I. M. Krichever, L. A. Takhtadzhyan, Algebraic de Rham theorem and Baker–Akhiezer function, *Izvestiya: Mathematics*, 2024, Volume 88, Issue 3, 506–514

DOI: 10.4213/im9533e

Izvestiya: Mathematics **88**:3 506–514 *Izvestiya RAN: Ser. Mat.* **88**:3 101–110
DOI: https://doi.org/10.4213/im9533e

Algebraic de Rham theorem and Baker–Akhiezer function

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Abstract. For the case of algebraic curves (compact Riemann surfaces), it is shown that de Rham cohomology group $H^1_{\mathrm{dR}}(X,\mathbb{C})$ of a genus g of the Riemann surface X has a natural structure of a symplectic vector space. Every choice of a non-special effective divisor D of degree g on X defines a symplectic basis of $H^1_{\mathrm{dR}}(X,\mathbb{C})$ consisting of holomorphic differentials and differentials of the second kind with poles on D. This result, which is the algebraic de Rham theorem, is used to describe the tangent space to Picard and Jacobian varieties of X in terms of differentials of the second kind, and to define a natural vector fields on the Jacobian of the curve X that move points of the divisor D. In terms of the Lax formalism on algebraic curves, these vector fields correspond to the Dubrovin equations in the theory of integrable systems, and the Baker–Akhierzer function is naturally obtained by the integration along the integral curves.

Keywords: Riemann surface, divisor, line bundle, Riemann–Roch theorem, differentials of the second kind, algebraic de Rham theorem, Picard and Jacobian varieties, vector field on the Jacobian variety, Lax representation, Dubrovin equation, Baker–Akhiezer function.

§ 1. Introduction

Let X be a smooth algebraic variety over $\mathbb{C}$ with the classical topology of a complex manifold. According to Atiyah and Hodge [1], a closed meromorphic p-form φ on X is called *differential of a second kind* if it has zero residues on open subsets $U = X \setminus D$ for sufficiently large divisors D. A far-reaching generalization of Atiyah and Hodge results was given by Grothendieck [2]. The quotient groups

$$\frac{\{p\text{-forms of the second kind}\}}{\{\text{exact forms}\}}$$

have a natural interpretation in terms of a spectral sequence of certain complex of sheaves of meromorphic forms on X (see [3], Ch. 3, § 5). In particular,

$$H^1_{\mathrm{dR}}(X,\mathbb{C}) \simeq \frac{\{1\text{-forms of the second kind}\}}{\{\text{exact forms}\}}.$$

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AMS 2020 *Mathematics Subject Classification*. 14F40, 14H40, 14H70.

When X is a smooth algebraic curve of genus g, this isomorphism follows from the Riemann–Roch theorem. It turns out that in this case the space of differentials of the second kind carries a natural skew-symmetric bilinear form, which is non-degenerate when taking a quotient by the subspace of exact forms. Using this bilinear form, in Theorem 1 we give a more explicit formulation of the algebraic de Rham theorem. Specifically, we show that each non-special effective divisor D of degree g on the Riemann surface X defines a symplectic basis of $H^1_{\mathrm{dR}}(X, \mathbb{C})$, which makes it possible to explicitly describe a complement to the Lagrangian subspace of holomorphic 1-forms on X as the subspace of differentials of the second kind with the poles in D.

In § 4, it will be shown that every non-special effective divisor D of degree g on X defines an explicit isomorphism between the vector space $H^{0,1}(X, \mathbb{C})$ and the Lagrangian subspace of differentials of the second kind with poles in D. This allows us to explicitly describe the tangent space to the Picard variety (and its "incarnations", Albanese and Jacobian varieties) in pure algebro-geometric terms.

It is quite remarkable that this formalism is connected with the theory of integrable systems. In the standard approach (see, for example, [4]), an integrable system is described by the zero curvature equation

$$\frac{\partial L}{\partial t} - \frac{\partial M}{\partial x} + LM - ML = 0,$$

where $L(x, t, \lambda)$ and $M(x, t, \lambda)$ are certain $r \times r$ matrix-valued rational functions of the spectral parameter $\lambda \in \mathbb{CP}^1$ which also depend on extra arguments x and t (physical space and time variables). In [5], [6] by the first author, the zero curvature formalism was extended to the case when spectral parameter varies on an algebraic curve.

