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Erratum: Supersymmetry and trace formulas. Part I. Compact Lie groups

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1. We emphasize that the insertion of χ_μ saturates the zero modes in the path integral side of eq. (2.9) and makes it non-zero, while the insertion of local operators $\hat{\chi}_\mu$ saturates the fermion number operator $(-1)^F$ in the spectral side of eq. (2.9). It is important to observe that the operators $\hat{\chi}_\mu$ are not naive quantization of the zero modes, since the latter are generically non-local in time and have no counterpart in the Hamiltonian formalism. The local operators $\hat{\chi}_\mu$ are just $\hat{\psi}^\mu(0)$, where fermion operators are normalized to satisfy eq. (2.8).

2. The identification in section 2.2 between the product of fermionic zero modes has a missing overall numerical factors to be consistent with $((-1)^F)^2 = 1$, and therefore the correct equality is $(-1)^F = c_n 2^{n/2}\hat{\psi}^1 \cdots \hat{\psi}^n$ with $c_n = \pm i^{n(n-1)/2}$. Two choices of c_n simply reflects the $\mathbb{Z}_2$ ambiguity in the definition of the total fermion parity. The choice of c_n is reflected in the overall sign factor in the fermionic path integral measure.

3. In section 4, it is more natural to use the Cartan-Weyl basis (see appendix A) for the description of the fermion number operator $(-1)^F$. And this will identify the correct fermionic measures in the path integrals. The following is to replace from the beginning of section 4.2 to the end of page 19.

   Namely, we have
$$\psi(0) = \psi^a(0)T_a = \sum_j \psi_j \cdot iH_j + \sum_{\alpha\in R} \psi_\alpha E_\alpha, \tag{1}$$

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where $i=\sqrt{-1}$ and $\bar{\psi}_j=\psi_j$, $\bar{\psi}_\alpha=-\psi_{-\alpha}$. Corresponding fermion operators satisfy anti-commutation relations

$$[\hat{\psi}_i,\hat{\psi}_j]=\delta_{ij},\quad [\hat{\psi}_\alpha,\hat{\psi}_\beta]=-\delta_{\alpha,-\beta},$$

where $\hat{\psi}_i^\dagger=\hat{\psi}_i$ and $\hat{\psi}_\alpha^\dagger=-\hat{\psi}_{-\alpha}$. Introduce Hermitian fermion operators $\hat{\chi}_j=\hat{\psi}_j$ and $\hat{\chi}_\alpha$, $\hat{\chi}_{-\alpha}$ by the following formulas

$$\hat{\chi}_\alpha=\frac{1}{\sqrt{2}}(\hat{\psi}_\alpha-\hat{\psi}_{-\alpha}),\quad \hat{\chi}_{-\alpha}=\frac{1}{i\sqrt{2}}(\hat{\psi}_\alpha+\hat{\psi}_{-\alpha}),\quad \text{where}\quad \alpha\in R_+.$$

The operators $\{\hat{\chi}_j,\hat{\chi}_\alpha,\hat{\chi}_{-\alpha}\}$ satisfy canonical relations (2.8). For each two-dimensional fermion Hilbert subspace generated by $\hat{\psi}_\alpha$ and $\hat{\psi}_{-\alpha}$, we have a natural fermion number operator

$$(-1)^{F_\alpha}=2i\hat{\chi}_\alpha\hat{\chi}_{-\alpha},\quad \alpha\in R_+.$$

Thus the fermion number naturally associated with the Cartan-Weyl basis is

$$c_n\hat{\psi}^1\dots\hat{\psi}^n=c_r\hat{\chi}_1\dots\hat{\chi}_r\prod_{\alpha\in R_+}i\hat{\chi}_\alpha\hat{\chi}_{-\alpha}=2^{-n/2}(-1)^F, \tag{2}$$

where the phase c_r is such that $\{(-1)^F\}^2=\hat{I}$ holds. So according to (4.22), bosonic and fermionic degrees of freedom are totally decoupled and

$$\operatorname{Str}\Big(c_r\hat{\chi}_1\dots\hat{\chi}_r\prod_{\alpha\in R_+}i\hat{\chi}_\alpha\hat{\chi}_{-\alpha}e^{-\beta\hat{H}}\Big)=e^{-\frac{1}{12}\beta R}\operatorname{Tr}e^{-\frac{1}{2}\beta\Delta}.$$

