MAT 314: HOMEWORK 5

DUE TH, MARCH 9

Throughout this problem set, all representations are finite-dimensional and complex. Unless otherwise stated, G is a finite group.

We denote by $\mathcal{F}(G)$ be the vector space of complex-valued fucntions on G; it has an inner product given by

$$(f_1, f_2) = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}$$

1. (a) Show that the formula

$$(g.f)(h) = f(g^{-1}h), \qquad g, h \in G, f \in \mathcal{F}(G)$$

defines on $\mathcal{F}(G)$ a structure of representation of G. This is called the left regular representation of G.

(b) Show that the formula

$$(g.f)(h) = f(hg), \qquad g, h \in G, f \in \mathcal{F}(G)$$

defines on $\mathcal{F}(G)$ a structure of representation of G. This is called the right regular representation of G.

(c) Show that the linear map $i\colon \mathcal{F}(G) \to \mathcal{F}(G)$, given by $(if)(g) = f(g^{-1})$ is an isomorphism of left and right regular representations of G. (Thus, it is common to talk about regualr representation of G, omitting words "left" or "right").

2. Recall that for a representation V of G, we defined the character $\chi_V \in \mathcal{F}(G)$ by

$$\chi_V(g) = \mathrm{tr}_V(g).$$

(a) Show that each character is constant on conjugacy classes in G: if g_1, g_2 are conjugate in G, then $\chi_V(g_1) = \chi_V(g_2)$. [Such functions are called central; it is known that characters of irreducible representations form a basis in the space of central functions, but we didn't prove it in class.]

(b) Let $G = S_3$, $V = \mathbb{C}^3$. Make the table, showing the value of χ_V on each of the conjugacy classes of G.

(c) Use the previous part to give another proof that V is a direct sum of two exactly two irreducible representations: $V = \mathbb{C} \oplus U$, where C is the trivial representation and U is a 2-dimensional representation. [Hint: by orthogonality of characters discussed in class, if $V = n_1V_1 \oplus n_2V_2 \oplus \dots$, where V_i are pairwise non-isomorphic irreducible representations, then $(\chi_V, \chi_V) = \sum n_i^2$.]

(d) Can you do the same computation for $G = S_4, V = \mathbb{C}^4$?

3. Let C be the standard cube in $\mathbb{R}^3$: $C = \{|x_i| \le 1\}$, and let S be the set of faces of C (thus, S consists of 6 elements). Consider the 6-dimensional complex vector space V of functions on S, and define $A\colon V \to V$ by

$$(Af)(\sigma) = \frac{1}{4}\sum_{\sigma'} f(\sigma')$$

where the sum is taken over all faces σ' which are neighbors of σ (i.e., have a common edge with σ). The goal of this problem is to diagonalize A.

(a) Let $G = \{g \in GL(3,\mathbb{R}) \mid g(C) = C\}$ be the group of symmetries of C. Show that A commutes with the natural action of G on V.

(b) Let $z = diag(-1,-1,-1) \in G$ be the diagonal matrix with -1 on the diagonal. Show that as a representation of G, V can be decomposed in the direct sum

$$V = V_+ \oplus V_-, \qquad V_\pm = \{f \in V \mid zf = \pm f\}.$$

(c) Show that as a representation of G, V_+ can be further decomposed in the direct sum

$$V_+ = V_+^0 \oplus V_+^1, \quad V_+^0 = \{f \in V_+ \mid \sum_\sigma f(\sigma) = 0\}, \quad V_+^1 = \mathbb{C}\cdot 1$$

where 1 denotes the constant function on S whose value at every $\sigma \in S$ is 1.

(d) Find the eigenvalues of A on V_-, V_+^0, V_+^1.

[Note: in fact, each of V_-, V_+^0, V_+^1 is an irreducible representation of G, but you do not need this fact.]