MAT 314: FINAL EXAM

MAY 21, 2019

Name:	**ID #**:

	1	2	3	4	5	**Total**
Grade						

1. Let M be the module over the ring $R = \mathbb{C}[x]$, which is 4-dimensional as a vector space over $\mathbb{C}$, and with action of x given by the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Write M in the form

$$M = \bigoplus_i \mathbb{C}[x]/(p_i^{n_i}),$$

where $p_i \in \mathbb{C}[x]$ are irreducible.

2. Let G be the abelian group generated by three generators x_1, x_2, x_3 with relations

$$3x_1 + x_2 + 2x_3 = 0$$
$$x_1 + x_2 + x_3 = 0$$
$$2x_1 + 3x_2 + 6x_3 = 0$$

Describe G as a product of cyclic groups.

3. Let D_n be the dihedral group, i.e. the group of all symmetries of a regular n-gon. It is known that this group is generated by two elements x (rotation by $2\pi/n$) and y (reflection), with the following relations:

$$x^n = y^2 = 1, \qquad yxy^{-1} = x^{-1}.$$

In parts (a)–(c) below, V is a finite-dimensional complex representation of G; we denote by $\rho(x)$, $\rho(y)$ the action of $x, y \in D_n$ in V.

(a) Show that in any representation V, the operator $\rho(x)$ is diagonalizable. What are the possible eigenvalues of this operator? [Hint: subgroup generated by x is the cyclic group...]

(b) Let $v \in V$ be an eigenvector for $\rho(x)$ with eigenvalue λ. Show that then $v' = \rho(y)v$ is also an eigenvector for $\rho(x)$, with eigenvalue λ^{-1}, and that the subspace V' generated by v, v' is a subrepresentation.

(c) Classify all irreducible representations of D_n.

Continued on back

4. Let $\alpha = \sqrt{10 + 5\sqrt{2}}$.

(a) Find the degree of α over $\mathbb{Q}$ and the minimal polynomial.

(b) Determine the Galois group of that minimal polynomial over $\mathbb{Q}$.

(c) Can α be written in the form $\sqrt{a} + \sqrt{b}$, with $a, b \in \mathbb{Q}$? [Hint: this would imply $\mathbb{Q}(\alpha) \subset \mathbb{Q}(\sqrt{a}, \sqrt{b})$.]

5. Let L be the splitting field of the polynomial $x^6 + 1$ over $\mathbb{Q}$.

(a) What is the degree $[L : \mathbb{Q}]$?

(b) Describe the Galois group $G = Gal(L/\mathbb{Q})$, both as an abstract group (e.g. by generators and relations) and as a subgroup in S_6.

(c) Now let K be the splitting field of the polynomial $x^2 + 1$ over $\mathbb{Q}$. Prove that then $\mathbb{Q} \subset K \subset L$, and describe the corresponding subgroup $H \subset G$ as prescribed by the main theorem of Galois theory.