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Berezin Quantization, Conformal Welding and the Bott–Virasoro Group

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Dedicated to the memory of our old friend Krzysztof Gawedzki.

Abstract. Following Nag–Sullivan, we study the representation of the group $\mathrm{Diff}^+(S^1)$ of diffeomorphisms of the circle on the Hilbert space of holomorphic functions. Conformal welding provides triangular decompositions for the corresponding symplectic transformations. We apply Berezin formalism and lift this decomposition to operators acting on the Fock space. This lift provides quantization of conformal welding, gives a new representative of the Bott–Virasoso cocycle class, and leads to a surprising identity for the Takhtajan–Teo energy functional on $\mathrm{Diff}^+(S^1)$.

1. Introduction

Coadjoint orbits of the canonical central extension

$$1 \to S^1 \to \widehat{\mathrm{Diff}^+(S^1)} \to \mathrm{Diff}^+(S^1) \to 1$$

of the group $\mathcal{G} = \mathrm{Diff}^+(S^1)$ of orientation-preserving diffeomorphisms of the circle (also called Virasoro coadjoint orbits) attracted attention in both the mathematics and physics literature since long time, see, e.g., [2,6,8,13,17]. The coadjoint action on the hyperplane corresponding to the coordinate c (dual to $\mathrm{Lie}(S^1) \cong \mathbb{R}$) is defined on the space of quadratic differentials on the circle $T(x)dx^2$, and it is given by formula

$$\chi : T(x)dx^2 \mapsto T^\chi(x)dx^2 = \left(T(\chi(x))\chi'(x)^2 + \frac{c}{12}\,\mathrm{Sch}(\chi) \right) dx^2,$$

where $\mathrm{Sch}(\chi)$ is the Schwarzian derivative

$$\mathrm{Sch}(\chi) = \frac{\chi'''(x)}{\chi'(x)} - \frac{3}{2}\left(\frac{\chi''(x)}{\chi'(x)} \right)^2.$$

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For $c \neq 0$, one of the Virasoro coadjoint orbits is of special importance. It corresponds to $T(x) = \frac{c}{24} \, \mathrm{d}x^2$, and it is the unique orbit with the stabilizer isomorphic to the group $\mathrm{PSL}(2, \mathbb{R})$. This orbit (also called the Teichmüller orbit) naturally embeds in the universal Teichmüller space $T(1)$[1]:

$$O_{\mathrm{Teich}} \cong \mathrm{Diff}^+(S^1)/\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{QS}(S^1)/\mathrm{PSL}(2, \mathbb{R}),$$

where $\mathrm{QS}(S^1)$ is the group of quasi-symmetric mappings of the circle.

Consider the space $\mathrm{Hyp}(\mathbb{D})$ of geodesically complete hyperbolic metrics on the unit disk $\mathbb{D} \subset \mathbb{C}$. A typical example in this class is the standard Poincaré metric. The group of orientation-preserving diffeomorphisms $\mathrm{Diff}^+(\mathbb{D})$ acts transitively on $\mathrm{Hyp}(\mathbb{D})$. Now consider the group $\mathrm{Diff}^+(\mathbb{D}, \partial \mathbb{D}) \subset \mathrm{Diff}^+(\mathbb{D})$ which fixes the boundary of the disk $\partial \mathbb{D} \cong S^1$. It was argued in the physics literature (see [12]) that O_{Teich} is symplectomorphic to the following moduli space:

$$O_{\mathrm{Teich}} \cong \mathrm{Hyp}(\mathbb{D})/\mathrm{Diff}^+(\mathbb{D}, \partial \mathbb{D}).$$

Formal Duistermaat–Heckman integrals over this space were defined and studied in [3,14].

Recall that for $\chi \in \mathrm{QS}(S^1)$ there exist two univalent holomorphic functions $f_+ : \mathbb{D} \to \mathbb{C}, f_- : \mathbb{D}^* \to \mathbb{C}$ such that

$$f_+(e^{i\chi(x)}) = f_-(e^{ix})$$

for $x \in \mathbb{R}$. Here, $\mathbb{D}^*$ is the unit disk centered at infinity, and we identify $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. The functions $f_+(z)$ and $f_-(z)$ are called components of conformal welding of χ. The Kähler potential of the Weil–Petersson metric on O_{Teich} is given by the Takhtajan–Teo (TT) energy functional (see [15][2]):

$$S(\chi) = \int_{\mathbb{D}} \left| \frac{f_+''(z)}{f_+'(z)} \right|^2 \mathrm{d}^2 z + 4\pi \log(|f_+'(0)|) + \int_{\bar{\mathbb{D}}} \left| \frac{f_-''(z)}{f_-'(z)} \right|^2 \mathrm{d}^2 z - 4\pi \log(|f_-'(\infty)|).$$

In this paper, we focus our attention on the subgroup $\mathrm{Diff}^+_{\mathrm{hol}}(S^1) \subset \mathrm{Diff}^+(S^1)$ which is characterized by the property that the map $z = e^{ix} \mapsto e^{i\chi(x)}$ extends to a holomorphic function on an annulus $\mathcal{A}_{r,R} = \{z \in \mathbb{C}; r < |z| < R\}$ for some $r < 1 < R$. For this subgroup, following Nag–Sullivan [9] we define a group homomorphism to the group of restricted symplectic transformations acting on the Hilbert space $H = H_+ \oplus H_-$ of holomorphic functions (modulo constants):

$$\mathrm{Diff}^+_{\mathrm{hol}}(S^1) \to \mathrm{Sp}^{\mathrm{res}}(H_+ \oplus H_-).$$

Here, H_+ is spanned by z^n for $n \geq 1$, and H_- by z^n for $n \leq -1$, and $||z^n||^2 = |n|$. We then use the metaplectic representation of $\mathrm{Sp}^{\mathrm{res}}(H_+ \oplus H_-)$ defined by Berezin formalism of normal symbols (see [5]) to construct operators

$$N_\chi = N_{f_+^{-1}} * N_{f_-}. \tag{1}$$

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[1]Teichmüller spaces for curves of all finite genera naturally embed in $T(1)$.

[2]In fact, in [15] a new Hilbert manifold structure on $T(1)$ was introduced, and it was shown that $S(\chi)$ is a Kähler potential for the Weil–Petersson metric on the Hilbert submanifold $T_0(1) \subset T(1)$.

Here, $*$ is the product of operators acting on the Fock space $\mathcal{F}$ defined by the polarization $H = H_+ \oplus H_-$. In this sense, Eq. (1) defines a quantization of conformal welding. Intriguingly, the relation between Berezin quantization and the action of $\mathrm{Diff}^+(S^1)$ on the space of univalent holomorphic functions was pioneered in [1] in the framework of probability theory.

   Our first main result is as follows:

Theorem 1.1. *For* $\chi, \phi \in \mathrm{Diff}^+_{\mathrm{hol}}(S^1)$, *we have*

$$N_\chi N_\phi = C(\chi, \phi) N_{\chi \circ \phi},$$

where $C(\chi, \phi) \in \mathbb{C}^*$ *is a multiplicative group 2-cocycle with the property that*

$$C(\chi, \phi) = C(f_-, g_+), \qquad (2)$$

and $f_\pm$ *define a conformal welding of* χ, *and* $g_\pm$ *define a conformal welding of* ϕ. *Furthermore, for* $\chi \in \mathrm{Diff}^+_{\mathrm{hol}}(S^1)$

$$\log(|C(f_-^{-1}, f_+)|) = -\frac{S(\chi)}{24\pi}.$$

   In Eq. (2), the components of conformal welding $f_\pm, g_\pm$ of diffeomorphisms χ, ϕ are related to each other. This relation is relaxed in Theorem 5.4 which addresses the question of more general triangular decompositions of holomorphic maps. From this perspective, it is surprising that the cocycle C_N depends only on the components f_- and g_+ (and not on f_+ and g_-).

   Our second main result is the following theorem (see also Theorem 5.16):

Theorem 1.2. *For* $\chi \in \mathrm{Diff}^+_{\mathrm{hol}}(S^1)$, *the operator*

$$U_\chi = e^{-S(\chi)/48\pi} N_\chi$$

is unitary, and the cocycle $C(\chi, \phi)$ *satisfies the equality*

$$\log(|C(\chi, \phi)|) = \frac{S(\chi) + S(\phi) - S(\chi \circ \phi)}{48\pi}. \qquad (3)$$

   The left-hand side and the right-hand side of (3) have rather different analytic forms. Equation (3) follows from Berezin formalism, but at this point we are not aware of its direct proof.

   We believe that our findings admit extensions to other representations of the group of diffeomorphisms of the circle defined in terms of free fields. In particular, this applies to representations of affine Kac–Moody algebras on Wakimoto modules. We also believe that our results may find applications in Theoretical Physics. Our original motivation comes from the work [10] which introduced conformal welding in the study of Fermions in a gravitational field in 2 dimensions (see also the analysis of the gravitational Wess–Zumino functionals in [4]).

   The structure of the paper is as follows: in Sect. 2, we recall the definition of the Bott–Virasoro 2-cocycle on $\mathrm{Diff}^+(S^1)$, and we extend it to the groupoid of conformal maps. In Sect. 3, we explain how holomorphic maps define symplectic transformations on the space H, and we define their Grunsky coefficients. In Sect. 4, we set up the Berezin formalism for normal and unitary

symbols of operators on the Fock space $\mathcal{F}$. Finally, in Sect. 5 we describe quantization of conformal welding, the cocycles for normal and unitary symbols and their relation to the TT functional.

2. Group Cocycles

In this section, we recall the notion of a group 2-cocycle. We then focus our attention on the Bott–Virasoro cocycle on the group of orientation-preserving diffeomorphisms of the circle $\text{Diff}^+(S^1)$ and on its extension to holomorphic maps.

2.1. Group 2-Cocycles: Definition and Basic Properties

Let G be a group and $\mathbb{K}$ be the basic field (in this article, $\mathbb{R}$ or $\mathbb{C}$) viewed as a trivial G-module. A map $c : G \times G \to \mathbb{K}$ is an additive group 2-cocycle if

$$c(f,g) + c(fg,h) = c(f,gh) + c(g,h) \tag{4}$$

for all $f, g, h \in G$. The definition implies $c(e,g) = c(e,e) = c(g,e)$ for all $g \in G$. Also, for all $k \in \mathbb{K}$ the assignment $c(f,g) = k$ is a 2-cocycle.

For every map $b : G \to \mathbb{K}$ one defines a trivial 2-cocycle

$$\delta b(f,g) = b(f) - b(fg) + b(g).$$

Note that for $b(f) = k \in \mathbb{K}$ we obtain $c(f,g) = k$. Hence, for any 2-cocycle c there is a cohomologous normalized cocycle

$$\tilde{c}(f,g) = c(f,g) - c(e,e)$$

which has the property $\tilde{c}(e,e) = 0$. We will use the following cyclic property of 2-cocycles:

Proposition 2.1. *Assume that a 2-cocycle c has the property $c(f, f^{-1}) = 0$ for all $f \in G$. Then,*

$$c(f,g) = c(g,h) = c(h,f), \tag{5}$$

where $fgh = e$.

