MAT 314: HOMEWORK 1

DUE TH, FEB 2, 2023

Throughout this assignment, R is an arbitrary associative ring with unit (not necessarily commutative). All modules are modules over R. Letter $\mathbb{F}$ always stands for a field. As usual, we call an ideal $I \subset R$ *proper* if $I \neq R$, and similarly for submodules of a module. PID stands for Principal Ideals Domain.

1. In this problem, R is a commutative ring, and for an element $a \in R$, we denote $(a) = Ra \subset R$ – the ideal generated by a.
  (a) Show that $a \in R$ is invertible iff $(a) = R$.
  (b) Show that R is a field iff it has no nonzero proper ideals.
  (c) Show that $(a) \subset (b)$ iff b is a divisor of a, i.e. $a = bc$ for some c.
  (d) Let R be a PID. Show that the ideal (a) is a maximal proper ideal if and only if a is not invertible and irreducible: in any factorization $a = xy$ either x or y is invertible.

2. Let R be a commutative ring, let $I \subset R$ be a proper ideal. Denote by $R' = R/I$ the quotient ring and let $f\colon R \to R'$ be the obvious map.
  (a) Show that for any ideal $K' \subset R/I$, the preimage $f^{-1}(K') \subset R$ is an ideal in R. Show that it gives a bijection:

$$(\text{Ideals } K' \subset R/I) \leftrightarrow (\text{Ideals } K \subset R \text{ which contain } I)$$

  (b) Deduce from this that R/I is a field iff I is a maximal proper ideal.
  (c) Show that if R is a PID, $R/(a)$ is a field iff a is not invertible and irreducible.

3. As discussed in class, a module over the ring $\mathbb{F}[x]$ can be described as a pair (V, A), where V is a vector space over $\mathbb{F}$ and $A\colon V \to V$ is a linear operator.
  (a) Give such a description for the module $M = \mathbb{F}[x]/(x^2 + 2x + 2)$, by constructing an explicit basis in the corresponding vector space V and writing the operator A as a matrix in this basis.
  (b) Show that for $\mathbb{F} = \mathbb{C}$, the module M constructed in the previous part is isomorphic to a direct sum $M_1 \oplus M_2$, where M_1, M_2 are $\mathbb{C}[x]$ modules which are one-dimensional vector spaces over $\mathbb{C}$. [Hint: is A diagonalizable?]. Is the same true when $\mathbb{F} = \mathbb{R}$?

4. Let M be a an R-module and $N \subset M$, an R-submodule. Let $M' = M/N$ and denote by $f\colon M \to M'$ the obvious homomorphism of modules.
  (a) Show that if $K' \subset M'$ is a submodule, then $K = f^{-1}(K') \subset M$ is also a submodule, and M/K is isomorphic to M'/K'.
  (b) Show that the construction of the previous part gives a bijection

$$(\text{Submodules } K' \subset M/N) \leftrightarrow (\text{Submodules } K \subset M \text{ which contain } N)$$

5. Let M be an R-module and $M_1, M_2 \subset M$ be submodules such that $M = M_1 + M_2$, i.e. every element in M can be written as a sum $m = m_1 + m_2$, $m_1 \in M_1$, $m_2 \in M_2$ (possibly not uniquely). Prove that then one has an isomorphism
$M \simeq (M_1 \oplus M_2)/(M_1 \cap M_2)$.