepsilon_2 - Aw_0 + Bw_1)$. Let S be the area of the largest face of the convex hull of $\tilde{W}$. The first two columns of $\tilde{W}$ have bounded norms whereas the third has norm of order $O(b)$. Hence $S \leq Kb$. Since all columns of W have bounded norm and since the determinant is continuous and alternating $\det \tilde{W} = \det W + o(b^{-n})$. From Lemma 31 we get that

$$\|D_{u,v,c}\xi(\bar{f})x\| = \|\tilde{W}\xi\| \geq \frac{|\det W| - |o(b^{-n})|}{Kb}\|\xi\|. \tag{67}$$

In the construction of $\mathcal{D}_b$ we were free to choose u' and v' arbitrarily close to 1. This fact together with Proposition 10 and (1) show that λ_0' and λ_1' can be taken arbitrarily close to α/c' and $\lambda_1' \to \alpha/(1-c')$, respectively. Using Lemma 24 we get

$$\det W = -\frac{2\alpha(2\alpha-1)(\alpha-1)c'(1-c')}{(\alpha-c')^2(\alpha-1+c')^2} + \varepsilon,$$

for arbitrarily small ε. This is uniformly bounded away from 0 since $c' \in \Delta$ for all $\bar{f} \in \bar{\mathcal{D}}_b$, and $\alpha > 1$. Hence 67 implies the first inequality of (66).

This concludes the proof of (66). We will now show how these bounds give us an invariant expanding cone field and a local unstable manifold.

Note that $\|\hat{x}\|/\|x\| = t_1/|U| + t_2/|V| + t_3/|C|$, where $\sum |t_i| = 1$. We may assume $|U|, |V| < |C|$ by choosing b large enough, so $\|\hat{x}\|/\|x\| \geq |C|^{-1}$. Using this and (66) it follows that (63) is satisfied for

$$\mu = \inf_{\bar{f}\in\bar{\mathcal{D}}_b} \frac{1}{Kb|C|}, \qquad \nu = Kb^2, \qquad \lambda = o(b^{-n}), \qquad 1-\tau = o(b^{-n}).$$

By Lemma 5, $\mu \to \infty$ as $b \to \infty$. From the above it follows that the conditions of Lemma 26 are satisfied. The angles of Lemmas 26 and 27 satisfy $\theta_0 \leq o(b^{-n})$ whereas $\theta_i \geq 1/(Kb^2)$ for $i = 1, 2, 3$. This shows that we may choose $\theta_b = 1/b^3$ and get that the cone H_{θ_b} is invariant and expanded. Note the differences to Theorem 28: the angle of the invariant cone cannot be chosen independently of b but instead we may choose it arbitrarily small. We also get a local unstable manifold W_{loc} which is the graph of a θ_b–Lipschitz map $\gamma_{\mathrm{loc}} : B \to E_b$ for some open neighborhood $B \subset Q \times \Delta_b$ of $(u_b^\star, v_b^\star, c_b^\star)$, where $\bar{f}_b^\star = (u_b^\star, v_b^\star, c_b^\star, \bar{s}_b^\star)$.

Step 2. We will now define the graph transform on graphs over $Q \times \Delta$. The arguments use step 2 of the proof of Theorem 28 but they are somewhat different.

Let G be the set of θ_b–Lipschitz maps $\gamma : Q \times \Delta \to E_b$ such that $\bar{f}_b^\star \in \operatorname{graph}\gamma$, where θ_b is as in step 1. We claim that $\xi|_W$ is a diffeomorphism to its image and that $\xi(W) \supset Q \times \Delta$, where $W = \bar{\mathcal{D}}_b \cap \operatorname{graph}\gamma$. This implies that $\mathcal{R}(W)$ is a graph over $Q \times \Delta$ and from the invariant cone field it follows that it is θ_b–Lipschitz. Hence the graph transform on G is well-defined. We will now prove the claim.

Every value of $\xi|_W$ is regular. This follows from the cone invariance, which shows that every tangent plane of the graph of γ is in H_{θ_b} so its (u, v, c)–projection is onto, and from Lemma 24, which shows that $D_{u,v,c}\xi$ has non-zero determinant.

By the above we can pull back $(u, v) \times (0, 1)$ by $\xi|_W$ and get a (nonempty) curve $\ell(u, v) \subset W$, $\forall (u, v) \in Q$. This curve is a graph over Δ_b by step 2 of the proof of Theorem 28. To see this use that $(u, v) \mapsto (u, v, c_0, \gamma(u, v, c_0))$ is a δ_b–Lipschitz graph so its intersection with W is a diffeomorphic copy of Q under $\mathcal{R}$ followed by a (u, v)–projection, $\forall c_0 \in \Delta_b$.

The ξ–image of $\ell(u, v)$ is in $(u, v) \times (0, 1)$ and by Lemma 15 it contains $(u, v) \times \Delta$. Furthermore $D_{u,v,c}\xi$ is of the form $(0, 0, t)$ for tangent vectors of $\ell(u, v)$ and $t \neq 0$ since each value of $\xi|_W$is regular. Thus ξ is injective on $\ell(u, v)$. This finishes the proof of the claim, since by the above W is foliated by $\{\ell(u, v)\}_{(u,v)\in Q}$.

Step 3. This is identical to step 3 in the proof of Theorem 28.

