# MAT 314: HOMEWORK 8

DUE TH, APRIL 6, 2023

Throughout this problem set, all fields have characteristic zero. Recall that for a normal extension $K \subset L$, we define its Galois group to be

$$Gal(L/K) = \mathrm{Aut}(L/K) = \{\varphi \in \mathrm{Aut}(L) \mid \varphi|_K = \mathrm{id}\}$$

**1.** Show that any extension of degree 2 is always normal.

**2.** Let $F \subset E \subset K$ be a chain of extensions such that $K$ is a finite normal extension of $F$. Show that then $K$ is also a normal extension of $E$. Is it true that $E$ is always a normal extension of $F$?

**3.** Describe the Galois group of the extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

**4.** For each of the following polynomials $p(x) \in \mathbb{Q}[x]$, describe its splitting field $L$, writing it in the form $L = \mathbb{Q}(\alpha_1, \ldots, \alpha_k)$ and the Galois group $G = Gal(L/\mathbb{Q})$. For each generator of the Galois group, describe how it permutes the roots of $p$:

(a) $x^3 - 2$
(b) $x^4 - 2$
(c) $x^4 + 1$

**5.** Let $F \subset E$ be an algebraic extension. We say that elements $\alpha, \alpha' \in E$ are conjugate (over $F$) if they are roots of the same irreducible polynomial $f \in F[x]$. We define $\deg_F(\alpha) = \deg f$.

(a) Show that an element $\alpha \in E$ can have no more than $n = \deg_F(\alpha)$ conjugates in $E$ (including itself); if $E$ is algebraically closed, then $\alpha$ has exactly $n$ conjugates.

(b) Show that if $\alpha, \alpha' \in \bar{F}$ are conjugate, then there exists an automorphism $\varphi \in \mathrm{Aut}(\bar{F}/F)$ such that $\varphi(\alpha) = \alpha'$. Deduce from this that if $\varphi(\alpha) = \alpha$ for each $\varphi \in \mathrm{Aut}(\bar{F}/F)$, then $\alpha \in F$.

(c) Assume that $\alpha$ has exactly $n = \deg_F(\alpha)$ conjugates $\alpha_1, \ldots, \alpha_n$ in $E$. Define the elements

$$N = \prod_i \alpha_i$$
$$T = \sum_i \alpha_i$$

Show that then $N, T \in F$.