Namely, it was shown in [5] that a natural framework for this generalization is provided by the Hitchin system expressed in terms of Tyurin parameters for stable holomorphic vector bundles of rank r and degree rg on an algebraic curve. Correspondingly, the rational functions L on $\mathbb{CP}^1$ become special $r \times r$ meromorphic matrix 1-forms $L(z)\,dz$ on an algebraic curve, and the set of such matrices is parameterized by the moduli space (more precisely, by its Zariski open subset) of stable holomorphic vector bundles of rank r and degree rg (see [5] for details). A similar explicit description is given for $r \times r$ meromorphic matrix-valued functions $M(z)$.

It turns out that, in the simplest case $r = 1$, this formalism is still non-trivial and is naturally connected with Theorem 1 (see the related discussion in § 3). Namely, as shown in § 5, the meromorphic 1-forms $L(z)\,dz$ become differentials of the first kind on an algebraic curve X, while analogs of $M(z)$ are meromorphic functions f defined using two non-special effective divisors D and D_0 of degree g on X. The varying divisors D parameterize the Jacobian of X with the base point D_0, and the vector fields describing the motion of points of D are naturally expressed in terms of the meromorphic functions f.

Remarkably, in the case when X is a hyperelliptic curve, equations (5.3) for the integral curves of these vector fields coincide with the *Dubrovin equations* arising in the theory of finite-gap integration of the Korteweg–de Vries equation [7]. Moreover, integrating meromorphic functions f along these integral curves and using the Dubrovin equations, we naturally obtain the *Baker–Akhiezer function*, a fundamental object in the algebro-geometric approach to integrable systems, which was introduced by the first author in [8].

The second author is grateful to the referee for constructive remarks and suggestions.

§ 2. Differentials of the second kind

Let X be a connected compact Riemann surface of genus g with classical topology. We denote by $\mathcal{O}_X$ the sheaf of germs of holomorphic functions on X. Let $\mathcal{M}_X$ be the sheaf of germs of meromorphic functions on X, and let $\mathcal{M}$ be the vector space of meromorphic functions on X. For every divisor D on X, let $L = \mathcal{O}(D)$ be the holomorphic line bundle associated with D, and let $H^0(X, L)$ be the vector space of holomorphic sections of L over X. The isomorphism

$$H^0(X, L) \simeq \mathcal{L}_D = \{f \in \mathcal{M} \colon (f) + D \geqslant 0\}.$$

is very useful.

From the Riemann–Roch theorem together with the Kodaira–Serre duality it follows that

$$h^0(L) - h^0(K_X - L) = \deg L + 1 - g,$$

where $h^0(L) = \dim_{\mathbb{C}} H^0(X, L)$, $\deg L$ is the degree of L, and K_X is the canonical class of X (the holomorphic cotangent bundle to X).

Let d be the exterior derivative on X. The sheaf $\mathrm{d}\mathcal{M}_X$ is a sheaf of germs of differentials of the second kind on X and $\Omega^{(\mathrm{2nd})} = H^0(X, \mathrm{d}\mathcal{M}_X)$ is the infinite-dimensional vector space of the differentials of the second kind (the meromorphic 1-forms on X with zero residues).

The infinite-dimensional vector space $\Omega^{(\mathrm{2nd})}$ has the natural skew-symmetric bilinear form[1]

$$\omega_X(\theta_1, \theta_2) = \sum_{P \in X} \operatorname*{Res}_P(\mathrm{d}^{-1}\theta_1 \theta_2), \qquad \theta_1, \theta_2 \in \Omega^{(\mathrm{2nd})},$$

where $\mathrm{d}^{-1}\theta_1$ denotes any locally defined function f such that $\mathrm{d}f = \theta_1$ (the local antiderivative). The ambiguity in the choice of f does not matter.

Indeed, it is clear that the bilinear form ω_X is defined by a finite sum and the choice of an additive constant in the definition of a local antiderivative is irrelevant. The skew-symmetry of ω_X follows from the basic property

$$\operatorname*{Res}_P(f_1\,\mathrm{d}f_2) = -\operatorname*{Res}_P(f_2\,\mathrm{d}f_1),$$

where the meromorphic functions f_1 and f_2 are the local antiderivatives of θ_1 and θ_2 in a neighbourhood of $P \in X$.