Proof. In Eq. (4), put $fgh = e$ to obtain

$$c(f,g) + c(h^{-1},h) = c(f,f^{-1}) + c(g,h).$$

By assumption, $c(h^{-1},h) = c(f,f^{-1}) = 0$. Hence, we get $c(f,g) = c(g,h)$. The last equality follows since $fgh = e$ implies $hfg = e$. $\qquad\square$

Assume that the group G possesses a Lie algebra $\mathfrak{g} = \text{Lie}(G)$, denote by $\exp : \mathfrak{g} \to G$ the exponential map, and assume that finite products of the type $\exp(u_1) \dots \exp(u_m)$ cover G. We define a map $\beta : \mathfrak{g} \times G \to \mathbb{K}$ by formula

$$\beta(u,g) = \frac{\mathrm{d}}{\mathrm{d}t} c(\exp(tu),g)|_{t=0}. \tag{6}$$

The condition $c(g,e) = c(e,e)$ implies that $\beta(u,e) = 0$ for all $u \in \mathfrak{g}$.

Proposition 2.2. *The map $\beta : \mathfrak{g} \times G \to \mathbb{K}$ uniquely determines a normalized group 2-cocycle c.*

Proof. Put $f = \exp(tu)$ in Eq. (4) and differentiate in t at $t = 0$. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} c(\exp(tu)g, h) = \beta(u, gh) - \beta(u, g).$$

Hence, if $\beta(u, g) = 0$ then the normalized cocycle $c(f, g)$ vanishes, as required. $\square$

Define

$$\alpha(u, v) = \frac{\mathrm{d}}{\mathrm{d}s} \beta(u, \exp(sv))|_{s=0} = \frac{\partial^2}{\partial s \partial t} c(\exp(tu), \exp(sv))|_{s=t=0}$$

and

$$a(u, v) = \frac{1}{2} \left(\alpha(u, v) - \alpha(v, u) \right).$$

Recall that $a \in \wedge^2 \mathfrak{g}^*$ is a Lie algebra 2-cocycle, and that it satisfies the equation

$$a(u, [v, w]) + a(w, [u, v]) + a(v, [w, u]) = 0. \tag{7}$$

Note that there is no analog of Proposition 2.2 which would allow to reconstruct maps β and c starting from the map a (or the map α). Indeed, adding a trivial cocycle δb such that $\mathrm{d}/\mathrm{d}t\, b(\exp(tu))|_{t=0} = 0$ does not affect the maps α and a, but it changes β and c, in general.

Let $\rho : G \to \mathrm{End}(V)$ be a projective representation, and assume that

$$\rho(f)\rho(g) = e^{ic(f,g)}\rho(fg),$$

where $c : G \times G \to \mathbb{C}$ is a complex-valued function. Then, c verifies the identity (4) modulo $2\pi\mathbb{Z}$. This is a direct consequence of associativity of the product in $\mathrm{End}(V)$. If G is a connected topological group, and the function c is a continuous function, then it is actually a 2-cocycle. Indeed, in this case the defect in Eq. (4) is also a continuous function of f, g, h which vanishes for $f = g = h = e$. Hence, it vanishes for all $f, g, h \in G$. Furthermore, assume that V is a Hilbert space and that $\rho : G \to U(V)$ is a unitary representation. Then, $c(f, g) \in \mathbb{R}$ is a real-valued 2-cocycle.

Every 2-cocycle $c : G \times G \to \mathbb{C}$ defines a group law on $\hat{G} = G \times \mathbb{C}^*$ defined by formula

$$(f, z) \cdot (g, w) = (fg, zw \exp(ic(f, g))).$$

This group fits into a short exact sequence

$$1 \to \mathbb{C}^* \to \hat{G} \to G \to 1$$

and defines a central extension of G. If the cocycle c is real valued, this central extension is by the circle S^1 (instead of $\mathbb{C}^*$).

2.2. The Bott–Virasoro Cocycle

Consider the group $G = \mathrm{Diff}^+(S^1)$ of orientation-preserving diffeomorphisms of the circle. We recall the following basic fact:

Theorem 2.3. (Bott–Virasoro cocycle) *The map $c_{\mathrm{BV}} : G \times G \to \mathbb{R}$ defined by formula*

$$c_{\mathrm{BV}}(\chi, \phi) = \int_0^{2\pi} \log(\chi'(\phi(x))) \log(\phi'(x))' \, dx$$

is a normalized real-valued group 2-cocycle. Furthermore, it satisfies the cyclic property (5).

Proof. For convenience of the reader, we give a proof of this statement. The left-hand side of Eq. (4) is as follows:

$$
\begin{aligned}
c_{\mathrm{BV}}(\chi, \phi) + c_{\mathrm{BV}}(\chi \circ \phi, \psi) &= \int_0^{2\pi} \big(\log(\chi'(\phi(x))(\log(\phi'(x)))' \\
&\quad + \log((\chi \circ \phi)'(\psi(x)))(\log(\psi'(x)))' \big) \, dx \\
&= \int_0^{2\pi} \big(\log(\chi'(\phi(\psi(x)))) \log(\psi'(x))' \\
&\quad + \log(\phi'(\psi(x))) \log(\psi'(x))' \big) \, dx \\
&\quad + \int_0^{2\pi} \log(\chi'(\phi(x))(\log(\phi'(x)))' \, dx,
\end{aligned}
$$

and the right-hand side has the following form:

$$
\begin{aligned}
c_{\mathrm{BV}}(\chi, \phi \circ \psi) + c_{\mathrm{BV}}(\phi, \psi) &= \int_0^{2\pi} \big(\log(\chi'(\phi(\psi(x)))) \log((\phi \circ \psi)'(x))' \\
&\quad + \log(\phi'(\psi(x))) \log(\psi'(x))' \big) \, dx \\
&= \int_0^{2\pi} \big(\log(\chi'(\phi(\psi(x)))) \log(\psi'(x))' \\
&\quad + \log(\phi'(\psi(x))) \log(\psi'(x))' \big) \, dx \\
&\quad + \int_0^{2\pi} \log(\chi'(\phi(\psi(x))) \log(\phi'(\psi(x)))' \, dx.
\end{aligned}
$$

Note that the first and second lines in the two final expressions coincide term by term, and the third line of the right-hand side of (4) is obtained from the third line of the left-hand side by the change of variable $x \mapsto \psi(x)$.

For the cyclic property, put $\chi = \phi^{-1}$. Then, $\log(\chi'(\phi(x))) = -\log(\phi'(x))$ and

$$c_{\mathrm{BV}}(\phi^{-1}, \phi) = - \int_0^{2\pi} \log(\phi'(x)) \log(\phi'(x))' \, dx = -\frac{1}{2} \log(\phi'(x))^2 |_0^{2\pi} = 0.$$

Here, we have used the fact that $\phi'(x)$ is periodic. The cyclic property follows by Proposition 2.1. $\qquad\square$

It is instructive to compute the map β_{BV}:

$$\beta_{\mathrm{BV}}(u, \phi) = \int_0^{2\pi} u'(\phi(x)) \log(\phi'(x))' \, dx = - \int_0^{2\pi} u'(y) \log((\phi^{-1})'(y))' \, dy.$$

Here, we made a change of variables $y = \phi(x)$ in the integral. The map α_{BV} is given by the Gelfand–Fuchs cocycle:

$$\alpha_{\mathrm{BV}}(u, v) = \int_0^{2\pi} u'(x)v''(x)\mathrm{d}x,$$

which is skew-symmetric under exchange of u and v.

2.3. Extension to Holomorphic Maps

In this section, we extend the Bott–Virasoro cocycle to a certain class of holomorphic maps. In more detail, let $\mathcal{S}$ be a connected open subset of the complex plane with $\pi_1(\mathcal{S}) \cong \mathbb{Z}$. By the uniformization theorem, such a domain is holomorphically isomorphic to an annulus

$$\mathcal{A}_{r,R} = \{z \in \mathbb{C}; r < |z| < R\}.$$

We will consider triples $(\mathcal{S}, f, \mathcal{T})$, where $\mathcal{S}$ and $\mathcal{T}$ are two such domains, and $f : \mathcal{S} \to \mathcal{T}$ is a holomorphic isomorphism between $\mathcal{S}$ and $\mathcal{T}$. Sometimes it is convenient to label the domain and the range of f by $\mathcal{S}_f$ and $\mathcal{T}_f$, respectively. Note that the domain $\mathcal{S}_f$ contains a closed curve C_f which represents the generator of $\pi_1(\mathcal{S}_f)$. Its image $f(C_f)$ represents the generator of $\pi_1(\mathcal{T}_f) \cong \mathbb{Z}$. By the analytic continuation principle, the holomorphic function f is uniquely determined by its restriction to C_f. Furthermore, we will be assuming that $\log(f(z)/z)$ and $\log(f'(z))$ are univalued functions on C_f. The first condition follows from the fact that

$$\int_{C_f} \log(f(z)/z)'\mathrm{d}z = \int_{C_f} \left(\frac{f'(z)}{f(z)} - \frac{1}{z} \right) \mathrm{d}z = \int_{f(C_f)} \frac{\mathrm{d}w}{w} - \int_{C_f} \frac{\mathrm{d}z}{z} = 0.$$

Triples $(\mathcal{S}_f, f, \mathcal{T}_f)$ form a groupoid with composition law

$$(\mathcal{S}_f, f, \mathcal{T}_f) \circ (\mathcal{S}_g, g, \mathcal{T}_g) = (\mathcal{S}_g, f \circ g, \mathcal{T}_f).$$

Two triples are composable if $\mathcal{S}_f = \mathcal{T}_g$. The curves $g(C_g)$ and C_f are homotopic to each other in $\mathcal{S}_f = \mathcal{T}_g$. Note that conditions on the logarithms $\log(f(z)/z)$ and $\log(f'(z))$ are reflexive and transitive. Indeed, for the inverse function we have

$$\log(f^{-1}(w)/w) = -\log(f(z)/z), \quad \log((f^{-1})'(w)) = -\log(f'(z)),$$

where $w = f(z)$. Since the right-hand sides are univalued functions on C_f, so are the left-hand sides on $f(C_f) = C_{f^{-1}}$. Similarly, for the composition of functions we obtain

$$\log(f(g(z))/z) = \log(f(g(z))/g(z)) + \log(g(z)/z),$$
$$\log((f \circ g)'(z)) = \log(f'(g(z))) + \log(g'(z)).$$

Again, the right-hand sides are univalued functions on C_g. Hence, the left-hand sides are also univalued on C_g, as required.

We define a subset $\mathrm{Diff}^+_{\mathrm{hol}}(S^1) \subset \mathrm{Diff}^+(S^1)$ by the following property: $\chi \in \mathrm{Diff}^+_{\mathrm{hol}}(S^1)$ if the map

$$z = e^{ix} \mapsto e^{i\chi(x)}$$

extends to a univalent holomorphic map f on an annulus $\mathcal{A}_{r,R}$ with $r < 1 < R$. One can always choose the curve C_f to be the unit circle C_1. It is easy to see that $\mathrm{Diff}^+_{\mathrm{hol}}(S^1)$ is a subgroup of $\mathrm{Diff}^+(S^1)$. Furthermore, for $z = e^{ix}$ we have

$$\log(f(z)/z) = i(\chi(x) - x), \quad \log(f'(z)) = \chi'(x) + i(\chi(x) - x),$$

and the right-hand sides are univalued functions on C_1.