Step 4. Pick $c \in \Delta_b$ close to $c(\bar{f}_b^\star)$. Let $\gamma_c : Q \to \{c\} \times E_b$ be the map whose graph contains all points of $\mathcal{W}_b^{\mathrm{u}}$ with critical point c, i.e. $\gamma_c(u, v) = (c, \gamma_b^{\mathrm{u}}(u, v, c))$. Let $(\bar{\phi}_{u,v}, \bar{\psi}_{u,v}) = \gamma_b^{\mathrm{u}}(u, v, c)$, $\phi_{u,v} = \bigcirc\bar{\phi}_{u,v}$, $\psi_{u,v} = \bigcirc\bar{\psi}_{u,v}$, and define

$$F(u, v) = \left(\frac{\phi_{u,v}(u) - c}{1 - c}, \frac{c - \psi_{u,v}(1 - v)}{c} \right).$$

F takes each map in the graph of γ_c to its critical values, normalized to lie in $[0, 1]$. Extend F continuously to a rectangle set $\Lambda \supset Q$ such that $F(\Lambda) = [0, 1]^2$ and $F : \partial\Lambda \to \partial([0, 1]^2)$ has non-zero degree. For example, set $\bar{\phi}_{u,v}(\tau) = 0$ and $\bar{\psi}_{u,v}(\tau) = 0$ outside an ε–neighborhood of Q, $\forall\tau$, and interpolate linearly. This extension simply means that we get a family where if one branch is trivial or full then the critical value of the other branch runs through all possible values as we change (u, v). From [16, Prop. 2.1] it follows that there exists $(u, v) \in \Lambda$ such that $\bar{f} = (u, v, c, \bar{\phi}_{u,v}, \bar{\psi}_{u,v})$ is infinitely (a, b)–renormalizable ($\bar{f}$ can be approximated by $\bar{f}_n$ whose associated

Lorenz maps f_n have finite critical orbits, e.g. take $\bar{f}_n$ such that $\mathcal{R}^n \bar{f}_n$ is full). Maps outside $\bar{\mathcal{D}}_b$ are at most once renormalizable (see step 4 of the proof of Theorem 28), so $(u, v) \in Q$, and consequently $\bar{f} \in \mathcal{W}_b^{\mathrm{u}}$. □

Appendix A. Collected results

Lemma 30. *Let q be a power-law branch with critical point at 0. Given a triplet of points $0 < x < y < z < \infty$, we denote $L = [0, x]$, $M = [x, y]$, $R = [y, z]$, $LM = L \cup M$, $MR = M \cup R$, and $LR = L \cup M \cup R$. Then*

$$\frac{|q(L)|}{|q(LM)|} = \left(\frac{|L|}{|LM|}\right)^{\alpha}, \tag{68}$$

$$\frac{|M|}{|LM|} \leq \frac{|q(M)|}{|q(LM)|} \leq \alpha \frac{|M|}{|LM|}, \tag{69}$$

$$\frac{|q(M)|}{|q(MR)|} \leq \frac{|M|}{|MR|}. \tag{70}$$

Furthermore, for every $\tau > 0$ there exists $\rho \in (0, 1)$ such that if $|LR| \geq (1+\tau)|LM|$ and $|R| \geq \tau |M|$, then

$$\frac{|q(M)|}{|q(MR)|} \leq \rho \frac{|M|}{|MR|}. \tag{71}$$

Lemma 31. *Let M be a non-singular $n \times n$ matrix, let V be the volume of the n–simplex σ with vertices $\{0, e_1, \ldots, e_n\}$, and let A be the surface area of the largest face of the convex hull of $\{\pm m_i\}_{i=1}^n$. Here e_i are the standard basis vectors of $\mathbb{R}^n$ and m_i are the columns of M. Then*

$$\frac{\|Mx\|_1}{\|x\|_1} \geq V \frac{|\det M|}{A}, \qquad \forall x \in \mathbb{R}^n \setminus \{0\},$$

where $\|\cdot\|_p$ denotes the ℓ^p–norm.

Proof. Note that $\|Mx\|_1 / \|x\|_1$ is the ℓ^1–distance from the origin to a point on the boundary, H, of the convex hull of $\{\pm m_i\}_{i=1}^n$, for $x \neq 0$. Let $\xi \in H$ be a point at minimal ℓ^1–distance to the origin. By changing signs of some columns of M it is possible to obtain a matrix $\tilde{M}$ such that ξ is in the image the simplex σ under $\tilde{M}$. By the change of variables formula $\mathrm{vol}(\tilde{M}\sigma) = |\det M| V$, since $|\det \tilde{M}| = |\det M|$. At the same time $\mathrm{vol}(\tilde{M}\sigma) \leq \|\xi\|_2 A$, since $\|\xi\|_2 \geq \min\{\|x\|_2 \mid x \in H \cap \tilde{M}\sigma\}$. Hence

$$\frac{\|Mx\|_1}{\|x\|_1} \geq \|\xi\|_1 \geq \|\xi\|_2 \geq V \frac{|\det M|}{A}.$$

□

Lemma 32 ([15, Lemma 10.3])**.** *If $g \in \mathrm{Diff}^2$ (see §2), then*

$$e^{-|y-x|\|g\|} \leq \frac{Dg(y)}{Dg(x)} \leq e^{|y-x|\|g\|}, \qquad e^{-\|g\|} \leq Dg(x) \leq e^{\|g\|}.$$

Lemma 33 (Koebe Lemma [11, Lem. 2.4])**.** *If $g^{-1} \in \mathrm{Diff}^{\mathrm{S}}$ (see §2), then*

$$|Ng(x)| \leq 2 \min\{|x|, |1-x|\}^{-1}.$$

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Institute for Mathematical Sciences, Stony Brook University, Stony Brook NY 11794-3660, USA

E-mail address: marco@math.stonybrook.edu

E-mail address: bjorn.winckler@gmail.com