[1] Analogues of the skew-symmetric bilinear form ω_X and of the algebraic de Rham theorem for meromorphic quadratic differentials will be considered in a forthcoming paper by the second author.

§ 3. Algebraic de Rham theorem

In abstract form, the algebraic de Rham theorem reads as follows:

$$H^1_{\mathrm{dR}}(X,\mathbb{C}) \simeq \Omega^{(2\mathrm{nd})}/\mathrm{d}\mathcal{M}, \tag{3.1}$$

which is easily proved using the sheaf-theoretic de Rham isomorphism

$$H^1_{\mathrm{dR}}(X,\mathbb{C}) \simeq H^1(X,\underline{\mathbb{C}}),$$

where $\underline{\mathbb{C}}$ is the locally constant sheaf.

Indeed, consider the short exact sequence of sheaves

$$0 \to \underline{\mathbb{C}} \xrightarrow{i} \mathcal{M}_X \xrightarrow{\mathrm{d}} \mathrm{d}\mathcal{M}_X \to 0$$

and the corresponding exact cohomology sequence

$$H^0(X,\mathcal{M}_X) \xrightarrow{\mathrm{d}} H^0(X,\mathrm{d}\mathcal{M}_X) \xrightarrow{\delta} H^1(X,\underline{\mathbb{C}}) \to H^1(X,\mathcal{M}_X).$$

By the Riemann–Roch theorem,

$$H^1(X,\mathcal{O}(D)) = \{0\}$$

if $\deg D > 2g-2$, which implies (see, for example, [9], Ch. 2, § 17.7)

$$H^1(X,\mathcal{M}_X) = \{0\},$$

which proves (3.1).

Using the bilinear form ω_X, we can make isomorphism (3.1) more concrete. Namely, we have the following result (see [10], Ch. 6, § 8, [11], Ch. III, § 5.3, § 5.4, and [12], Theorem 4).

Theorem 1. *The following assertions hold.*

(i) *The restriction of the bilinear form ω_X to $\Omega^{(2\mathrm{nd})}/\mathrm{d}\mathcal{M}$ is non-degenerate and*

$$\dim_{\mathbb{C}} \Omega^{(2\mathrm{nd})}/\mathrm{d}\mathcal{M} = 2g.$$

(ii) *Each non-special effective divisor D of degree g on X defines the isomorphism*

$$\Omega^{(2\mathrm{nd})}/\mathrm{d}\mathcal{M} \simeq \Omega^{(2\mathrm{nd})} \cap H^0(X, K_X + 2D).$$

(iii) *Let $D = P_1 + \dots + P_g$ be a non-special divisor of degree g with distinct points. For every choice of local coordinates in the neighbourhoods of P_i, the vector space $\Omega^{(2\mathrm{nd})} \cap H^0(X, K_X + 2D)$ has the basis $\{\vartheta_i, \tau_i\}_{i=1}^g$ symplectic with respect to the bilinear from ω_X,*

$$\omega_X(\vartheta_i,\vartheta_j) = \omega_X(\tau_i,\tau_j) = 0, \quad \omega_X(\vartheta_i,\tau_j) = \delta_{ij}, \qquad i,j = 1,\dots,g.$$

This basis consists of differentials of the first kind ϑ_i and differentials of the second kind τ_i uniquely characterized by the conditions

$$\vartheta_i = \bigl(\delta_{ij} + O(z-z_j)\bigr)\,\mathrm{d}z \quad \textit{and} \quad \tau_i = \biggl(\frac{\delta_{ij}}{(z-z_j)^2} + O(z-z_j)\biggr)\mathrm{d}z,$$

where $z_j = z(P_j)$ for a local coordinate z at P_j, and $i,j = 1,\dots,g$.

Proof. Let $(\theta)_\infty = \sum_{i=1}^l n_i Q_i$ be the polar divisor of $\theta \in \Omega^{(2\mathrm{nd})}$, $n_i \geqslant 2$. Since D is non-special, $h^0(K_X - D) = 0$, and by Riemann–Roch formula we have $h^0(D + nQ_i) = n+1$ for $n \geqslant 0$. Thus if Q_i is not a point of D, there exists a meromorphic function $f_i \in \mathcal{L}_{D+(n_i-1)Q_i}$ such that

$$\operatorname*{ord}_{Q_i}(\theta - df_i) \geqslant 0.$$

If Q_i is a point of D, there is a function $f_i \in \mathcal{L}_{D+(n_i-1)Q_i}$ such that

$$\operatorname*{ord}_{Q_i}(\theta - df_i) \geqslant -2.$$

(In this case, $h^0(D) = 1$, and hence one can not adjust the principle part of df_i at Q_i to cancel possible second order pole of θ.) So, for $f = \sum_{i=1}^l f_i$, we have

$$(\theta - \mathrm{d}f) \geqslant -2D,$$

which proves assertion (ii).