Consider two diffeomorphisms $\chi, \phi \in \mathrm{Diff}^+_{\mathrm{hol}}(S^1)$ and the corresponding univalent holomorphic functions f and g such that

$$f(e^{ix}) = e^{i\chi(x)}, \quad g(e^{ix}) = e^{i\phi(x)}.$$

By making the annulus $\mathcal{S}_g$ smaller if needed, one can always achieve $\mathcal{T}_g \subset \mathcal{S}_f$. By restricting f to $\mathcal{T}_g$, one obtains a pair of composable holomorphic maps, and

$$f(g(e^{ix})) = f(e^{i\phi(x)}) = e^{i\chi(\phi(x))}.$$

Since the analytic function $f \circ g$ is uniquely determined by its values on the unit circle, we conclude that it corresponds to the diffeomorphism $\chi \circ \phi$.

Proposition 2.4. *Let $\chi, \phi \in \mathrm{Diff}^+_{\mathrm{hol}}(S^1)$ and f, g the corresponding composable holomorphic functions. Then,*

$$c_{\mathrm{BV}}(\chi, \phi) = \int_{C_1} \log\left(\frac{g(z)f'(g(z))}{f(g(z))}\right) \log\left(\frac{zg'(z)}{g(z)}\right)' \, \mathrm{d}z. \tag{8}$$

Proof. The proof is by a direct calculation. In particular, for $z = e^{ix}, g(z) = e^{i\phi(x)}$ we have $zg'(z)/g(z) = \phi'(x)$, and $g(z)f'(g(z))/f(g(z)) = \chi'(\phi(x))$. $\quad\square$

For a pair of composable univalent holomorphic maps f and g, one can use the right-hand side of Eq. (8) as a definition of a functional of a pair (f, g):

$$C_{\mathrm{BV}}(f, g) = \int_{C_g} \log\left(\frac{g(z)f'(g(z))}{f(g(z))}\right) \log\left(\frac{zg'(z)}{g(z)}\right)' \, \mathrm{d}z. \tag{9}$$

Here, the integration is over the curve C_g on which both holomorphic functions g and $f \circ g$ are well defined. Note that the logarithms in Eq. (9) are univalued. Indeed,

$$\log\left(\frac{zf'(z)}{f(z)}\right) = \log(f'(z)) - \log(f(z)/z),$$

and the two logarithms on the right-hand side are univalued by assumptions. The choice of a branch of the logarithms does not affect the value of C_{BV} because of the derivative on the second factor in Eq. (9). In general, C_{BV} is complex valued. This is in contrast to the cocycle c_{BV} which takes values in $\mathbb{R}$.

Proposition 2.5. *The map C_{BV} is a groupoid 2-cocycle. That is, for all composable triples (f, g, h), it satisfies the equation*

$$C_{\mathrm{BV}}(f, g) + C_{\mathrm{BV}}(fg, h) = C_{\mathrm{BV}}(f, gh) + C_{\mathrm{BV}}(g, h).$$

Furthermore, it satisfies the cyclic property (5).

Proof. The proof of the cocycle condition is analogous to the one of Theorem 2.3. The only nontrivial step in the proof is as follows: one needs to check that

$$\int_{C_h} \log \left(\frac{g(h(z)) f'(g(h(z)))}{f(g(h(z)))} \right) \log \left(\frac{h(z) g'(h(z))}{g(h(z))} \right)' \mathrm{d}z$$
$$= \int_{C_g} \log \left(\frac{g(w) f'(g(w))}{f(g(w))} \right) \log \left(\frac{w g'(w)}{g(w)} \right)' \mathrm{d}w.$$

Two integrals are related by the change of variable $w = h(z)$. After this change of variables, the integration contour on the left-hand side is C_h, and on the right-hand side it is $h^{-1}(C_g)$. Both these curves represent the generator of $\pi_1(\mathcal{S}_h)$, and therefore they are homotopic to each other.

For the cyclic property, put $f = g^{-1}$. Then,

$$\log \left(\frac{g(z) f'(g(z))}{f(g(z))} \right) = - \log \left(\frac{z g'(z)}{g(z)} \right)$$

and

$$C_{\mathrm{BV}}(g^{-1}, g) = -\frac{1}{2} \int_{C_g} \frac{\mathrm{d}}{\mathrm{d}z} \left(\log \left(\frac{z g'(z)}{g(z)} \right) \right)^2 \mathrm{d}z = 0.$$

The proof of Proposition 2.1 applies verbatim to the case of groupoids. This completes the proof. □

Note that the expression (9) can be rewritten using the change of variables $z = g^{-1}(w)$. We get

$$C_{\mathrm{BV}}(f, g) = - \int_{C_f} \log \left(\frac{w f'(w)}{f(w)} \right) \log \left(\frac{w (g^{-1})'(w)}{g^{-1}(w)} \right)' \mathrm{d}w.$$

Holomorphic functions $f : \mathcal{S}_f \to \mathcal{T}_f$ and $h : \mathcal{T}_g \to \mathcal{S}_g$ are actually defined on the same domain $\mathcal{S}_f = \mathcal{T}_g$ which contains the curve C_f. We compute the expression β for C_{BV}. By putting $f(w) = w + tu(w) + O(t^2)$, we obtain

$$\beta_{\mathrm{BV}}(u, g) = - \int_{C_1} \left(u'(w) - \frac{u(w)}{w} \right) \log \left(\frac{w (g^{-1})'(w)}{g^{-1}(w)} \right)' \mathrm{d}w.$$

Remark 2.6. Yet another groupoid cocycle which has the cyclic property is given by formula

$$\tilde{C}(f, g) = \int_{C_g} \log(f'(g(z))) \log(g'(z))' \mathrm{d}z. \tag{10}$$

The proof is similar to those of Theorem 2.3 and of Proposition 2.5. By assumptions, the logarithms in Eq. (10) are univalued. Again, the choice of a branch of the logarithms doesn't influence the value of $\tilde{C}(f, g)$ because of the derivative on the second term in Eq. (10).

3. Symplectic Transformations

In this section, we recall the notion of symplectic transformations associated with holomorphic maps.

3.1. Symplectic Transformations in Finite Dimensions

Recall the following standard setup: let U be a complex vector space, and $\omega \in \wedge^2 U$ be a non-degenerate (symplectic) 2-form. A linear map $A \in \mathrm{End}(U)$ is called symplectic if it preserves ω:

$$\omega(A(u_1), A(u_2)) = \omega(u_1, u_2)$$

for all $u_1, u_2 \in U$.

The following set of examples is of special interest for us: let V be a finite-dimensional complex vector space. Then, one can equip the direct sum $U = V \oplus V^*$ (here V^* is the dual space of V) with the symplectic form

$$\omega(a + a^*, b + b^*) = \langle b^*, a \rangle - \langle a^*, b \rangle,$$

where $a, b \in V, a^*, b^* \in V^*$ and $\langle \cdot, \cdot \rangle$ is the canonical pairing between V^* and V. A splitting $U = V \oplus V^*$ is also called a polarization of the symplectic space U.

Consider a transformation $A \in \mathrm{End}(V \oplus V^*)$ defined by formula

$$\begin{pmatrix} a \\ a^* \end{pmatrix} \mapsto \begin{pmatrix} \tilde{a} \\ \tilde{a}^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} a \\ a^* \end{pmatrix}. \tag{11}$$

Here $\alpha : V \to V, \beta : V^* \to V, \gamma : V \to V^*, \delta : V^* \to V^*$. This transformation is symplectic, $A \in \mathrm{Sp}(V \oplus V^*)$, if and only if the following conditions are verified:

$$\beta\alpha^t = (\beta\alpha^t)^t, \ \alpha^t\gamma = (\alpha^t\gamma)^t, \ \alpha\delta^t - \beta\gamma^t = 1,$$
$$\delta^t\beta = (\delta^t\beta)^t, \ \gamma\delta^t = (\gamma\delta^t)^t, \ \alpha^t\delta - \gamma^t\beta = 1.$$

Note that if α and δ are invertible, the following operators are symmetric: $(\alpha^{-1}\beta), (\beta\delta^{-1}), (\gamma\alpha^{-1}), (\delta^{-1}\gamma)$. In this case, one can also express α and δ in terms of three other operators:

$$\delta = (\alpha^t)^{-1} + \gamma\alpha^{-1}\beta, \quad \alpha = (\delta^t)^{-1} + \beta\delta^{-1}\gamma.$$

If V is equipped with a Hermitian scalar product, one can identify $V^* \cong V$ and define the unitary subgroup $\mathrm{USp}(V \oplus V^*) \subset \mathrm{Sp}(V \oplus V^*)$ which has the following property: $\tilde{a}_i^*$ is the Hermitian conjugate of $\tilde{a}_i$ for all i. This condition imposes an extra requirement on the components $\alpha, \beta, \gamma, \delta$ of A:

$$\gamma = \overline{\beta}, \quad \delta = \overline{\alpha}.$$

Here $\overline{\alpha}, \overline{\beta}$ are complex conjugate of α and β, respectively. In particular, we obtain

$$\alpha\alpha^* = \alpha\delta^t = 1 + \beta\gamma^t = 1 + \beta\beta^*.$$

This implies that α is invertible, and that so is $\delta = \overline{\alpha}$. Furthermore, we have the following useful identity:

$$\alpha^{-1}(\alpha^*)^{-1} = \alpha^{-1}(\alpha\alpha^* - \beta\beta^*)(\alpha^*)^{-1} = 1 - (\alpha^{-1}\beta)(\alpha^{-1}\beta)^*. \qquad (12)$$

3.2. Symplectic Transformations and Holomorphic Maps

We now pass to the infinite-dimensional context and apply the theory of symplectic transformations to holomorphic functions. As in the previous section, let $\mathcal{S} \subset \mathbb{C}$ be a connected domain with $\pi_1(\mathcal{S}) = \mathbb{Z}$, and let $C \subset \mathcal{S}$ be a closed oriented curve which realizes the generator of $\pi_1(\mathcal{S})$. We consider the space $\mathcal{H}_\mathcal{S}$ of holomorphic functions on $\mathcal{S}$ and define the following 2-form

$$\omega_\mathcal{S} = \frac{1}{4\pi} \int_C (\delta\phi(z)) \partial_z (\delta\phi(z)) \, \mathrm{d}z.$$

Here δ is the de Rham differential on $\mathcal{H}_\mathcal{S}$ and ∂_z is the z-derivative. It is clear that the definition of $\omega_\mathcal{S}$ is independent of the choice of the curve C.

