1. $y^{(3)} - 2y'' + 2y' = 6\sin(3x)$

(a) Complementary eqn: $y^{(3)} - 2y'' + 2y' = 0$

Charact. equation:

$$r^3 - 2r^2 + 2r = 0$$

$$r(r^2 - 2r + 2) = 0$$

$$r = 0 \quad \text{or} \quad r^2 - 2r + 2 = 0$$

$$(r-1)^2 + 1 = 0$$

$$(r-1)^2 = -1$$

$$r = 1 \pm i$$

thus, $y_c = C_1 + C_2 \cdot e^x \sin(x) + C_3\, e^x \cdot \cos(x)$

(b) Particular solution: look for solution in the form $y_p = A\sin(3x) + B\cos(3x)$

$$y_p' = 3A\cos(3x) - 3B\sin(3x)$$

$$y_p'' = -9A\sin(3x) - 9B\cos(3x)$$

$$y_p''' = -27A\cos(3x) + 27B\sin(3x)$$

$$y_p''' - 2y_p'' + 2y_p' = -27A\cos(3x) + 27B\sin(3x) + 18A\sin(3x) + 18B\cos(3x) + 6A\cos(3x) - 6B\sin(3x)$$

$$= \cos(3x)\cdot(-27A + 18B + 6A) + \sin(3x)\cdot(18A + 27B - 6B)$$

$$= \cos(3x)\,(-21A + 18B) + \sin(3x)(18A + 21B)$$

$$\begin{cases} -21A + 18B = 0 \\ 18A + 21B = 6 \end{cases}$$

$$B = \frac{21}{18}A = \frac{7}{6}A$$

$$18A + \frac{21 \cdot 7}{6}A = 6$$

$$\frac{85}{2}A = 6$$

$$A = \frac{12}{85}, \quad B = \frac{7}{6}A = \frac{14}{85}$$

$$y_p = \frac{12}{85}\sin(3x) + \frac{14}{85}\cos(3x)$$

(c) Full solution

$$y = y_p + y_c = \frac{12}{85}\sin(3x) + \frac{14}{85}\cos(3x) + C_1 + C_2 e^x \sin(x)$$
$$+ C_3 e^x \cos(x)$$

2 $y'' - 3y' + 2y = x + e^x, \quad y(0) = 0, \quad y'(0) = 1$

- Complementary:

$$y'' - 3y' + 2y = 0$$

$$r^2 - 3r + 2 = 0$$

$$(r-2)(r-1) = 0$$

$$r = 1, \quad r = 2$$

$$y = c_1 e^x + c_2 e^{2x}$$

- particular solution of

$$y'' - 3y' + 2y = x$$

Look for solution $\quad y = Ax + B$

$$y' = A$$

$$y'' = 0$$

$$y'' - 3y' + 2y = -3A + 2Ax + 2B = x$$

$$\begin{cases} 2B - 3A = 0 \\ 2A = 1 \end{cases}$$

$$A = \frac{1}{2} \qquad B = \frac{3}{2}A = \frac{3}{4}$$

$$y = \frac{1}{2}x + \frac{3}{4}$$

- particular solution of

$$y'' - 3y' + 2y = e^x$$

Since $r = 1$ is a root of char. eqn, we look for solution of the form

$$y = A \cdot x e^x$$

Then $y' = A\,(x+1)\,e^x$

$y'' = A\,(x+2)\,e^x$

$$y'' - 3y' + 2y = A\left((x+2) - 3(x+1) + 2x\right)e^x$$

$$= A \cdot (-1) \cdot e^x$$

$$A \cdot (-1) = 1$$

$$A = -1$$

$$y = -x e^x$$

- ~~Thus~~ Full solution

$$y = \left(\frac{1}{2}x + \frac{3}{4}\right) - x e^x + c_1 e^x + c_2 e^{2x}$$

$$y(0) = 0 \Rightarrow \frac{3}{4} + c_1 + c_2 = 0$$

$$y'(0) = 1 \Rightarrow \frac{1}{2} - 1 + c_1 + 2c_2 = 1$$

$$\begin{cases} c_1 + c_2 = -\frac{3}{4} \\ c_1 + 2c_2 = 1\frac{1}{2} \end{cases}$$

$$\begin{cases} c_2 = 2\frac{1}{4} = \frac{9}{4} \\ c_1 = -3 \end{cases}$$

$$y = \left(\frac{1}{2}x + \frac{3}{4}\right) - x e^x - 3e^x + \frac{9}{4}e^{2x}$$

3 $y'' + y = \frac{1}{\cos(x)}$

First, solve $y'' + y = 0$.

$$r^2 + 1 = 0$$
$$r = \pm i$$
$$y = C_1 \cdot \sin(x) + C_2 \cdot \cos(x)$$

Now, look for solution

$$y = u(x)\sin(x) + v(x)\cdot\cos(x)$$

$$y' = u'\cdot\sin(x) + u\cdot\cos(x) + v'\cdot\cos(x) - v\cdot\sin(x)$$

Let us require

$$u'\cdot\sin(x) + v'\cdot\cos(x) = 0$$

Then $y' = u\cdot\cos(x) - v\cdot\sin(x)$

$$y'' = u'\cdot\cos(x) - u\cdot\sin(x) - v'\cdot\sin(x) - v\cdot\cos(x)$$
$$= (u' - v)\cos(x) - (u + v')\sin(x)$$

$$y'' + y = u'\cdot\cos(x) - v'\cdot\sin(x)$$

Thus:

$$u'\cdot\sin(x) + v'\cdot\cos(x) = 0$$
$$u'\cdot\cos(x) - v'\cdot\sin(x) = \frac{1}{\cos(x)}$$