The dimension formula in (i) easily follows from assertion (ii) since

$$\begin{aligned}\dim_{\mathbb{C}} \Omega^{(2\mathrm{nd})} \cap H^0(X, K_X + 2D) &= h^0(X, K_X + 2D) - h^0(X, K_X + D) + h^0(X, K_X) \\ &= (3g-1) - (2g-1) + g = 2g.\end{aligned}$$

To prove assertion (iii) and the remaining part in assertion (i), consider the linear map

$$L \colon \Omega^{(2\mathrm{nd})} \cap H^0(X, K_X + 2D) \to \mathbb{C}^{2g},$$

defined as follows. For each $\theta \in \Omega^{(2\mathrm{nd})} \cap H^0(X, K_X + 2D)$, let $\alpha_i(\theta), \beta_i(\theta) \in \mathbb{C}$ be such that

$$\frac{\theta}{\mathrm{d}z} - \alpha_i(\theta) - \frac{\beta_i(\theta)}{(z - z_i)^2} = O(z - z_i)$$

near P_i. We also set

$$L(\theta) = \big(\alpha_1(\theta), \beta_1(\theta), \dots, \alpha_g(\theta), \beta_g(\theta)\big).$$

The divisor D is non-special, and hence the map L is injective, and therefore, is an isomorphism, and we define ϑ_i and τ_i to have only non-zero components of L to be, respectively, $\alpha_i = 1$ and $\beta_i = 1$. This proves Theorem 1.

Remark 1. The choice of a non-special effective divisor D on X with g distinct points P_i and local coordinates is as an algebraic analogue of the choice of a-cycles on a Riemann surface. Correspondingly, the differentials τ_i are analogues of differentials of the second kind with second-order poles, zero a-periods and normalized b-periods. The symplectic property of the basis $\{\vartheta_i, \tau_i\}_{i=1}^g$ is an analogue of the reciprocity laws for differentials of the first kind and the second kind (see [13], Ch. 5, § 1, and [14], Ch. VI, § 3).

Remark 2. Let $\Omega^{(2\mathrm{nd})}(2D)$ be the subspace in $\Omega^{(2\mathrm{nd})}$ spanned by τ_i,

$$\Omega^{(2\mathrm{nd})}(2D) = \mathbb{C}\tau_1 \oplus \cdots \oplus \mathbb{C}\tau_g.$$

Then $\Omega^{(2\mathrm{nd})}(2D)$ and $H^0(X, K_X)$ are Lagrangian subspaces in $\Omega^{(2\mathrm{nd})}/\mathrm{d}\mathcal{M}$ dual with respect to the pairing given by the symplectic form ω_X.

§ 4. Tangent space to the Picard variety

We have the decomposition

$$H^1_{\mathrm{dR}}(X,\mathbb{C}) = H^{1,0}(X,\mathbb{C}) \oplus H^{0,1}(X,\mathbb{C}) \tag{4.1}$$

with natural pairing

$$H^{1,0}(X,\mathbb{C}) \otimes H^{0,1}(X,\mathbb{C}) \ni \alpha\otimes\beta \mapsto (\alpha,\beta) = \int_X \alpha\wedge\beta \in \mathbb{C}.$$

The period map

$$H^{1,0}(X,\mathbb{C}) \ni \vartheta \mapsto \int_c \vartheta \in \mathbb{C},$$

where $c\in H_1(X,\mathbb{Z})$, defines the canonical inclusion of the lattice $H_1(X,\mathbb{Z})$ into the vector space $H^{1,0}(X,\mathbb{C})^\vee$, which is the dual space of $H^{1,0}(X,\mathbb{C})$, and defines the Albanese variety

$$\mathrm{Alb}(X) = H^{1,0}(X,\mathbb{C})^\vee / H_1(X,\mathbb{Z}).$$

Using the Dolbeault isomorphism and the exponential exact sequence of sheaves on X, we have, for the Picard variety of line bundles of degree 0 on X,

$$\mathrm{Pic}^0(X) = H^{0,1}(X,\mathbb{C})/H^1(X,\mathbb{Z}).$$

Thus, the holomorphic tangent space to $\mathrm{Pic}^0(X)$ can be identified with the vector space $H^{0,1}(X,\mathbb{C})$.