Let $f : \mathcal{S} \to \mathcal{T}$ be a holomorphic isomorphism. It induces an isomorphism $f^* : \mathcal{H}_\mathcal{T} \to \mathcal{H}_\mathcal{S}$ by composition: $\phi \mapsto f^*\phi(z) = \phi(f(z))$. In turn, the map f^* induces a pullback map of differential forms that we denote by $(f^*)^*$.

Proposition 3.1. $(f^*)^*\omega_\mathcal{S} = \omega_\mathcal{T}$.

Proof. We compute,

$$\begin{aligned} 4\pi(f^*)^*\omega_\mathcal{S} &= \int_{C_\mathcal{S}} (\delta\phi(f(z)))\partial_z(\delta\phi(f(z))) \, \mathrm{d}z \\ &= \int_{f(C_\mathcal{S})} (\delta\phi(w))\partial_w(\delta\phi(w)) \, \mathrm{d}w \\ &= \int_{C_\mathcal{T}} (\delta\phi(w))\partial_w(\delta\phi(w)) \, \mathrm{d}w \\ &= 4\pi\omega_\mathcal{T}. \end{aligned}$$

Here, we made a change of variables $z = f^{-1}(w)$, and then used the fact that $f(C_\mathcal{S})$ is homotopic of $C_\mathcal{T}$ in $\mathcal{T}$. □

For an annulus

$$\mathcal{A}_{r,R} = \{z \in \mathbb{C}; r < |z| < R\}$$

with $r < 1 < R$, one can choose $C_\mathcal{S}$ to be the unit circle. We will consider the space $H = \mathcal{H}_\mathcal{A}/\mathbb{C}$ of holomorphic functions modulo constants. Using the Fourier transform,

$$\phi(z) = \sum_{n=1}^\infty \frac{a_n}{\sqrt{n}} z^n + \sum_{n=1}^\infty \frac{a_n^*}{\sqrt{n}} z^{-n} + a_0,$$

we obtain a formula for $\omega_\mathcal{A}$:

$$\omega_\mathcal{A} = \frac{i}{2} \sum_{n=1}^\infty \delta a_n \wedge \delta a_n^*.$$

This form is symplectic on H. In what follows, it will be more convenient to work with functions

$$\psi(z) = \phi'(z) = \sum_{n=1}^\infty \sqrt{n} \, a_n z^{n-1} - \sum_{n=1}^\infty \sqrt{n} \, a_n^* z^{-n-1}$$

which do not contain the superfluous constant a_0. One can view Fourier components $\{a_n, a_n^*\}$ as coordinates on the infinite-dimensional symplectic space of holomorphic functions.

The space H admits a polarization

$$H = H_+ \oplus H_-,$$

where H_+ is spanned by monomials z^n with $n \geq 1$ and H_- by monomials z^n with $n \leq -1$. In particular, $\omega_{\mathcal{A}}(\phi_1, \phi_2) = 0$ for $\phi_{1,2} \in H_+$ and for $\phi_{1,2} \in H_-$.

By Proposition 3.1, holomorphic maps induce symplectic transformations

$$f : \psi \mapsto \tilde{\psi}(z) = \psi(f(z)) f'(z).$$

In more detail,

$$\sum_{n=1}^{\infty} \sqrt{n} \left(\tilde{a}_n z^{n-1} - \tilde{a}_n^* z^{-n-1} \right) = f'(z) \sum_{n=1}^{\infty} \sqrt{n} \left(a_n (f(z))^{n-1} - a_n^* (f(z))^{-n-1} \right). \tag{13}$$

This equation implies

$$\alpha_{m,n} = \frac{1}{2\pi i} \sqrt{\frac{n}{m}} \int_C \frac{f(z)^{n-1} f'(z)}{z^m} \, \mathrm{d}z, \tag{14}$$

$$\beta_{m,n} = -\frac{1}{2\pi i} \sqrt{\frac{n}{m}} \int_C \frac{f(z)^{-n-1} f'(z)}{z^m} \, \mathrm{d}z, \tag{15}$$

$$\gamma_{m,n} = -\frac{1}{2\pi i} \sqrt{\frac{n}{m}} \int_C f(z)^{n-1} f'(z) z^m \, \mathrm{d}z, \tag{16}$$

$$\delta_{m,n} = \frac{1}{2\pi i} \sqrt{\frac{n}{m}} \int_C f(z)^{-n-1} f'(z) z^m \, \mathrm{d}z, \tag{17}$$

where $\alpha_{m,n}, \beta_{m,n}, \gamma_{m,n}, \delta_{m,n}$ are infinite-dimensional matrices representing operators $\alpha, \beta, \gamma, \delta$.

The map from holomorphic maps to symplectic transformations is a group anti-homomorphism:

Proposition 3.2. *Let f, g be two composable holomorphic maps and A_f, A_g be the corresponding symplectic transformations. Then,*

$$A_{f \circ g} = A_g A_f.$$

Proof. The proof is by a direct computation. $\qquad\square$

We recall the following simple fact:

Proposition 3.3. *Let $\chi \in \mathrm{Diff}_{\mathrm{hol}}^+(S^1)$ and f be the corresponding holomorphic map. Then, the transformation $A_f \in \mathrm{USp}(H_+ \oplus H_-)$ is unitary.*

Proof. We can choose C to be the unit circle. Then, for $z = e^{ix}$ we have $f(z) = e^{i\chi(x)}$. By making a change of variables from z to x in (14), (15), (16), (17), we obtain $\delta_{m,n} = \bar{\alpha}_{m,n}$ and $\gamma_{m,n} = \bar{\beta}_{m,n}$, as required. $\qquad\square$

Remark 3.4. Let $m(z)$ be a Möbius transformation preserving the unit circle. That is,

$$m(z) = \frac{az + b}{\bar{b}z + \bar{a}},$$

with $|a|^2 - |b|^2 = 1$. Then, the integrand in Eq. (15) is holomorphic on $\mathbb{D}^*$, the integrand in Eq. (16) is holomorphic on $\mathbb{D}$, and $\beta = \gamma = 0$. By Proposition 3.3, A_m is unitary. Hence, the operators α δ are unitary on H_+ and H_-, respectively.

3.3. Grunsky Coefficients and Symplectic Transformations

We will need the following simple properties of holomorphic functions.

Let $f(z)$ be a univalent holomorphic function on neighborhood of zero with $f(0) = 0$. Then, $f(z) = \sum_{n=1}^{\infty} f_n z^n$ with $f_1 \neq 0$. Recall that the function

$$\log\left(\frac{f(z) - f(w)}{z - w}\right) = \sum_{m,n=0}^{\infty} F_{m,n} z^m w^n \tag{18}$$

is regular in z and w. Here $F_{m,n}$ are the Grunsky coefficients of $f(z)$ (see [11] for details).

In a similar fashion, let $f(z)$ be a univalent holomorphic function on a neighborhood of infinity with $f(\infty) = \infty$. Then, $f(z) = \sum_{n=-\infty}^{1} f_n z^n$ with $f_1 \neq 0$, and

$$\log\left(\frac{f(z) - f(w)}{z - w}\right) = \log(f_1) + \sum_{m,n=1}^{\infty} F_{-m,-n} z^{-m} w^{-n}. \tag{19}$$

The following proposition will be important for the rest of the paper:

Proposition 3.5. *Let $f = \sum_{n=1}^{\infty} f_n z^n$ with $f_1 \neq 0$ be a univalent holomorphic map on a neighborhood of zero. Then, the corresponding symplectic transformation is upper-triangular, the operator γ vanishes, and the symmetric operator $(\alpha^{-1}\beta)$ is of the following form:*

$$\sum_{m,n=1}^{\infty} \sqrt{mn}\, (\alpha^{-1}\beta)_{m,n} u^{m-1} w^{-1} n = \frac{(f^{-1})'(u)(f^{-1})'(w)}{(f^{-1}(z) - f^{-1}(w))^2} - \frac{1}{(u - w)^2}. \tag{20}$$

Let $f = \sum_{n=-\infty}^{1} f_n z^n$ with $f_1 \neq 0$ be a univalent holomorphic map on a neighborhood of infinity. Then, the corresponding symplectic transformation is lower triangular, the operator β vanishes, and the symmetric operator $(\gamma\alpha^{-1})$ is of the following form:

$$-\sum_{m,n=1}^{\infty} \sqrt{mn}\, (\gamma\alpha^{-1})_{m,n} z^{-m-1} w^{-n-1} = \frac{f'(z)f'(w)}{(f(z) - f(w))^2} - \frac{1}{(z - w)^2} \tag{21}$$

Proof. Let $f = \sum_{n=1}^{\infty} f_n z^n$ with $f_1 \neq 0$. Observe that the expression $f(z)^{n-1} f'(z) z^m$ in Eq. (16) is regular at zero and its integral over C vanishes. Hence, the operator γ vanishes. Furthermore, for $n > m$ the function $f(z)^{n-1} f'(z)/z^m$ in Eq. (14) is also regular at zero which implies $\alpha_{m,n} = 0$. Therefore, α is an

upper-triangular (infinite) matrix. A similar argument shows that δ is also upper-triangular.

Let h be the inverse function of f. Equation (14) implies

$$\alpha_{m,n}^{-1} = \frac{1}{2\pi i} \sqrt{\frac{n}{m}} \int_{C'} \frac{h(w)^{n-1} h'(w)}{w^m} \mathrm{d}w,$$

where C' is some circle around zero in the domain of h. Without loss of generality, choose C and C' in such a way that $h(C')$ is contained in the interior of C. Combining with Eq. (15), we obtain

$$(\alpha^{-1}\beta)_{m,n} = \sum_{k=1}^{\infty} \alpha_{m,k}^{-1} \beta_{k,n}$$

$$= -\frac{1}{(2\pi i)^2} \sqrt{\frac{n}{m}} \int_{C \times C'} \frac{h'(w) f'(z)}{w^m f(z)^{n+1}} \sum_{k=1}^{\infty} \frac{h(w)^{k-1}}{z^k} \mathrm{d}w \mathrm{d}z.$$

Summing up the geometric series and making a substitution $z = h(u)$ yields

$$(\alpha^{-1}\beta)_{m,n} = -\frac{1}{(2\pi i)^2} \sqrt{\frac{n}{m}} \int_{C' \times C'} \frac{h'(w)}{w^m u^{n+1}} \frac{1}{h(u) - h(w)} \mathrm{d}u \mathrm{d}w$$

Finally, integration by parts over u gives rise to

$$(\alpha^{-1}\beta)_{m,n} = \frac{1}{(2\pi i)^2} \frac{1}{\sqrt{mn}} \int_{C' \times C'} \frac{1}{w^m u^n} \frac{h'(u) h'(w)}{(h(u) - h(w))^2} \mathrm{d}u \mathrm{d}w.$$

The function

$$\frac{h'(u) h'(w)}{(h(u) - h(w))^2} - \frac{1}{(u - w)^2} = \frac{\partial^2}{\partial u \partial w} \log \left(\frac{h(u) - h(w)}{u - w} \right)$$

is regular in u, w. Hence, it is given by the Taylor series (20).

Proof of equation (21) is similar. $\square$

4. Metaplectic Representation and Berezin Formalism

In this section, we recall the metaplectic representation of the symplectic group and Berezin formalism in finite and infinite dimensions.

