MAT 314: HOMEWORK 10

DUE TH, APRIL 20, 2023

Throughout this problem set, F is a field of characteristic zero.

1. Consider the cyclotomic field $\mathbb{Q}(\zeta)$ where ζ is the primitive root of 1 of order 15.

(a) Describe explicitly the Galois group $G = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. Is it cyclic?

(b) Construct a sequence of subgroups

$$G \supset G_1 \supset G_2 \dots$$

such that $|G_i/G_{i+1}| = 2$.

(c) Construct a tower of subfields

$$K_0 = \mathbb{Q} \subset K_1 \subset \cdots \subset K_n = \mathbb{Q}(\zeta)$$

so that $[K_{i+1} : K_i] = 2$. Try to describe these subfields explicitly, by writing the generators for each of K_{i+1} over K_i.

2. Let L be the splitting field of a polynomial $f(x)$ over a field K of characteristic zero, and let $x_1, \dots, x_n \in L$ be all roots of f, so that $f(x) = c(x - x_1) \dots (x - x_n)$. Let $G = \mathrm{Gal}(L/K)$.

(a) Let $D = \prod_{i<j}(x_i - x_j)^2$. Prove that $D \in L^G = K$.

(b) Prove that $K \subset K(\sqrt{D}) \subset L$ and $\mathrm{Gal}(L/K(\sqrt{D})) \subset A_n$ (the alternating group).

(c) Prove that if f is an irreducible cubic polynomial, then:
- if D is a square in K, then $[L : K] = 3$, $G = \mathbb{Z}_3$.
- if D is not a square in K, then $[L : K] = 6$, $G = S_3$. In this case, $L^{\mathbb{Z}_3} = K(\sqrt{D})$, where $\mathbb{Z}_3 = A_3$ is the subgroup in S_3 generated by a 3-cycle.

(It can be shown that up to a constant, D coincides with $R(f) = \gcd(f, f')$.)

3. Let $f(x) = x^3 + px + q$. Let x_1, x_2, x_3 be the roots of $f(x)$ (in the splitting field) and let D be as in the previous problem.

(a) Show that $f'(x_1) = (x_1 - x_2)(x_1 - x_3)$. Deduce from this that $D = -f'(x_1)f'(x_2)f'(x_3)$.

(b) Prove that $D = -4p^3 - 27q^2$.

4. Use the previous two problems to compute the following Galois groups:

(a) of the polynomial $x^3 - 3x + 1$ over $\mathbb{Q}$.

(b) of the polynomial $x^3 - 3x + 3$ over $\mathbb{Q}$.

(c) Of the polynomial $x^3 - 10$ over $\mathbb{Q}(\sqrt{-3})$.