MAT 314: HOMEWORK 3

DUE TH, FEB 16, 2023

Throughout the assignment, R is a principal ideal domain. All modules are assumed to be finitely generated and with finite set of relations.

Most questions will be about the structure theorem: every such module is isomorphic to a module of the form

$$M \simeq R^r \oplus \Big(\bigoplus_i R/(p_i^{k_i})\Big) \qquad (1)$$

where p_i are irreducible (not necessarily distinct).

1. In this problem, we consider $\mathbb{Z}_n$ as a ring, not just as a group. Let $\mathbb{Z}_n^\times = \{a \in \mathbb{Z}_n \mid a \text{ is invertible mod } n\}$; this is a group with respect to multiplication (but not with respect to addition).

Use Chinese Remainder Theorem to show that if m, n are relatively prime, then $\mathbb{Z}_{mn}^\times \simeq \mathbb{Z}_m^\times \times \mathbb{Z}_n^\times$ (isomorphism of groups).

2. Let $R = \mathbb{R}[x]$, $M = R/(x^3 + x - 10)$. Write M in the form (1).

3. Let $R = \mathbb{C}[x]$, $M = R/(x^3 + x - 10)$. Write M in the form (1).

4. For an R-module M and an element $a \in R$, denote

$$M_{(a)} = \{m \in M \mid am = 0\} \subset M$$

(this notation is not standard). Prove that if a, b are relatively prime, then $M_{(ab)} = M_{(a)} \oplus M_{(b)}$

5. Let M be a module over R and let $a \in R$, $a \neq 0$ be such that $am = 0$ for any element $m \in M$.

(a) Show that all irreducibles p_i appearing in the canonical form (1) of M must be divisors of a. [Hint: if a is relatively prime with p, then a is invertible mod p^k]

(b) Show that if $a = q_1 \dots q_m$, where q_i are *distinct* irreducibles, then all powers k_i appearing in the canonical form (1) of M must be 1.

6. Use the previous problem to show that if $A\colon V \to V$ is a linear operator in a finite-dimensional vector space over $\mathbb{C}$, and $A^k = I$ for some k, then A is diagonalizable. Is the same true over $\mathbb{R}$?