MAT 314: HOMEWORK 4

DUE TH, MARCH 2, 2022

Throughout this problem set, all representations are complex and finite-dimensional. Unless specified otherwise, G is a finite group. Given two representations V, W, we denote by $\mathrm{Hom}_G(V, W)$ the space of homomorphisms from V to W, i.e. the space of linear operators $V \to W$ which commute with action of G.

1. For a representation V of G, denote

$$V^G = \{v \in V \mid g.v = v \text{ for all } g \in G\}.$$

Show that if V is irreducible, then $V^G = V$ if $V = \mathbb{C}$ is the trivial representation, and $V^G = 0$ for all other irreducible representations. (Hint: V^G is a subrepresentation.)

2. Let V, W be representations of G. Denote by $L(V, W)$ the space of all linear operators $V \to W$.

(a) Show that this space is naturally a representation of G, with the action given by

$$(g.f)(v) = g(f(g^{-1}v))$$

(b) Show that

$$L(V, W)^G = \mathrm{Hom}_G(V, W).$$

3. Let V be a representation of G. Define the operator $\mathrm{Sym}\colon V \to V$ by

$$\mathrm{Sym} = \frac{1}{|G|} \sum_{g \in G} \rho(g)$$

(a) Show that for any $v \in V$, the vector $w = \mathrm{Sym}(v)$ is invariant under action of G:

$$\rho_h(w) = w \quad \forall g \in G.$$

(b) Show that Sym is a projector: $(\mathrm{Sym})^2 = \mathrm{Sym}$. What is the subspace it is projecting onto?

4. (a) Let $U \subset \mathbb{C}^3$ be the subspace defined by

$$U = \{x \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\}.$$

Prove that U is an irreducible representation of the symmetric group S_3 (where S_3 acts on $\mathbb{C}^3$ by permuting the coordinates, as described in class).

*(b) Can you prove the similar result for a subspace $U \subset \mathbb{C}^n$ and the group S_n?

5. Consider $V = \mathbb{C}^n$ with the natural action of $\mathbb{Z}_n = \langle a, a^n = 1 \rangle$ by cyclic permutation of coordinates:

$$a \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

Write $\mathbb{C}^n$ as a direct sum of irreducible representations of $\mathbb{Z}_n$. [Hint: what are eigenvalues of a in $\mathbb{C}^n$?]

6. This problem is the baby model of how group symmetry can be used to help solve various mathematical problems.

Let $A\colon \mathbb{C}^n \to \mathbb{C}^n$ be the linear operator given by

$$(Ax)_i = \tfrac{1}{2}(x_{i-1} + x_{i+1}),$$

where $i-1$, $i+1$ are taken modulo n. The goal is to diagonalize A. Straightforward approach, by writing characteristic polynomial and finding its roots, is difficult. A better way is using $\mathbb{Z}_n$ symmetry.

(a) Show that A commutes with the natural action of $\mathbb{Z}_n$ on $\mathbb{C}^n$ (see problem 5).

(b) Let

$$\mathbb{C}^n = \bigoplus V_i$$

be the decomposition of $\mathbb{C}^n$ into a direct sum of irreducible representations of $\mathbb{Z}_n$ which you found in problem 5. Use Shur's lemma to show that A preserves each of V_i's and $A|_{V_i}$ is a scalar.

(c) Find all eigenvalues and eigenvectors of A .

(d) Is it true that for any vector $x \in \mathbb{C}^n$ we have

$$\lim_{N\to\infty} A^N x = c \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad ?$$