Multiply 1st eqn by $\sin(x)$, 2nd by $\cos(x)$ and add:

$$\begin{cases} u' = 1 \\ v' = -\dfrac{\sin(x)}{\cos(x)} \cdot u' = -\tan(x) \end{cases}$$

$u = x$

$$v = -\int \tan(x)\,dx = -\int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{du}{u} = \ln(u)$$

$$\left. u = \cos(x),\ du = -\sin(x)\,dx \right| = \ln \cos(x)$$

$$y = x \cdot \sin(x) + \ln(\cos(x)) \cdot \cos(x)$$

4 $y^{(4)} - 2y'' + y = 0$

$r^4 - 2r^2 + 1 = 0$

$(r^2-1)^2 = 0$

$(r-1)^2 \cdot (r+1)^2 = 0$

$r = 1$ (multtplicity 2),

$r = 2$ (multiplicity 2)

$y = (C_1 + C_2 x)e^x + (C_3 + C_4 x)e^{2x}$

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$$x'' = 6x + 2y$$
$$y'' = 3x + 7y$$

$$y = \frac{x'' - 6x}{2}$$

$$y'' = \frac{x^{(4)} - 6x''}{2} = 3x + 7\left(\frac{x'' - 6x}{2}\right)$$

$$x^{(4)} - 6x'' = 6x + 7x'' - 42x$$

$$x^{(4)} - 13x'' + 36x = 0$$

$$r^4 - 13r^2 + 36 = 0$$

$$(r^2 - 9)(r^2 - 4) = 0$$

$$\cancel{r4} \quad r = \pm 3, \ \pm 2$$

$$x = C_1 e^{3t} + C_2 e^{-3t} + C_3 e^{2t} + C_4 e^{-2t}$$

$$y = C_1 \cdot \frac{3}{2} e^{3t} + C_2 \cdot \frac{3}{2} e^{-3t} + C_3 \cdot e^{2t} - C_4 e^{-2t}$$

⑥

$$x'' = -x + y$$
$$y' = -3x - x' + 3y$$

Let $x_1 = x$, $x_2 = x'$, $x_3 = y$

$$x_1' = x_2$$
$$x_2' = -x_1 + x_3$$
$$x_3' = -3x_1 - x_2 + 3x_3$$

Writing $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ -3 & -1 & 3 \end{bmatrix} \vec{x}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ -3 & -1 & 3-\lambda \end{bmatrix}$$

$$= \lambda^2(3-\lambda) - 3 + (3-\lambda) - \lambda = \lambda^2(3-\lambda) - 2\lambda$$
$$= -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda^2 - 3\lambda + 2)$$
$$= -\lambda(\lambda - 1)(\lambda - 2)$$

Eigenvalues $\lambda = 1$, $\lambda = 2$, $\lambda = 0$

$\lambda = 0$: eigenvector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (computation skipped)

$\lambda = 1$: eigenvector $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\lambda = 2$: eigenvector $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

General solution:

$$\vec{X} = C_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_2 \cdot e^{t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_3 e^{2t} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

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$$x' = \begin{pmatrix} 1 & -5 \\ 1 & 5 \end{pmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Eigenvalues:

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -5 \\ 1 & 5-\lambda \end{pmatrix}$$

$$= (1-\lambda)(5-\lambda) + 5 =$$

$$= \lambda^2 - 6\lambda + 5 + 5 = \lambda^2 - 6\lambda + 10$$

$$= (\lambda - 3)^2 + 1$$

$$\lambda = 3 \pm i$$

For $\lambda = 3 + i$: eigenvector

$$\begin{pmatrix} -2-i & -5 \\ 1 & 2-i \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$a + (2-i) b = 0$$

$$\cancel{a = 2+i, \; b = 1}$$

$$a = 2 - i, \quad b = -1$$

$$v = \begin{bmatrix} 2-i \\ -1 \end{bmatrix}$$

Solution:

$$x = e^{(3+i)t} \cdot \begin{bmatrix} 2-i \\ -1 \end{bmatrix} = e^{3t} \left(\cos(t) + i \sin(t) \right) \begin{bmatrix} 2-i \\ -1 \end{bmatrix}$$

$$= e^{3t} \begin{bmatrix} 2\cos(t) + \sin(t) - i\cos(t) + 2i\sin(t) \\ -\cos(t) - i\sin(t) \end{bmatrix}$$

$$= e^{3t}\begin{bmatrix} 2\cos(t)+\sin(t) \\ -\cos(t) \end{bmatrix} + i\, e^{3t}\begin{bmatrix} 2\sin(t)-\cos(t) \\ -\sin(t) \end{bmatrix}$$ 12

Thus, general solution is

$$X = C_1 \cdot e^{3t} \cdot \begin{bmatrix} 2\cos(t)+\sin(t) \\ -\cos(t) \end{bmatrix} + C_2\, e^{3t}\begin{bmatrix} 2\sin(t)-\cos(t) \\ -\sin(t) \end{bmatrix}$$

~~x(0) = [0]~~ $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives

$$C_1 \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C_1 = 0, \quad C_2 = -1$$

$$X = -e^{3t}\begin{bmatrix} 2\sin(t)-\cos(t) \\ -\sin(t) \end{bmatrix}$$