However, Theorem 1 allows us to describe the tangent space to the Picard variety in purely algebro-geometric terms. Namely, the following simple result holds.

Proposition 1. *Each non-special effective divisor D of degree g on a Riemann surface X defines an isomorphism*

$$H^{0,1}(X,\mathbb{C}) \simeq \Omega^{(2\mathrm{nd})}(2D).$$

Proof. It follows from assertion (iii) of Theorem 1 that the mapping

$$H^{0,1}(X,\mathbb{C}) \ni \beta \mapsto \psi(\beta) = \sum_{i=1}^{g} (\vartheta_i,\beta)\tau_i \in \Omega^{(2\mathrm{nd})}(2D)$$

satisfies

$$(\vartheta,\beta) = \omega_X(\vartheta,\psi(\beta))$$

for any $\vartheta\in H^0(X,K_X)$, and is an isomorphism. This proves the proposition.

By identifying $H^{1,0}(X,\mathbb{C})^\vee$ with $\Omega^{(2\mathrm{nd})}(2D)$, we get an inclusion of the lattice $H_1(X,\mathbb{Z})$ into the vector space $\Omega^{(2\mathrm{nd})}(2D)$, which is defined as follows. Let θ_c be the $(0,1)$-component of the Poincaré dual of a cycle $c\in H_1(X,\mathbb{Z})$, so that

$$\int_c \vartheta = \int_X \vartheta\wedge\theta_c = (\vartheta,\theta_c) \quad \text{for all} \quad \vartheta\in H^{1,0}(X,\mathbb{C}).$$

Hence

$$H_1(X,\mathbb{Z}) \ni c \mapsto \tau_c = \psi(\theta_c) = \sum_{i=1}^{g} \int_c \vartheta_i \cdot \tau_i \in \Omega^{(2\mathrm{nd})}(2D), \tag{4.2}$$

and, therefore,

$$\mathrm{Alb}(X) = \Omega^{(2\mathrm{nd})}(2D)/H_1(X,\mathbb{Z}). \tag{4.3}$$

Thus, by choosing a non-special effective divisor D of degree g one can identify the holomorphic tangent spaces to the manifolds

$$\mathrm{Alb}(X) \simeq \mathrm{Pic}^0(X) \simeq \mathrm{Jac}(X)$$

with vector space $\Omega^{(2\mathrm{nd})}(2D)$ of the differentials of the second kind with poles in D. Correspondingly, the holomorphic cotangent space is naturally identified with the vector space of $H^{1,0}(X,\mathbb{C})$ of differentials of the first kind, and the pairing with $\Omega^{(2\mathrm{nd})}(2D)$ is given by the symplectic form ω_X.

§ 5. The Baker–Akhiezer function

Given a fixed non-special effective divisor $D_0 = Q_1 + \dots + Q_g$ of degree g on X, let $\{\vartheta_i\}_{i=1}^{g}$ be the basis of $H^0(X,K_X)$ from Theorem 1 specialized to the divisor D_0. Consider the Abel–Jacobi map

$$X^{(g)} \ni D \to \mu^{(g)}(D) \in \mathrm{Jac}(X),$$

where $\mu^{(g)}$ is the Abel sum: for the varying divisor $D = P_1 + \dots + P_g$,

$$\mu^{(g)}(D) = \biggl(\sum_{i=1}^{g} \int_{Q_i}^{P_i} \vartheta_1, \dots, \sum_{i=1}^{g} \int_{Q_i}^{P_i} \vartheta_g\biggr). \tag{5.1}$$

We choose local coordinates at the points P_i and put $z_i = z(P_i)$. It follows from (5.1) that 1-forms dz_i on $\mathrm{Jac}(X)$ at the base point $\mu^g(D_0)$ correspond to the differentials ϑ_i, and the vector fields $\partial/\partial z_i$ correspond to the differentials of the second kind τ_i from Theorem 1. If divisor D is also non-special, it follows from the group law on the Jacobian and Theorem 1 that $\mathrm{d}z_i$ and $\partial/\partial z_i$ at a point $\mu^{(g)}(D) \in \mathrm{Jac}(X)$ are given by the symplectic basis of $\Omega^{(2\mathrm{nd})} \cap H^0(X, K_X + 2D)$ from Theorem 1.

