MAT 314: HOMEWORK 11

DUE TH, MAY 4, 2023

Throughout this problem set, F is a field of characteristic zero.

1. Let $f(x) = x^4 + ax^2 + b \in \mathbb{Q}[x]$. Assume that $f(x)$ is irreducible and denote by G its Galois group over $\mathbb{Q}$. Let $\pm\alpha$, $\pm\beta$ be the roots of $f(x)$. Prove the following:
   (a) If $\alpha\beta \in \mathbb{Q}$, then $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is the Klein 4-group.
   (b) If $\alpha\beta \notin \mathbb{Q}$ but $\alpha\beta \in \mathbb{Q}(\alpha^2)$, then $G = \mathbb{Z}_4$
   (c) If $\alpha\beta \notin \mathbb{Q}(\alpha^2)$, then $G = D_8$, the dihedral group.

2. (a) Make a change of variables that reduces equation $x^3 + x^2 - 2x - 1 = 0$ to the form $y^3 + py + q = 0$, with $p, q \in \mathbb{Z}$.
   (b) Use Cardano's formulas to solve this equation

3. Let $\zeta = e^{2\pi i/7} \in \mathbb{Q}$ be the primitive 7th root of unity, and let $\alpha = \zeta + \zeta^{-1}$.
   (a) Show that ζ satisfies a quadratic equation with coefficients in $\mathbb{Q}(\alpha)$.
   (b) Show that α is a root of the polynomial $x^3 + x^2 - 2x - 1$
   (c) Using the previous problem, express ζ is radicals.