MAT 312: HOMEWORK 8

1. Consider the set $\mathrm{Aff}(\mathbb{R}^n)$ of all functions $\mathbb{R}^n \to \mathbb{R}^n$ which have the form

$$\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}, \quad \mathbf{x} \in \mathbb{R}^n$$

where A is an invertible $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$. Thus, every such function is described by a pair $(A, \mathbf{b})$

Such functions are called *affine transformations*.

(a) Show that $\mathrm{Aff}(\mathbb{R}^n)$ is a group. Write explicitly the group law: $(A_1, \mathbf{b}_1) \circ (A_2, \mathbf{b}_2) = ?$.

(b) Let $H_1 = \{(A, \mathbf{b}) \mid A = I_n\} \subset \mathrm{Aff}(\mathbb{R}^n)$, where I_n is the $n \times n$ identity matrix; thus, H_1 consists of transformations $\mathbf{x} \mapsto \mathbf{x} + \mathbf{b}$. Show that H_1 is a subgroup of $\mathrm{Aff}(\mathbb{R}^n)$.

(c) Let $H_2 = \{(A, \mathbf{b}) \mid \mathbf{b} = 0\} \subset \mathrm{Aff}(\mathbb{R}^n)$; thus, H_2 consists of transformations $\mathbf{x} \mapsto A\mathbf{x}$. Show that H_2 is also a subgroup of $\mathrm{Aff}(\mathbb{R}^n)$.

(d) Which of the subgroups H_1, H_2 is normal? can you describe the quotient group $\mathrm{Aff}(\mathbb{R}^n)/H$ for that subgroup?

2. Let $A_4 \subset S_4$ be the group of even permutations, and let $H \subset A_4$ be the subgroup generated by elements $x = (12)(34)$, $y = (13)(24)$, i.e. the subgroup consisting of all permutations you can get from x, y by taking products and inverses.

(a) Show that $x^2 = e$, $y^2 = e$, $xy = yx$.

(b) Show that H has 4 elements, but is not isomorphic to $\mathbb{Z}_4$.

(c) Prove that H is normal in A_4. Can you describe the quotient?

3. Let $H \subset G$ be a subgroup such that $|G/H| = 2$. Prove that H is normal.

4. Describe all subgroups of symmetric group S_3. For each of them, say whether it is normal; if it is, describe the quotient S_3/H.

5. An element z in a group G is called *central* if $zx = xz$ for any $x \in G$. Denote by $Z(G)$ the set of all central elements in G. (this is called the *center* of G).

(a) Show that $Z(G)$ is a normal subgroup of G.

(b) Compute the center of the group D_n of all symmetries of regular n-gon. [Hint: you can use the fact mentioned in class: if R_α is rotation by angle α around the origin, and s is reflection across a line going through the origin, then $sR_\alpha s^{-1} = R_{-\alpha}$].