Since Δ commutes with left and right translations, one can also express the heat kernel on G as a supertrace. Namely, we recall that the heat kernel is a fundamental solution $K_\tau(g_1,g_2)$ of the heat equation on G

$$\frac{\partial K}{\partial\tau}=-\frac{1}{2}\Delta K$$

with respect to g_1, satisfying

$$\lim_{\tau\to 0}K_\tau(g_1,g_2)=\delta_G(g_1g_2^{-1}),$$

where δ_G is the Dirac delta-function on G with respect to the Cartan-Killing volume form. Fix a Cartan subgroup T in G and corresponding Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{g}$, $\dim\mathfrak{t}=r$, the rank of $\mathfrak{g}$. It follows from the bi-invariance of the heat kernel that it only depends on $g_1g_2^{-1}\in T$, and we will denote it by $k_t(e^h)$, where $h\in\mathfrak{t}$ and $e^h\in T$. Using Dirac notation,

$$k_\tau(e^h)=K_\tau(e^h,1)=\langle e^h|e^{-\frac{1}{2}\tau\Delta}|1\rangle,$$

where 1 is the identity element in G.

Correspondingly, $\operatorname{Tr}e^{-\frac{1}{2}\beta\Delta}=V_Gk_\beta(1)$, where V_G is the volume of G, and more generally

$$\operatorname{Tr}e^{-\frac{1}{2}\beta\Delta+i(h,\hat{r})}=V_Gk_\beta(e^h),$$

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where $\hat{r} = \hat{r}^a T_a$. The extra term $i(h, \hat{r})$ in the exponent can be thought as a (imaginary) 'chemical potential' added to the Hamiltonian $\hat{H}$. Since the operators $\hat{r}^a$ commute with $\hat{Q}$, we have

$$\mathrm{Str}\left(c_r \hat{\chi}_1 \dots \hat{\chi}_r \prod_{\alpha \in R_+} i\hat{\chi}_\alpha \hat{\chi}_{-\alpha} e^{-\beta \hat{H} + i(h, \hat{r})}\right) = V_G e^{-\frac{1}{12}\beta R} k_\beta(e^h). \tag{3}$$

Here the supertrace is given by the following path integral

$$\int\limits_{\Pi TLG} \chi_1 \cdots \chi_r \prod_{\alpha \in R_+} i\chi_\alpha \chi_{-\alpha} \, e^{-S_E^h} \, \mathscr{D}g \mathscr{D}\psi \,, \tag{4}$$

with the Euclidean action

$$S_E^h = \frac{1}{2}\int_0^\beta \left((J, J) + (\psi, \dot{\psi})\right) d\tau + \frac{1}{\beta}\int_0^\beta (\mathrm{Ad}_{g^{-1}} h, J) d\tau + \frac{1}{2\beta}(h, h) \tag{5}$$

and G-invariant 'measure' $\mathscr{D}g$. The fermion 'measure' $\mathscr{D}\psi$ is determined from the condition

$$\begin{aligned} 1 &= \mathrm{Tr}_{\mathscr{H}_F}\left(c_r \hat{\chi}_1 \dots \hat{\chi}_r \prod_{\alpha \in R_+} i\hat{\chi}_\alpha \hat{\chi}_{-\alpha} \, (-1)^F\right) \\ &= \int\limits_{\Pi TLG} \chi_1 \cdots \chi_r \prod_{\alpha \in R_+} i\chi_\alpha \chi_{-\alpha} \, e^{-\frac{1}{2}\int_0^\beta (\psi, \dot{\psi}) d\tau} \, \mathscr{D}\psi, \end{aligned} \tag{6}$$

which follows from (2).

Indeed, using the Cartan-Weyl basis we have,

$$\psi(\tau) = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{r} \psi_{j,n} \cdot iH_j e^{i\omega_n \tau} + \sum_{n=-\infty}^{\infty} \sum_{\alpha \in R} \psi_{\alpha,n} E_\alpha e^{i\omega_n \tau}, \tag{7}$$

where $\overline{\psi}_{j,n} = \psi_{j,-n}$, $\overline{\psi}_{\alpha,n} = -\psi_{-\alpha,-n}$, so

$$-\frac{1}{2}\int_0^\beta (\psi, \dot{\psi}) d\tau = \sum_{n=1}^{\infty} \sum_{j=1}^{r} i\beta\omega_n \psi_{j,n} \overline{\psi}_{j,n} + \sum_{n=-\infty}^{\infty} \sum_{\alpha \in R_+} i\beta\omega_n \psi_{\alpha,n} \overline{\psi}_{\alpha,n}$$