4.1. Heisenberg Lie Algebra and Normal Symbols

To a symplectic vector space $V \oplus V^*$, one can naturally associate a Heisenberg Lie algebra with generators $\hat{a}, \hat{a}^*$ for $a \in V, a^* \in V^*$ defined by canonical commutation relations

$$[\hat{a}, \hat{b}] = [\hat{a}^*, \hat{b}^*] = 0, \quad [\hat{a}, \hat{b}^*] = \omega(b^*, a) = \langle b^*, a \rangle.$$

Choose a Hermitian scalar product $(\cdot, \cdot)$ on V. Then, the symmetric algebra SV^* also carries a Hermitian product, and it can be completed to a Fock space

$$\mathcal{F} = \overline{SV^*} \cdot v_0.$$

Here, v_0 is the cyclic (vacuum) vector which corresponds to $1 \in \overline{SV^*}, 1 \cdot v_0 = v_0$.

The Fock space $\mathcal{F}$ carries a natural action (by unbounded operators) of the Heisenberg algebra, where $\hat{a} \cdot v_0 = 0$ for all $a \in V$, and for all $a \in V, b^* \in V^*$ we have

$$(\hat{a} + \hat{b}^*) \cdot (f v_0) = (\partial_a f + b^* f) \cdot v_0.$$

Here, ∂_a is the first-order differential operator on SV^* uniquely determined by the following properties: $\partial_a(1) = 0, \partial_a(b^*) = \langle b^*, a \rangle$. One can also view the actions of ∂_a as an action of a constant vector field on V on the space of polynomial functions SV^* (or formal power series $\mathcal{F}$) on V. By abuse of notation, we denote the Hermitian product on $\mathcal{F}$ again by $(\cdot, \cdot)$. We normalize the vacuum vector v_0 such that $(v_0, v_0) = 1$.

Introduce an orthonormal basis $\{a_i\}$ of V and the dual basis $\{a_i^*\}$ of V^*. In this basis, ω takes the canonical form

$$\omega(a_i, a_j) = \omega(a_i^*, a_j^*) = 0, \quad \omega(a_i, a_j^*) = \delta_{ij}.$$

Then, operators $\hat{a}_i, \hat{a}_i^*$ on $\mathcal{F}$ are conjugate to each other under the Hermitian structure on $\mathcal{F}$. To multi-indices $I = (i_1, \ldots, i_m), J = (j_i, \ldots, j_n)$, we associate monomials

$$a_I = a_{i_1} \ldots a_{i_m}, \quad a_J^* = a_{j_1}^* \ldots a_{j_n}^*.$$

To a power series in formal variables a_i, a_i^*

$$N_q(a, a^*) = \sum_{I,J} q_{I,J} a_I a_J^* \in \overline{S(V \oplus V^*)}$$

one associates an operator

$$\hat{q} = \sum_{I,J} q_{I,J} \hat{a}_J^* \hat{a}_I.$$

If the sum is finite, this operator is well defined, and $N_q(a, a^*)$ is called its normal symbol. Sometimes, $\hat{q}$ is well defined even for infinite series $N_q(a, a^*)$.

Note that the matrix element $(v_0, \hat{q}\, v_0)$ is given by

$$(v_0, \hat{q}\, v_0) = q_{0,0}. \tag{22}$$

Furthermore, note that the normal symbol of the adjoint operator $\hat{q}^*$ is given by

$$N_{q^*}(a, a^*) = \sum_{I,J} \overline{q_{I,J}} a_J a_I^*.$$

The operator product $\hat{q} \cdot \hat{r}$ is represented by a formal Gaussian integral in terms of normal symbols (see equation (2.31) in Section I.2.7 of [5]):

$$N_q * N_r = N_{q \cdot r}(a, a^*) = \int N_q(a + b, a^*)\, e^{-\langle b^*, b \rangle}\, N_r(a, a^* + b^*)\, \mathrm{d}b \mathrm{d}b^*. \tag{23}$$

Note that this formal integral in defined modulo sign since in general it involves a square root of the determinant (see below for a more detailed discussion).

4.2. Berezin Formalism

In the finite-dimensional context, the group of symplectic transformations $\mathrm{Sp}(V \oplus V^*)$ has a double cover

$$1 \to \mathbb{Z}_2 \to \mathrm{Mp}(V \oplus V^*) \to \mathrm{Sp}(V \oplus V^*) \to 1$$

called the metaplectic group. We will also need the associated central extension of $\mathrm{Sp}(V \oplus V^*)$ by $\mathbb{C}^*$:

$$1 \to \mathbb{C}^* \to \widehat{\mathrm{Sp}}(V \oplus V^*) = \mathrm{Mp}(V \oplus V^*) \times_{\mathbb{Z}_2} \mathbb{C}^* \to \mathrm{Sp}(V \oplus V^*) \to 1.$$

It comes with a natural representation on the Fock space $\mathcal{F}$ which can also be viewed as a projective representation of $\mathrm{Sp}(V \oplus V^*)$. For a given symplectic transformation

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}(V \otimes V^*)$$

one says that an invertible operator $\hat{A}$ on $\mathcal{F}$ implements it if it represents a lift of A in $\widehat{\mathrm{Sp}}(V \oplus V^*)$. In more detail, it means that

$$\hat{A} \begin{pmatrix} \hat{a} \\ \hat{a}^* \end{pmatrix} \hat{A}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^* \end{pmatrix} \tag{24}$$

for all $a \in V, a^* \in V^*$.

We will call a symplectic transformation $A \in \mathrm{Sp}(V \oplus V^*)$ admissible if its components $\alpha : V \to V$ and $\delta : V^* \to V^*$ are invertible. The following theorem summarizes a result of Berezin[3]:

Theorem 4.1. *Let $A \in \mathrm{Sp}(V \oplus V^*)$ be an admissible symplectic transformation. Then, A is implemented by a unique operators $\hat{A}$ with a normal symbol N_A whose constant term is equal to 1. This normal symbol is given by formula*

$$N_A(a, a^*) = \exp\left(\langle a^*, (\alpha^{-1} - 1)a \rangle - \frac{1}{2} \langle a^*, (\alpha^{-1}\beta)a^* \rangle + \frac{1}{2} \langle (\gamma\alpha^{-1})a, a \rangle \right) \tag{25}$$

We will call a pair of admissible symplectic transformations A_1, A_2 composable if $A_1 A_2$ is also an admissible transformation. We use a similar terminology for triples. The following proposition gives a product rule in terms of normal symbols:

Proposition 4.2. *Let $A_1, A_2 \in \mathrm{Sp}(V \oplus V^*)$ be a composable pair. Then,*

$$N_{A_1} * N_{A_2} = \frac{1}{\det^{1/2}(1 + (\alpha_2^{-1}\beta_2)(\gamma_1\alpha_1^{-1}))} N_{A_2 A_1}. \tag{26}$$

Proof. The proof is by a direct calculation of the Gaussian integral (23). $\square$

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[3] See Theorem 4.3 in [5] for the proof in the case of $A \in \mathrm{USp}(V \oplus V^*)$. The proof in the case of $A \in \mathrm{Sp}(V \oplus V^*)$ is similar.

Note that the product rule for normal symbols (26) is not quite well defined because of the square root of the determinant. In fact, the subset of admissible elements in the metaplectic group admits the following description:

$$\mathrm{Mp_{adm}}(V \oplus V^*) = \{(A, z) \in \mathrm{Sp_{adm}}(V \oplus V^*) \times \mathbb{C}^*; \det(\alpha) = z^2\}.$$

Observe that

$$\det(1 + (\alpha_2^{-1}\beta_2)(\gamma_1\alpha_1^{-1})) = \frac{\det(\alpha)}{\det(\alpha_1)\det(\alpha_2)},$$

where α is associated with $A = A_2 A_1$. Hence, we can rewrite the product rule of normal symbols in terms of the metaplectic group as follows:

$$N_{(A_1, z_1)} * N_{(A_2, z_2)} = \frac{z_1 z_2}{z} N_{(A, z)}.$$

One can summarize the properties of the product rule as follows:

Proposition 4.3. *The expression*

$$C_N(A_2, z_2; A_1, z_1) = \frac{1}{\det^{1/2}(1 + (\alpha_2^{-1}\beta_2)(\gamma_1\alpha_1^{-1}))} = \frac{z_1 z_2}{z} \qquad (27)$$

is a multiplicative 2-cocycle. That is, for all composable triples A_1, A_2, A_3 we have

$$C_N(A_2, A_1)C_N(A_3, A_2 A_1) = C_N(A_3 A_2, A_1)C_N(A_3, A_2)$$

Proof. The statement follows from the fact that $C_N = z_1 z_2 / z$ is a trivial 2-cocycle. This fact reflects associativity of the operator product $*$. □

For operators with normal symbols $G_{(A, z)} = z^{-1} N_A$, we obtain a group anti-homomorphism:

$$G_{(A_2, z_2)} * G_{(A_1, z_1)} = G_{(A_1 A_2, z_1 z_2)}. \qquad (28)$$

Recall that for $A \in \mathrm{USp}(V \oplus V^*)$ the operators α and $\delta = \bar{\alpha}$ are invertible. This implies that all transformations $A \in \mathrm{USp}(V \oplus V^*)$ are admissible, all pairs A_1, A_2 are composable, and Theorem 4.1 and Proposition 4.2 apply without further assumptions. Also, the map $(A, z) \mapsto G_{(A, z)}$ defines a group anti-homomorphism from the corresponding subgroup of the metaplectic group $\mathrm{MUSp}(V \oplus V^*)$ to unitary operators on the Fock space $\mathcal{F}$.

4.3. The Infinite-Dimensional Case

Most of the facts reviewed in the previous Section generalize to the infinite-dimensional setup. Let V be a Hilbert space. This allows to identify $V^* \cong V$. In what follows, we list special features which distinguish the infinite-dimensional situation from the finite-dimensional one.

Instead of the symplectic group $\mathrm{Sp}(V \oplus V^*)$, one considers the restricted symplectic group

$$\mathrm{Sp^{res}}(V \oplus V^*) = \{A \in \mathrm{Sp}(V \oplus V^*); \alpha, \delta \text{ are Fredholm}, \beta, \gamma \text{ are Hilbert–Schmidt}\}.$$

For admissible elements of this subgroup, Theorem 4.1 and Proposition 4.2 hold true verbatim. In particular, the determinant

$$\det\big(1 + \big(\alpha_2^{-1}\beta_2\big)\big(\gamma_1\alpha_1^{-1}\big)\big)$$

is well defined since the operator

$$1 + \left(\alpha_2^{-1}\beta_2\right)\left(\gamma_1\alpha_1^{-1}\right) = \alpha_2^{-1}\alpha\alpha_1^{-1}$$

is invertible, the operators $(\alpha_2^{-1}\beta_2)$ and $(\gamma_1\alpha_1^{-1})$ are Hilbert–Schmidt, and hence the operator $(\alpha_2^{-1}\beta_2)(\gamma_1\alpha_1^{-1})$ is of trace class. The product formula (26) still makes sense on the metaplectic double cover.

