MAT 314: HOMEWORK 2

DUE TH, FEB 9, 2023

Problems 3-6 in this assignment are about modules over $\mathbb{Z}$, also known as abelian groups. Unless stated otherwise, all modules will be assumed to have finite set of generators and finite set of relations.

For an $m \times n$ matrix A with integer entries, we denote by M_A the $\mathbb{Z}$-module

$$M_A = \mathbb{Z}^n / N \tag{1}$$

wher $N \subset \mathbb{Z}^n$ is the submodule generated by rows of the matrix A (we consider every row as an element of $\mathbb{Z}^n$).

Problems marked by asterisk (*) are optional.

1. A module M over a (not necessarily commutative) ring R is called *simple* if it has no nonzero proper submodules.
   (a) Prove that every simple module is generated by a single element.
   (b) Prove that every simple module is isomorphic to a module of the form R/I, where $I \subset R$ is a maximal left ideal.
   (c) Desribe all simple modules over $\mathbb{Z}$ (i.e., abelian groups).
   *(d) Describe all simple modules over $\mathbb{C}[x]$.

2. Let $R = Mat_n(\mathbb{F})$ be the ring of $n \times n$ matrices with entries in a field $\mathbb{F}$. Then $\mathbb{F}^n$ is naturally a module over R. Show that it is simple.

Hint: show that for any nonzero vector $v \in \mathbb{F}^n$, the subspace Rv contains the basis vector

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Similarly, show that Rv contains each of the basis vectors e_i.

3. Consider the abelian group with generators e_1, e_2, e_3 and relations

$$\begin{aligned} -2e_1 + e_2 &= 0 \\ e_1 - 2e_2 + e_3 &= 0 \\ e_2 - 2e_3 &= 0 \end{aligned}$$

Write the corresponding matrix A and use it to describe this group as direct sum of cyclic groups. What is the order of this group?

4. Let M be a $\mathbb{Z}$-module (abelian group). Let $T \subset M$ be the subset of elements of finite order (also called torsion elements):

$$T = \{m \in M \mid nm = 0 \text{ for some } n \in \mathbb{Z}, n \neq 0\}$$

   (a) Prove that T is a subgroup.
   (b) Prove that M/T is a free abelian group, i.e. it is isomorphic to $\mathbb{Z}^r$ for some r.

5. Let A be an $n \times n$ integer matrix, and let M_A be defined by (1). Prove that M_A is finite iff $\det A \neq 0$; if it is finite, then $|M_A| = |\det A|$.

[Hint: any invertible integer matrix has determinant ± 1, so left multiplication by an invertible matrix doesn't change $|\det(A)|$.]

6. Let $P, Q \subset \mathbb{R}^n$ be subgroups defined as follows:

Q is the subgroup generated by elements of the form $e_i - e_j$, $i \neq j$. (Here e_i are the standard generators of $\mathbb{Z}^n$: $e_i = (0, \ldots, 1, \ldots 0)$, with 1 in the i^{th} place).

$$P = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum x_i = 0, \quad x_i - x_j \in \mathbb{Z} \quad \forall i, j\}$$

(a) Show that P, Q are free abelian groups of rank $n-1$, by producing a basis (set of free generators) of each of them. [Hint: start with small values of n, e.g. $n = 2$, $n = 3$.]

*(b) (Optional.) Show that $Q \subset P$ and describe the quotient P/Q.