MAT 312: HOMEWORK 11

Throughout this problem set, $\mathbb{F}$ is an arbitrary field.

1. Textbook, p. 278, problem 3

2. Textbook, p. 278, problem 4

3. We say that number $a \in \mathbb{R}$ is a root of multiplicity $m \geq 1$ for a polynomial $p(x)$ if m is the highest power of $(x-a)$ which is a divisor of p: $p(x) = (x-a)^m q(x)$, and $q(x)$ is not divisible by $(x-a)$.

Show that if the polynomial $p(x)$ has roots $a_1, \dots, a_k$, with multiplicities $m_1, \dots, m_k$ respectively, then

$$\sum m_i \leq \deg p.$$

In other words, total number of roots (counted with multiplicities) can not be more than degree of p.

4. Define the operator $D \colon \mathbb{F}[x] \to \mathbb{F}[x]$ by

$$\begin{aligned} D(c) &= 0, c \in \mathbb{F} (\text{considered as polynomial of degree } 0) \\ D(x^n) &= nx^{n-1}, \ n \geq 1 \\ D(f+g) &= D(f) + D(g) \\ D(cf) &= cD(f) \end{aligned}$$

(as you can see, it is the usual operator of differentiation, only defined purely algebraically, wihtout any reference to limits).

(a) Show (without using results from calculus) that $D(fg) = (Df)g + f(Dg)$.

(b) Show that for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, if $\deg f = n$, then $\deg Df = n-1$.

(c) Show that for $\mathbb{F} = \mathbb{Z}_p$, it is not necessarily so. (Hint: consider $f(x) = x^p - 1$.)

(d) Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let $a \in \mathbb{F}$ be a root of $p(x)$ of multiplicity $m > 1$. Show that then, a is a root of Df of multiplicity $m-1$. In particular, f and Df have a common divisor $(x-a)^{m-1}$.

(e) Derive from the previous part that if $\gcd(f, Df) = 1$, then f has no roots of multiplicity greater than 1. In particular, any irreducible polynomial has no multiple roots.