However, the determinant $\det(\alpha)$ is not well defined, in general. Therefore, the admissible part of the metaplectic group does not allow for a simple description using the equation $z^2 = \det(\alpha)$, and the cocycle $C_N(A_1, A_2)$ is a priori nontrivial. It is instructive to write the corresponding Lie algebra cocycle $a_N(x_1, x_2)$ on a pair of elements of the symplectic Lie algebra:

$$x_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad i = 1, 2,$$

where a_i and d_i are bounded operators and b_i and c_i are Hilbert–Schmidt operators. An easy calculation shows that

$$a_N(x_1, x_2) = \text{Tr}(b_1 c_2 - c_1 b_2).$$

The right-hand side is well defined because both terms $b_1 c_2$ and $c_1 b_2$ are of trace class.

The restricted group $\text{USp}^{\text{res}}(V \oplus V^*)$ is defined as before:

$$\text{USp}^{\text{res}}(V \oplus V^*) = \{A \in \text{Sp}^{\text{res}}(V \oplus V^*); \delta = \overline{\alpha}, \gamma = \overline{\beta}\}.$$

Again, all elements $A \in \text{USp}^{\text{res}}(V \oplus V^*)$ are admissible and all pairs A_1, A_2 are composable. An important result of Berezin (see Theorem 4.3 in [5]) is the following theorem:

Theorem 4.4. *Let $A \in \text{USp}^{\text{res}}(V \oplus V^*)$. Then, the normal symbol*

$$U_A = \pm\frac{1}{\det(\alpha\alpha^*)^{1/4}} \, N_A = \pm\det{}^{1/4}(1 - (\alpha^{-1}\beta)(\alpha^{-1}\beta)^*) \, N_A$$

defines a unitary operator on $\mathcal{F}$.

Here, we have used Eq. (12). Note that the resulting Fredholm determinant is well defined since the operator $(\alpha^{-1}\beta)(\alpha^{-1}\beta)^*$ is of trace class.

Let $A \in \text{USp}(V \oplus V^*)$, and $\hat{A}$ be its unitary lift acting on $\mathcal{F}$, for instance the one defined by U_A. Consider a function

$$\tau(A) = |(v_0, \hat{A}v_0)|.$$

This function is well defined since different unitary lifts are related by $\hat{A}' = z\hat{A}$ with $|z| = 1$. The function $\tau(A)$ has the following properties:

Proposition 4.5. *For all $A \in \text{USp}^{\text{res}}(V \oplus V^*)$,*

$$\tau(A) = \frac{1}{\det{}^{1/4}(\alpha\alpha^*)}, \quad \tau(A^{-1}) = \tau(A). \tag{29}$$

Proof. The first equality is a direct consequence of Theorem 4.4. Indeed, since we can use any unitary lift of A in the definition of $\tau(A)$, it is convenient to choose U_A:

$$\tau(A) = |(v_0, U_A v_0)| = \frac{1}{\det^{1/4}(\alpha\alpha^*)} |(v_0, N_A v_0)| = \frac{1}{\det^{1/4}(\alpha\alpha^*)}.$$

Here, we have used the fact that the constant term in N_A is equal to 1. For the second property, we compute

$$\tau(A^{-1}) = |(v_0, U_{A^{-1}} v_0)| = |(U_{A^{-1}}^* v_0, v_0)| = |(U_{A^{-1}}^{-1} v_0, v_0)| = \tau(A).$$

Here, we have used the fact that $U_{A^{-1}}^{-1} = z U_A$ for some $z \in \mathbb{C}, |z| = 1$. □

Operators U_A satisfy the product rule

$$U_{A_1} * U_{A_2} = C_U(A_1, A_2) U_{A_2 A_1},$$

where the cocycle $C_U(A_1, A_2)$ is given by formula

$$C_U(A_1, A_2) = \left(\frac{\det(\alpha\alpha^*)}{\det(\alpha_1\alpha_1^*)\det(\alpha_2\alpha_2^*)} \right)^{\frac{1}{4}} \cdot \frac{1}{\det^{1/2}(1 + (\alpha_2^{-1}\beta_2)(\gamma_1\alpha_1^{-1}))}.$$

It is defined on the metaplectic double cover, and it takes values in $S^1 \cong \{z \in \mathbb{C}; |z| = 1\}$ instead of $\mathbb{C}^*$. This cocycle is nontrivial, in general. The corresponding Lie algebra cocycle is the same as for $C_N(A_1, A_2)$ (up to second order, the normalization factor is symmetric in A_1, A_2):

$$a(x_1, x_2) = \operatorname{Tr}(b_1\bar{b}_2 - \bar{b}_1 b_2).$$

5. Quantization of Conformal Welding

In this section, we apply Berezin quantization to triangular decomposition of symplectic transformations induced by holomorphic maps, and in particular to conformal welding.

5.1. Triangular Decomposition and Conformal Welding

In this section, we discuss an analog of triangular decomposition for holomorphic maps.

Let $f : \mathcal{A}_{r,R} \to \mathbb{C}$ be a univalent holomorphic map defined on the annulus $\mathcal{A}_{r,R}$ such that its range is contained in another annulus: $f(\mathcal{A}_{r,R}) \subset \mathcal{A}_{r',R'}$. We say that f admits a triangular decomposition if there exist univalent holomorphic maps $f_+ : \mathbb{D}_{R'} \to \mathbb{C}, f_- : \mathbb{D}_r^* \to \mathbb{C} \cup \{\infty\}$ such that $f_+(0) = 0, f_+'(0) \neq 0$, $f_-(\infty) = \infty$, $f_-'(\infty) \neq 0$ and

$$f_- = f_+ \circ f \tag{30}$$

on $\mathcal{A}_{r,R}$. Here,

$$\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}, \quad \mathbb{D}_r^* = \{z \in \mathbb{C}; |z| > r\} \cup \{\infty\}$$

are the disks centered at 0 and ∞, respectively. Holomorphic functions f_+ and f_- admit Taylor expansions at 0 and ∞:

$$f_+(z) = \sum_{k=1}^{\infty} (f_+)_k z^k, \quad f_-(z) = \sum_{k=-\infty}^{1} (f_-)_k z^k, \qquad (31)$$

where $(f_+)_1 \neq 0, (f_-)_1 \neq 0$. Note that the logarithms $\log(f_\pm(z)/z), \log(f'_\pm(z))$ are univalued functions on $\mathcal{A}_{r,R}$. Indeed, for $r < \rho < R$ we have

$$\int_{C_\rho} \log(f'_\pm(z))' \mathrm{d}z = \int_{C_\rho} \frac{f''_\pm(z)}{f'_\pm(z)} \, \mathrm{d}z = 0$$

because the function $f''_+(z)/f'(z)$ is holomorphic on $\mathbb{D}_R$, and the function $f''_-(z)/f'_-(z)$ is holomorphic on $\mathbb{D}^*_r$. Similarly,

$$\int_{C_\rho} \log(f_\pm(z)/z)' \mathrm{d}z = \int_{C_\rho} \left(\frac{f'_\pm(z)}{f_\pm(z)} - \frac{1}{z} \right) \mathrm{d}z = 0.$$

Here, we have used the facts that the function $f'_+(z)/f_+(z) - z^{-1}$ is holomorphic on $\mathbb{D}_R$ and that the function $f'_-(z)/f_-(z) - z^{-1}$ is holomorphic on $\mathbb{D}^*_r$.

A special case of triangular decomposition is given by conformal welding of diffeomorphisms of the circle. Let f be a holomorphic function which corresponds to $\chi \in \mathrm{Diff}^+_{\mathrm{hol}}(S^1)$. That is,

$$f(e^{ix}) = e^{i\chi(x)}.$$

In this case, one can choose the domain of f to be an annulus $\mathcal{A}_{r,R}$ with $r < 1 < R$. The following theorem follows from results of [7] on conformal welding for elements of $\mathrm{Diff}^+(S^1)$. We will only be interested in the subgroup $\mathrm{Diff}^+_{\mathrm{hol}}(S^1) \subset \mathrm{Diff}^+(S^1)$.

Theorem 5.1. *Let $f : \mathcal{A}_{r,R} \to \mathbb{C}$ be a univalent holomorphic map which corresponds to a diffeomorphism of the circle χ. Then, it admits a unique triangular decomposition $f = f_+^{-1} \circ f_-$ with $(f_+)'(0) = 1$.*

Remark 5.2. One says that the univalent holomorphic functions $f_\pm$ provide a conformal welding of the diffeomorphism χ. Note that the normalization $f'_+(0) = (f_+)_1 = 1$ (here $f_+(z) = \sum_{n=1}^{\infty}(f_+)_n z^n$) can be replaced by the normalization $f'_-(\infty) = (f_-)_1 = 1$ (here $f_-(z) = \sum_{n=-\infty}^{1}(f_-)_n z^n$. This is achieved by dividing both f_+ and f_- by $f'_-(\infty)$. In what follows, we will use both normalizations.

Equation (30) can also be rewritten in the form

$$f = f_+^{-1} \circ f_-$$

which resembles the Gauss decomposition of matrices. It turns out that it induces a Gauss decomposition on the corresponding elements of the infinite-dimensional symplectic group. Indeed, by Proposition 3.5, symplectic transformations A_{f_+} and A_{f_-} are upper and lower triangular, respectively. And

by Proposition 3.2, we have a Gauss-type decomposition (recall that the map $f \mapsto A_f$ is a group anti-homomorphism):

$$A_f = A_{f_-} A_{f_+^{-1}}.$$

It is convenient to introduce the notation

$$A_f = \begin{pmatrix} \alpha_f & \beta_f \\ \gamma_f & \delta_f \end{pmatrix}$$

for components of the symplectic transformation A_f. Then, we have

$$A_f = \begin{pmatrix} \alpha_{f_-} & 0 \\ \gamma_{f_-} & \delta_{f_-} \end{pmatrix} \begin{pmatrix} \alpha_{f_+^{-1}} & \beta_{f_+^{-1}} \\ 0 & \delta_{f_+^{-1}} \end{pmatrix} = \begin{pmatrix} \alpha_{f_-}\alpha_{f_+^{-1}} & \alpha_{f_-}\beta_{f_+^{-1}} \\ \gamma_{f_-}\alpha_{f_+^{-1}} & \gamma_{f_-}\beta_{f_+^{-1}} + \delta_{f_-}\delta_{f_+^{-1}} \end{pmatrix}.$$

Assuming that α_{f_-} and $\alpha_{f_+^{-1}}$ are invertible, this implies

$$\alpha_f^{-1}\beta_f = \alpha_{f_+^{-1}}^{-1}\beta_{f_+^{-1}}, \quad \gamma_f\alpha_f^{-1} = \gamma_{f_-}\alpha_{f_-}^{-1}. \tag{32}$$

5.2. Normal Symbols of Holomorphic Maps

In this section, we apply Berezin theory of normal symbols to holomorphic maps and the corresponding symplectic transformations.