Equivalently, these vector fields on $\mathrm{Jac}(X)$ can be described using the formalism of Lax equations on algebraic curves, developed by the first author in [5] and [6]. The main ingredients in [5] and [6] are stable vector bundles of rank r and degree rg, Lax operators, certain meromorphic $r \times r$ matrix-valued 1-forms $L(z)\,dz$ on a Riemann surface X, and $r \times r$ meromorphic matrix-valued functions $M(z)$.

Specialization to the Jacobian corresponds to the case $r = 1$ and substantially simplifies construction in [5] and [6]. Namely, the meromorphic 1-forms $L(z)\,\mathrm{d}z$ become differentials of the first kind $\vartheta \in H^0(X,K_X)$, while analogs of the meromorphic functions $M(z)$ are defined as follows.

Consider the vector space

$$\mathcal{L}_{D+D_0} = \{f \in \mathcal{M} \colon (f) + D + D_0 \geqslant 0\}.$$

It follows from the Riemann–Roch theorem that $\dim_{\mathbb{C}} \mathcal{L}_{D+D_0} = g+1$. Thus for any fixed choice of principal parts of f at the points of D_0, not all of them are equal to zero, there is a unique, up to an inessential additive constant, function $f \in \mathcal{L}_{D+D_0}$ satisfying

$$f(z) = \frac{\alpha_i}{z - z_i} + O(1), \qquad z_i = z(P_i), \tag{5.2}$$

at all points of the divisor $D = P_1 + \cdots + P_g$. Functions f, parametrized by their fixed principal parts at D_0, play the role of meromorphic functions $M(z)$ in case $r = 1$; coefficients α_i depend on the principal parts at D_0.

We have a unique decomposition

$$\mathrm{d}f = \tau - \tau_0,$$

where $\tau \in \Omega^{(2\mathrm{nd})}(2D)$ (see Remark 2) and $(\tau_0) + 2D_0 \geqslant 0$. By the residue theorem,

$$-\sum_{i=1}^{g} \operatorname*{Res}_{P_i}(f\vartheta) = \omega_X(\vartheta, \tau) = \omega_X(\vartheta, \tau_0), \qquad \vartheta \in H^0(X, K_X),$$

and so the pairing (2.22) in [5], as given by the Krichever–Phong form, coincides with the pairing given by the symplectic form ω_X.

A choice of a symplectic basis of

$$\Omega^{(2\mathrm{nd})} \cap H^0(X, K_X + 2D)$$

establishes the correspondence

$$f \mapsto \mathscr{L}_f = -\sum_{i=1}^{g} \alpha_i \frac{\partial}{\partial z_i}$$

between the rational functions $f \in \mathcal{L}_{D+D_0}$ and the vector fields on $\mathrm{Jac}(X)$. Along the integral curve $D(t) = P_1(t) + \cdots + P_g(t)$ of $\mathscr{L}_f$, where $D(0) = D$, we have

$$\dot{z}_i(t) = -\alpha_i(t), \qquad i = 1, \ldots, g, \tag{5.3}$$

where the dot stands for the t derivative. If X is a hyperelliptic curve, equations (5.3) are the classical *Dubrovin equations* arising in the theory of finite-gap integration for the Korteweg–de Vries equation [7], written in terms of the Abel transform. Using the Dubrovin equations, we see that, along the integral curve, equations (5.2) take the form

$$f_t(z) = -\frac{\dot{z}_i(t)}{z - z_i(t)} + O(1), \qquad i = 1, \ldots, g. \tag{5.4}$$

Thus integrating and introducing

$$\Psi(z) = \exp\left\{ \int_0^T f_t(z)\, dt \right\},$$

we see from (5.4) that Ψ is a meromorphic function on $X \setminus D_0$ with simple poles only at D, simple zeros only at $D(T)$, and essential singularities at D_0.

The function Ψ is nothing but the celebrated *Baker–Akhiezer function*, which was introduced by the first author in [8].

We leave it to the interested reader to describe by explicit formulas this connection between algebraic de Rham theorem and the integrable systems.

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Received 16/AUG/23
17/OCT/23
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