Now if the fermion measure is chosen to be

$$\mathscr{D}\psi = (-1)^{r(r-1)/2} \left(\prod_{j=1}^{r} d\psi_{j,0}\right) \left(\prod_{j=1}^{r} \prod_{n=1}^{\infty} i d\psi_{j,n} d\overline{\psi}_{j,n}\right) \left(\prod_{\alpha \in R_+} \prod_{n \in \mathbb{Z}} i d\psi_{\alpha,n} d\overline{\psi}_{\alpha,n}\right), \tag{8}$$

then it follows from (7) then the only terms in $\chi_1 \dots \chi_r \prod_{\alpha \in R_+} i\chi_\alpha \chi_{-\alpha}$ that give non-zero contribution to the path integral come from $\psi_{j,0}$ and $\psi_{\alpha,0}$ so using the rules of the fermion integration and the formula

$$\prod_{n=1}^{\infty} (2\pi n) = 1, \quad i \prod_{n=1}^{\infty} (-2\pi n) = 1,$$

where the 2nd identity comes from

$$\prod_{n=1}^{\infty} z = e^{\zeta(0)\log z} = 1/\sqrt{z}, \quad -\pi < \arg z \le \pi, \tag{9}$$

we get (6).

Now let us return to the full path integral (4). Because of the additional term $(\mathrm{Ad}_{g^{-1}}h, J)$ in the action, it is easy to verify that S_E^h is invariant under the modified supersymmetry transformation δ_h, which in the Euclidean time for fixed $h \in \mathfrak{t}$ has the form

$$\begin{aligned}\delta_h g &= g\psi,\\ \delta_h \psi &= -J^h - \psi\psi,\\ \delta_h J^h &= (\partial_\tau + \mathrm{ad}_{J^h})\psi,\end{aligned}$$

where $J^h = J + \dfrac{1}{\beta}\mathrm{Ad}_{g^{-1}}h$.

4. As a consquence, eq. 4.33 and 4.34 also has to be modified as follows

$$\begin{aligned}&\int_{\Pi TLG} \chi_1\cdots\chi_r \prod_{\alpha\in R_+} i\chi_\alpha\chi_{-\alpha} e^{-S_E^h}\mathscr{D}g\mathscr{D}\psi\\ &= \int_{\Pi TLG} \chi_1\cdots\chi_r \prod_{\alpha\in R_+} i\chi_\alpha\chi_{-\alpha} e^{-S_E^h - s\delta_h V}\mathscr{D}g\mathscr{D}\psi.\end{aligned} \tag{10}$$

$$\begin{aligned}&\int_{\Pi TLG} \chi_1\cdots\chi_r \prod_{\alpha\in R_+} i\chi_\alpha\chi_{-\alpha} e^{-S_E^h - s\delta_h V}\mathscr{D}g\mathscr{D}\psi\\ &= V_G\Big(\prod_{\alpha\in R_+} i\Big)\int_{\Pi T\Omega G} e^{-S_E^h - s\delta_h V}\mathscr{D}'g\mathscr{D}'\psi,\end{aligned} \tag{11}$$

5. Finally, the computation of the Pfaffians in page 34 should be changed as

$$\begin{aligned}&\frac{\mathrm{Pf}(-\partial_\tau^3 - \mathrm{ad}_{(h+\gamma)/\beta}\partial_\tau^2)}{\det\Big(-\partial_\tau^2 - \mathrm{ad}_{(h+\gamma)/\beta}\partial_\tau\Big)}\\ &= \Big(\prod_{n=1}^{\infty}\omega_n^3\Big)^r \prod_{\alpha\in R_+}\prod_{n\neq 0}\Big(\omega_n^3 + i\frac{\langle h+\gamma,\alpha\rangle}{\beta}\omega_n^2\Big) \Big/ \Big(\prod_{n=1}^{\infty}\omega_n^4\Big)^r \prod_{\alpha\in R_+}\prod_{n\neq 0}\Big(\omega_n^2 + i\frac{\langle h+\gamma,\alpha\rangle}{\beta}\omega_n\Big)^2\\ &= \Big(\prod_{n=1}^{\infty}\omega_n\Big)^{-r}\prod_{\alpha\in R_+}\prod_{n\neq 0}\Big(\omega_n + i\frac{\langle h+\gamma,\alpha\rangle}{\beta}\Big)^{-1}\\ &= \beta^{-\frac{n}{2}}\prod_{\alpha\in R_+}\frac{\frac{1}{2}\langle\alpha, h+\gamma\rangle}{i\sinh\frac{1}{2}\langle\alpha, h+\gamma\rangle},\end{aligned}$$

where factors of i's comes from (9).

Finally, together with the additional factor of i's in r.h.s. of (11) from the zero modes integrals, the final result for the Eskin trace formula is unchanged.

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