In order to do that, we equip the space of holomorphic functions (modulo constants) H with a structure of a Hilbert space (following [9]) by declaring $||z^n||^2 = |n|$ for all $n \neq 0$. Then, symplectic transformations induced by holomorphic maps admit a metaplectic projective representation on the corresponding Fock space.

In more detail, let f be a holomorphic map, and assume that the corresponding symplectic transformation A_f is admissible. Then, it is convenient to denote by N_f (instead of N_{A_f}) its normal symbol.

Proposition 5.3. *Let f and g be a pair of composable holomorphic maps, and assume that the corresponding symplectic transformations A_f and A_g are admissible. Then,*

$$N_f * N_g = \frac{1}{\det^{1/2}(1 + (\alpha_g^{-1}\beta_g)(\gamma_f\alpha_f^{-1}))} N_{f\circ g}$$

Furthermore, if $f = \sum_{n=1}^{\infty} f_n z^n$ with $f_1 \neq 0$, or if $g = \sum_{-\infty}^{1} g_n z^n$ with $g_1 \neq 0$, then

$$N_f * N_g = N_{f\circ g}.$$

Proof. The first statement follows from Proposition 4.2. For the second statement, note that if $f = \sum_{n=1}^{\infty} f_n z^n$, then by Proposition 3.5 $\gamma_f = 0$. Similarly, if $g = \sum_{-\infty}^{1} g_n z^n$, then $\beta_g = 0$. This completes the proof. $\square$

We are now ready to state one of our main results:

Theorem 5.4. *Let f and g be a pair of composable holomorphic maps, and assume that they admit triangular decompositions. Then,*

$$N_f * N_g = C_N(f, g)N_{f\circ g}, \tag{33}$$

where the 2-cocycle $C_N(f, g)$ is of the form

$$C_N(f, g) = \frac{1}{\det^{1/2}\left(1 + \left(\alpha_{g_+^{-1}}^{-1}\beta_{g_+^{-1}}\right)\left(\gamma_{f_-}\alpha_{f_-^{-1}}\right)\right)}. \tag{34}$$

The corresponding map β_N is given by formula

$$\beta_N(u, g) = -\frac{1}{(2\pi i)^2}\int_{C \times C} \frac{u_-(z) - u_-(w)}{z - w} \frac{\partial^2}{\partial z \partial w} \log\left(\frac{g_+(z) - g_+(w)}{z - w}\right) dz dw. \tag{35}$$

Proof. For the first statement, we use Proposition 5.3, and we observe that by formula (32)

$$\alpha_g^{-1}\beta_g = \alpha_{g_+}^{-1}\beta_{g_+}, \quad \gamma_f\alpha_f^{-1} = \gamma_{f_-}\alpha_{f_-}^{-1}.$$

In order to prove the formula for β_N, put $f(t) = \exp(tu)$, where $u = u(z)\frac{\partial}{\partial z}$ is a vector field with

$$u(z) = u_+(z) + u_-(z) = \sum_{n=1}^{\infty} u_n z^n + \sum_{n=-\infty}^{0} u_n z^n.$$

Note that $f(t) = f_+(t)^{-1}f_-(t)$ and $\log(f_-(t)) = tu_- + O(t^2)$. Hence,

$$\beta_N(u, g) = \frac{d}{dt} C(\exp(tu), g)|_{t=0} = \mathrm{Tr}\left(\alpha_{g_+^{-1}}^{-1}\beta_{g_+^{-1}}\right)\left(\frac{d}{dt}\left(\gamma_{\exp(tu_-)}\alpha_{\exp(tu_-)}^{-1}\right)\right)_{t=0}.$$

Recall that

$$\sum_{m,n}\sqrt{mn}\left(\gamma_{\exp(tu_-)}\alpha_{\exp(tu_-)}^{-1}\right)_{m,n} z^{-m-1}w^{-n-1}$$

$$= -\frac{\partial^2}{\partial z \partial w} \log\left(\frac{\exp(tu_-)(z) - \exp(tu_-)(w)}{z - w}\right),$$

where $\exp(tu_-)(z)$ is the image of z under the holomorphic map $\exp(tu_-)$. The derivative in t at $t = 0$ yields (after integrating over z and w)

$$\sum_{m,n}\frac{1}{\sqrt{mn}}\frac{d}{dt}\left(\gamma_{f_-}\alpha_{f_-}^{-1}\right)_{m,n}|_{t=0}z^{-m}w^{-n} = -\frac{u_-(z) - u_-(w)}{z - w}.$$

Also recall that

$$\sum_{m,n}\sqrt{mn}\left(\alpha_{g_+^{-1}}\beta_{g_+}\right)_{m,n} z^{m-1}w^{n-1} = \frac{\partial^2}{\partial z \partial w} \log\left(\frac{g_+(z) - g_+(w)}{z - w}\right).$$

Next, we convert the trace in m, n into a double contour integral. Since the factors $\sqrt{mn}$ and $1/\sqrt{mn}$ cancel out, we obtain the desired result. □

Surprisingly, the cocycle

$$C_N(f, g) = C_N\left(f_+^{-1}f_-, g_+^{-1}g_-\right)$$

has the following polarization property: it is independent of the components f_+ and g_- in triangular decompositions of f and g.

The following result of [15] (see Corollary 2.9 in Chapter 2) establishes important properties of symplectic transformations associated with conformal welding:

Theorem 5.5. *Let $\chi \in \mathrm{Diff}^+_{\mathrm{hol}}(S^1)$, f the corresponding holomorphic function, and $f_\pm$ the components of conformal welding of χ. Then, the maps $A_{f_+^{-1}}, A_{f_-}$ and $A_f = A_{f_-} A_{f_+^{-1}}$ belong to the restricted symplectic group $\mathrm{Sp}^{\mathrm{res}}(H_+ \oplus H_-)$. In particular, symmetric operators $(\alpha_{f_+^{-1}}^{-1} \beta_{f_+^{-1}})$ and $(\gamma_{f_-} \alpha_{f_-}^{-1})$ are Hilbert–Schmidt.*

Theorem 5.5 implies that conditions of Proposition 5.3 are verified for conformal welding, and we have

$$N_f = N_{f_+^{-1}} * N_{f_-}.$$

Furthermore, let $\chi, \phi \in \mathrm{Diff}^+_{\mathrm{hol}}(S^1)$. Recall that the corresponding conformal maps f and g are always composable. By Theorem 5.5, A_f and A_g are admissible. Hence, Theorem 5.4 applies, and we conclude that Eq. (33) holds true for the product of normal symbols $N_f * N_g$.

5.3. Takhtajan–Teo Energy Functional

In this section, we recall the definition and the main properties of the Takhtajan–Teo energy functional (see [15] for details).

Let $\mathcal{D} \subset \mathbb{C}$ be a simply connected domain which contains the unit disk $\mathbb{D}$ and $f_+ : \mathcal{D} \to \mathbb{C}$ be a univalent holomorphic function. Similarly, let $\mathcal{D}^*$ be a simply connected domain which contain $\mathbb{D}^*$ and $f_- : \mathcal{D}^* \to \mathbb{C} \cup \{\infty\}$ be a univalent holomorphic function. Introduce the functionals S_+ and S_- defined by formulas

$$S_+(f_+) = E_+(f_+) + 4\pi \log(|f_+'(0)|), \quad E_+(f_+) = \int_{\mathbb{D}} \left| \frac{f_+''(z)}{f_+'(z)} \right|^2 \mathrm{d}^2 z,$$

and

$$S_-(f_-) = E_-(f_-) - 4\pi \log(|f_-'(\infty)|), \quad E_-(f_-) = \int_{\mathbb{D}^*} \left| \frac{f_-''(z)}{f_-'(z)} \right|^2 \mathrm{d}^2 z.$$

The functionals S_+ and S_- possess the following important property:

Proposition 5.6. *Functionals S_+ and S_- are invariant under the $\mathrm{PSL}(2, \mathbb{R})$ action on the right by Möbius transformations preserving the unit circle: $f_+ \mapsto f_+ \circ m, f_- \mapsto f_- \circ m$.*

Proof. The proof is by a direct calculation, see the proof of Lemma 3.4 in Chapter 2 of [15]. □

Let $\chi \in \mathrm{Diff}^+_{\mathrm{hol}}$, f the corresponding holomorphic map and $f = f_+^{-1} \circ f_-$ the conformal welding of χ. The Takhtajan–Teo (TT) energy functional is defined as

$$S(\chi) = S_+(f_+) + S_-(f_-). \tag{36}$$

Remark 5.7. For an interesting alternative description of the TT functional, see [16].

Remark 5.8. Note that the TT functional is invariant under the simultaneous action of dilations $f_+(z) \mapsto \lambda f_+(z), f_-(z) \mapsto \lambda f_-(z)$ for all $\lambda \in \mathbb{C}^*$. In particular, the value of $S(\chi)$ is the same for conformal weldings with $f'_+(0) = 1$ and with $f'_-(\infty) = 1$. In what follows, it will be more convenient to use the second picture.

The following observation is crucial for interpretation of the TT functional as a Kähler potential on the Teichmüller coadjoint orbit:

Proposition 5.9. *The functional $S(\chi)$ is invariant under the left $\mathrm{PSL}(2, \mathbb{R})$ action of Möbius transformations preserving the unit circle: $S(m \circ \chi) = S(\chi)$.*

Proof. Let f_+, f_- be components of conformal welding of χ with the normalization $f'_-(\infty) = 1$. Note that $m(0) \in \mathbb{D}$ and let $c = f_+(m(0))$. Then,

$$\tilde{f}_+(z) = f_+(m^{-1}(z)) - c, \quad \tilde{f}_-(z) = f_-(z) - c$$

are components of conformal welding of $\tilde{\chi} = m \circ \chi$ with normalization $\tilde{f}_+(0) = 0, \tilde{f}_-(\infty) = \infty, \tilde{f}'_-(\infty) = 1$. Furthermore, observe that this transformation does not change the TT functional. Indeed, the invariance under Möbius transformations preserving the unit circle follows from Proposition 5.6, and invariance under shifts is obvious since the TT functional only depends on derivatives of f_+ and f_-. This completes the proof. $\qquad\square$

Finally, recall the following highly nontrivial property of the TT energy functional (see Theorem 3.8 in Chapter 2 of [15]):

Theorem 5.10. *For all $\chi \in \mathrm{Diff}^+(S^1)$, we have*

$$S(\chi^{-1}) = S(\chi).$$

In combination with Proposition 5.9, Theorem 5.10 implies that $S(\chi)$ is also invariant under the left action of Möbius transformations preserving the circle: $S(\chi \circ m) = S(\chi)$.

We observe the following interesting new property of the functional $S(\chi)$:

Proposition 5.11. *For $\chi \in \mathrm{Diff}^+_{\mathrm{hol}}(S^1)$, we have*

$$S(\chi) = \mathrm{Im}\, \tilde{C}(f_-^{-1}, f_+) - \int_C (\chi'(x) + 1) \log(\chi'(x)) \mathrm{d}x,$$

where $\tilde{C}$ is the groupoid 2-cocycle defined by Eq. (10).

