MAT 314: HOMEWORK 9

DUE TH, APRIL 13, 2023

Throughout this problem set, F is a field of characteristic zero.

1. Describe the Galois group of the extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Describe all subgroups of this Galois group and corresponding intermediate fields.

2. Let $z = e^{2\pi i/5}$. Recall that in one of the previous homeworks we have constructed a tower of extensions

$$\mathbb{Q} \subset \mathbb{Q}(t) \subset \mathbb{Q}(z), \qquad t = (z + z^{-1})/2.$$

Describe the Galois group $G = Gal(\mathbb{Q}(z)/\mathbb{Q})$ and the subgroup $H = Gal(\mathbb{Q}(z)/\mathbb{Q}(t) \subset G$.

3. For each of the following polynomials $p(x) \in \mathbb{Q}[x]$, describe its splitting field L, writing it in the form $L = \mathbb{Q}(\alpha_1, \ldots, \alpha_k)$ and the Galois group $G = Gal(L/\mathbb{Q})$. Also, describe subgroups of G corresponding to given subfields of L.

(a) $p(x) = x^3 - 2$; subfields generated by
  (i) $\sqrt[3]{2}$
  (ii) $e^{2\pi i/3}$

(b) $p(x) = x^4 - 2$; subfields generated by
  (i) $\sqrt[4]{2}$
  (ii) $\sqrt{2}$
  (iii) i

4. Let $K = \mathbb{Q}(\sqrt{2 + \sqrt{2}})$. Show that $Gal(K/\mathbb{Q}) = \mathbb{Z}_4$. Describe all subfields of K.

Hint: the roots of the correspoding irreducible polynomial are

$$\alpha_1 = \sqrt{2 + \sqrt{2}}$$
$$\alpha_2 = \sqrt{2 - \sqrt{2}}$$
$$\alpha_3 = -\alpha_1$$
$$\alpha_4 = -\alpha_2$$

Note that $\alpha_1\alpha_2 = \sqrt{2}$