Proof. First, observe that

$$E_+(f_+) = \frac{i}{2} \int_C \log(f'_+) \mathrm{d}\log(\overline{f'_+}), \quad E_-(f_-) = -\frac{i}{2} \int_C \log(f'_-) \mathrm{d}\log(\overline{f'_-}).$$

Also, note that

$$\mathrm{Im}\, \tilde{C}(f_-^{-1}, f_+) = \mathrm{Im}\, \tilde{C}(f_+, f) = -\frac{i}{2} \int_C (\log(f'_+(f)) d\log(f') - \log(\overline{f'_+(f)}) \mathrm{d}\log(\overline{f'})).$$

We consider

$$
\begin{aligned}
A_1(\chi) &= E_+(f_+) + E_-(f_-) - \operatorname{Im} \tilde{C}(f_-^{-1}, f_+) \\
&= \tfrac{i}{2} \int_C \log(f'_+) \mathrm{d} \log(\overline{f'_+}) \\
&\quad - \tfrac{i}{2} \int_C \left(\log(f'_+(f)) + \log(f') \right) \mathrm{d}(\log(\overline{f'_+(f)}) + \log(\overline{f'})) \\
&\quad + \tfrac{i}{2} \int_C (\log(f'_+(f)) d \log(f') - \log(\overline{f'_+(f)}) \mathrm{d} \log(\overline{f'}))
\end{aligned}
$$

Using that for $z = e^{ix}$

$$
\log(f'(z)) = \log(\chi'(x)) + i(\chi(x) - x),
$$

we obtain

$$
A_1(\chi) = - \int_C (\log(\chi'(x)) + \log(|f'_+(f(z))|^2)(\chi'(x) - 1) \mathrm{d}x.
$$

Next, we consider

$$
\begin{aligned}
A_2(\chi) &= 4\pi(\log(|f'_+(0)|) - \log(|f'_-(\infty)|)) \\
&= \int_C (\log(|f'_+(z)|^2) - \log(|f'_-(z)|^2)) \mathrm{d}x \\
&= \int_C (\log(|f'_+(z)|^2) - \log(|f'_+(f(z))|^2) - \log(|f'(z)|^2)) \mathrm{d}x \\
&= \int_C (\log(|f'_+(f(z))|^2)(\chi'(x) - 1) - 2\log(\chi'(x))) \mathrm{d}x.
\end{aligned}
$$

Adding up the expressions $A_1(\chi)$ and $A_2(\chi)$, we conclude

$$
S_+(f_+) + S_-(f_-) - \operatorname{Im} \tilde{C}(f_-^{-1}, f_+) = - \int_C (\chi'(x) + 1) \log(\chi'(x)) \mathrm{d}x,
$$

as required. $\qquad\qquad\square$

The statement of Proposition 5.11 can be rewritten as

$$
\operatorname{Im} \tilde{C}(f_-^{-1}, f_+) = S(\chi) + \int_C (\log(\chi'(x)) - \log((\chi^{-1})'(x))) \mathrm{d}x. \qquad (37)
$$

By Theorem 5.10, the first term on the right-hand side is invariant under the involution $\chi \mapsto \chi^{-1}$, while the second term is anti-invariant.

5.4. Quantization of Conformal Welding

In this section, we apply Berezin formalism to quantization of holomorphic maps f corresponding to elements $\chi \in \operatorname{Diff}_{\mathrm{hol}}^+(S^1)$. In particular, we use conformal welding to introduce a triangular decomposition of the corresponding unitary operators U_f.

Let $\chi \in \operatorname{Diff}_{\mathrm{hol}}^+(S^1)$ and f the corresponding holomorphic map. Recall that the symplectic transformation $A_f \in \operatorname{USp}^{\mathrm{res}}(H_+ \oplus H_-)$ belongs to the restricted unitary symplectic group. This implies that α_f is invertible, $\gamma_f = \bar{\beta}_f$, and

$$
\alpha_f^{-1}(\alpha_f^*)^{-1} = \alpha_f^{-1}(\alpha_f \alpha_f^* - \beta_f \beta_f^*)(\alpha_f^*)^{-1} = 1 - (\alpha_f^{-1}\beta_f)(\alpha_f^{-1}\beta_f)^*.
$$

Since the symmetric operator $\alpha_f^{-1}\beta_f = \alpha_{f_+^{-1}}^{-1}\beta_{f_+^{-1}}$ is Hilbert–Schmidt, the operator $(\alpha_f^{-1}\beta_f)(\alpha_f^{-1}\beta_f)^*$ is of trace class, and $\alpha_f^{-1}(\alpha_f^*)^{-1}$ possesses a Fredholm

determinant. The following result is an adaptation of Theorem 3.8 in Chapter 2 of [15]:

Theorem 5.12. *Let $\chi \in \mathrm{Diff}_{\mathrm{hol}}^{+}(S^1)$ and f the corresponding holomorphic map. Then, the operator α_f is bounded and invertible. The Fredholm determinant of the self-adjoint operator $\alpha_f^{-1}(\alpha_f^{*})^{-1}$ is given by*

$$\det\left(\alpha_f^{-1}\left(\alpha_f^{*}\right)^{-1}\right) = e^{-S(\chi)/12\pi}, \tag{38}$$

where $S(\chi)$ is the TT energy functional.

Theorem 5.12 allows to reprove several properties of the TT functional $S(\chi)$. Indeed, by Theorem 4.4 the unitary operator representing the symplectic transformation A_f is given by:

$$U_f = \frac{1}{\det^{1/4}(\alpha_f \alpha_f^{*})} \, N_f = e^{-S(\chi)/48\pi} \, N_f = e^{-S(\chi)/48\pi} \, N_{f_+^{-1}} * N_{f_-}. \tag{39}$$

By Proposition 4.5, we have

$$\tau(A_f) = \frac{1}{\det^{1/4}(\alpha_f \alpha_f^{*})} = e^{-S(\chi)/48\pi}.$$

The identity $\tau(A_f) = \tau(A_f^{-1}) = \tau(A_{f^{-1}})$ implies $S(\chi) = S(\chi^{-1})$ and gives a new proof of Theorem 5.10 based on Theorem 5.12.

Furthermore, let m be a Möbius transformation preserving the unit circle. Then, $\beta_m = \gamma_m = 0$ and α_m and δ_m are unitary operators. This implies $U_m v_0 = v_0$. Indeed, the normal symbol acts trivially $N_m v_0 = v_0$ since $\alpha_m^{-1}\beta_m = 0$, and $\det(\alpha_m \alpha_m^{*}) = 1$. Therefore, for m_1, m_2 two Möbius transformations preserving the unit circle we have

$$e^{-S(m_1 \circ \chi \circ m_2)/48\pi} = \tau(A_{m_1 \circ f \circ m_2}) = |(v_0, U_{m_2} U_f U_{m_1} v_0)|$$
$$= |(v_0, U_f v_0)| = \tau(A_f) = e^{-S(\chi)/48\pi}$$

which implies $S(m_1 \circ \chi \circ m_2) = S(\chi)$.

Using Eq. (39), we obtain the following interesting relation between the cocycle C_N and the TT energy functional:

Theorem 5.13. *Let $\chi \in \mathrm{Diff}_{\mathrm{hol}}^{+}(S^1)$, f the corresponding holomorphic function, and $f_{\pm}$ the components of the conformal welding of χ. Then,*

$$\log(|C_N(f_-^{-1}, f_+)|) = -\frac{S(\chi)}{24\pi}. \tag{40}$$

Proof. On the one hand, Eq. (39) implies:

$$U_f^{-1} = e^{S(\chi)/48\pi} N_{f_-}^{-1} * N_{f_+^{-1}}^{-1} = e^{S(\chi)/48\pi} N_{f_-^{-1}} * N_{f_+}$$
$$= e^{S(\chi)/48\pi} C_N(f_-^{-1}, f_+) N_{f^{-1}}.$$

Here, we have used the facts that $N_{f_-^{-1}}$ is the operator inverse of N_{f_-} and $N_{f_+^{-1}}$ is the operator inverse of N_{f_+}. On the other hand, we have

$$U_{f^{-1}} = e^{-S(\chi^{-1})/48\pi} N_{f^{-1}} = e^{-S(\chi))/48\pi} N_{f^{-1}}.$$

Since both U_f and $U_{f^{-1}}$ are unitary operators, their product $U_f U_{f^{-1}} = z \cdot \mathrm{Id}$ which implements $f \circ f^{-1} = e$ is a multiple of the identity with $|z| = 1$. Hence,

$$U_{f^{-1}} = z\, U_f^{-1} = z\, e^{S(\chi)/48\pi}\, C_N(f_-^{-1}, f_+) N_{f^{-1}}.$$

By comparing the two expressions for $U_{f^{-1}}$, we obtain Eq. (40), as required. $\square$

Remark 5.14. Note that the constant z in the proof of Theorem 5.13 can be computed as follows:

$$z = e^{-S(\chi)/24\pi} C_N(f_-^{-1}, f_+).$$

Remark 5.15. The statement of Theorem 5.13 should be compared to Proposition 5.11 and Eq. (37). In particular, it would be interesting to find a more explicit relation between the cocycles $\log(|C_N|)$ and $\tilde{C}$.

The second main result of this article is the following theorem:

Theorem 5.16. *Let $\chi, \phi \in \mathrm{Diff}_{\mathrm{hol}}^+(S^1)$, and f and g be the corresponding holomorphic maps. Then,*

$$U_f * U_g = C_U(f, g)\, U_{f \circ g},$$

where

$$C_U(f, g) = e^{(S(\chi \circ \phi) - S(\chi) - S(\phi))/48\pi}\, C_N(f, g). \tag{41}$$

Furthermore,

$$\log(|C_N(f, g)|) = \frac{S(\chi) + S(\phi) - S(\chi \circ \phi)}{48\pi}. \tag{42}$$

Proof. Formula for $C_U(f, g)$ is a direct consequence of Theorems 5.4 and 5.12. Since U_f, U_g and $U_{f \circ g}$ correspond to unitary operators, $C_U(f, g)$ takes values in $S^1 \cong \{z \in \mathbb{C}; |z| = 1\}$ and the real part of its logarithm vanishes. $\square$

Remark 5.17. Explicit expressions for the left- and the right-hand sides of Eq. (42) are of very different nature. The determinant on the left-hand side uses Grunsky coefficients of the welding components f_- and g_-. On the right-hand side, the TT energy functional is an integral of local expressions in terms of all welding components $f_\pm, g_\pm, (f \circ g)_\pm$. It would be interesting to find a direct proof of the surprising equality (42) between these expressions.

Remark 5.18. The imaginary part $\mathrm{Im}\, \log(C_U(f, g))$ of the cocycle $C_U(f, g)$ is an additive real-valued group 2-cocycle on $\mathrm{Diff}^+(S^1)$. Assuming that

$$H^2(\mathrm{Diff}_{\mathrm{hol}}^+(S^1), \mathbb{R}) = H^2(\mathrm{Diff}^+(S^1), \mathbb{R}) \cong \mathbb{R},$$

this cocycle must be cohomologous to the Bott–Virasoro cocycle. It would be interesting to find an explicit expression for the coboundary which represents their difference.

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