Practice midterm 1 solutions.

Problem 1

(i) $y' + 2xy = x \quad , \quad y(0) = -2$

$y' = x - 2xy$

$y' = x(1-2y)$

$\frac{dy}{1-2y} = x\,dx$

$\int \frac{dy}{1-2y} = \int x\,dx + C$

$-\frac{1}{2} \ln |1-2y| = \frac{1}{2} x^2 + C$

$\ln |1-2y| = -x^2 + C$

$1 - 2y = \pm e^{-x^2 + C}$

$1 - 2y = C_1 \cdot e^{-x^2}$

$y = \frac{1 - C_1 \cdot e^{-x^2}}{2}$

$y(0) = -2 \Rightarrow \quad \frac{1 - C_1}{2} = -2$

$1 - C_1 = -4$

$C_1 = 5$

$\boxed{y = \frac{1 - 5e^{-x^2}}{2}}$

(ii) $t(t+y)\,y' = y(t-y)$

$$\frac{dy}{dt} = \frac{y(t-y)}{t(t+y)}$$

Homogeneous equation, use substitution

$$v = \frac{y}{t} \qquad y = tv \qquad y' = v + tv'$$

$$tv' + v = \frac{tv(t-tv)}{t(t+tv)} = \frac{v(1-v)}{1+v}$$

$$tv' = \frac{v(1-v)}{1+v} - v = \frac{v(1-v) - v(1+v)}{1+v} = \frac{-2v^2}{1+v}$$

$$\frac{1+v}{v^2}\,dv = -2\,\frac{dt}{t}$$

$$\int\left(\frac{1}{v^2}+\frac{1}{v}\right)dv = -2\int\frac{dt}{t} + C$$

$$-\frac{1}{v} + \ln|v| = -2\ln|t| + C$$

$$e^{-\frac{1}{v}} \cdot v = e^{-2\ln|t| + C} = C_1 \cdot t^{-2}$$

$$v \cdot e^{-1/v} = C_1 \cdot t^{-2}$$

At $t=1$, $v = \frac{y}{1} = 1$, so $C_1 = e^{-1}$

$$\boxed{v \cdot e^{-1/v} = \frac{t^{-2}}{e}} \qquad y = tv$$

In this case, we can't explicitly solve for v (or y), so the above relation is the best we can do.

(iii) $3xy^2y' = 3x^4 + y^3, \quad y(1) = 1$

$$3xy^2 \cdot y' - y^3 = 3x^4$$

$$y' - \frac{y}{3x} = \frac{x^3}{y^2}$$

$$y' - \frac{y}{3x} = x^3 \cdot y^{-2}$$

This is Bernoulli equation, with $n = -2$.

Use substitution $v = y^{1-n} = y^3$

$$v' = 3y^2 \cdot y'$$

so original equation can be rewritten as

$$x \cdot v' = 3x^4 + v$$

$$xv' - v = 3x^4$$

$$v' - \frac{v}{x} = 3x^3$$

Using integrating factor $\rho = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$:

$$\left(\frac{1}{x}v' - \frac{v}{x^2}\right) = 3x^2$$

$$\cancel{\left(\frac{v}{x}\right)'} \left(\frac{v}{x}\right)' = 3x^2$$

$$\frac{v}{x} = x^3 + C$$

$$v = x^4 + Cx$$

$$y = \sqrt[3]{x^4 + Cx}$$

$$y(1) = 1 \Rightarrow y = \sqrt[3]{x^4} = x^{4/3}$$

Problem 2

(i) $\frac{dx}{dt} = k \cdot x(M-x)$

$M = 10$ (hundreds)

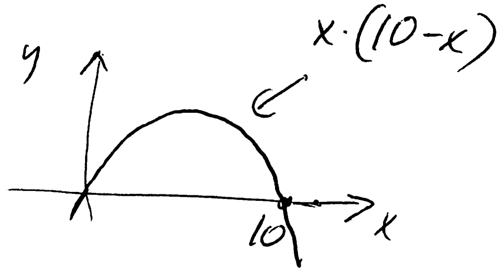
$k = 1$ [We assume that this is what "birth rate" refers to, because otherwise it makes no sense - we are not given death rate]

$\frac{dx}{dt} = x(10-x)$

(ii) Equilibrium points:

$x(10-x) = 0$

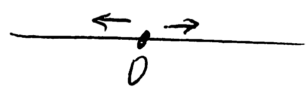
$x = 0$ or $x = 10$



For $x=0$: if x is close to 0, >0 then $x(10-x) > 0$

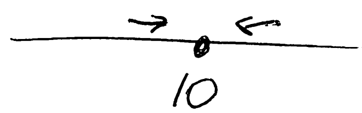
if is <0, $x(10-x) < 0$

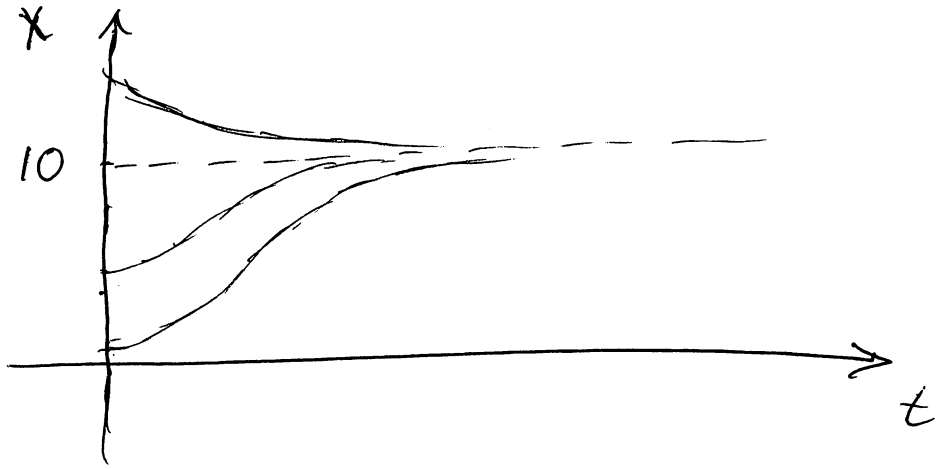
so $x = 0$ is unstable



Similarly, for $x=10$:

so $x=10$ is stable





(iii)

$$\frac{dx}{dt} = x(10-x) - \frac{h}{100}$$

(recall that x is population in <u>hundreds</u>)

Thus, righ-hand side is

$$x(10-x) - \frac{h}{100} = -x^2 + 10x - \frac{h}{100} = -\left(x^2 - 10x + \frac{h}{100}\right)$$

$D = 10^2 - 4 \cdot \frac{h}{100}$ – discriminant

If $D \geq 0$, $x_{1,2} = \frac{10 \pm \sqrt{D}}{2}$; for $D < 0$, no roots

~~So, for h~~

$$D \geq 0 \iff 100 - \frac{h}{25} \geq 0 \iff h \leq 2500$$

Thus, for $h < 2500$ there are 2 equilibrium pts

$h = 2500$ one equilibrium pt

$h > 2500$ no equilibrium pts.

Critical value is $h = 2500$: for larger h, all solutions will go to zero.

For this value of h, we have one equilibrium point, $x_{1,2} = \frac{10 \pm \sqrt{0}}{2} = 5$, i.e. 500 fish.

Problem 3

(i) Equation: $a = \frac{F}{m} = -\frac{GM}{r^2}$

$$\frac{dv}{dt} = -\frac{GM}{r^2}$$

$$\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} = v \cdot \frac{dv}{dt}, \text{ so}$$

$$v \cdot \frac{dv}{dr} = -\frac{GM}{r^2}$$

$$v \cdot dv = -GM \cdot \frac{dr}{r^2}$$

$$\frac{1}{2} v^2 = \frac{GM}{r} + C$$

$$v^2 = \frac{2GM}{r} + C$$

since $v(R) = v_0$, we get $v_0^2 = \frac{2GM}{R} + C$,

$$C = v_0^2 - \frac{2GM}{R},$$

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right)$$ (This formula is also in textbook)

$v_{escape} = \sqrt{\frac{2GM}{R}}$, so $v_0 = \frac{1}{2} v_{escape}$

$$v_0^2 = \frac{1}{4} \cdot \frac{2GM}{R}$$

$$v^2 = \frac{1}{4}\frac{2GM}{R} - \frac{2GM}{R} + \frac{2GM}{r} = -\frac{3}{4}\frac{2GM}{R} + \frac{2GM}{r}$$

$$= 2GM\left(\frac{1}{r} - \frac{3}{4R}\right)$$

$$v = \sqrt{2GM\left(\frac{1}{r} - \frac{3}{4R}\right)}$$

(ii) Maximal height is when $v=0$,

so $$\frac{1}{r}-\frac{3}{4R}=0$$

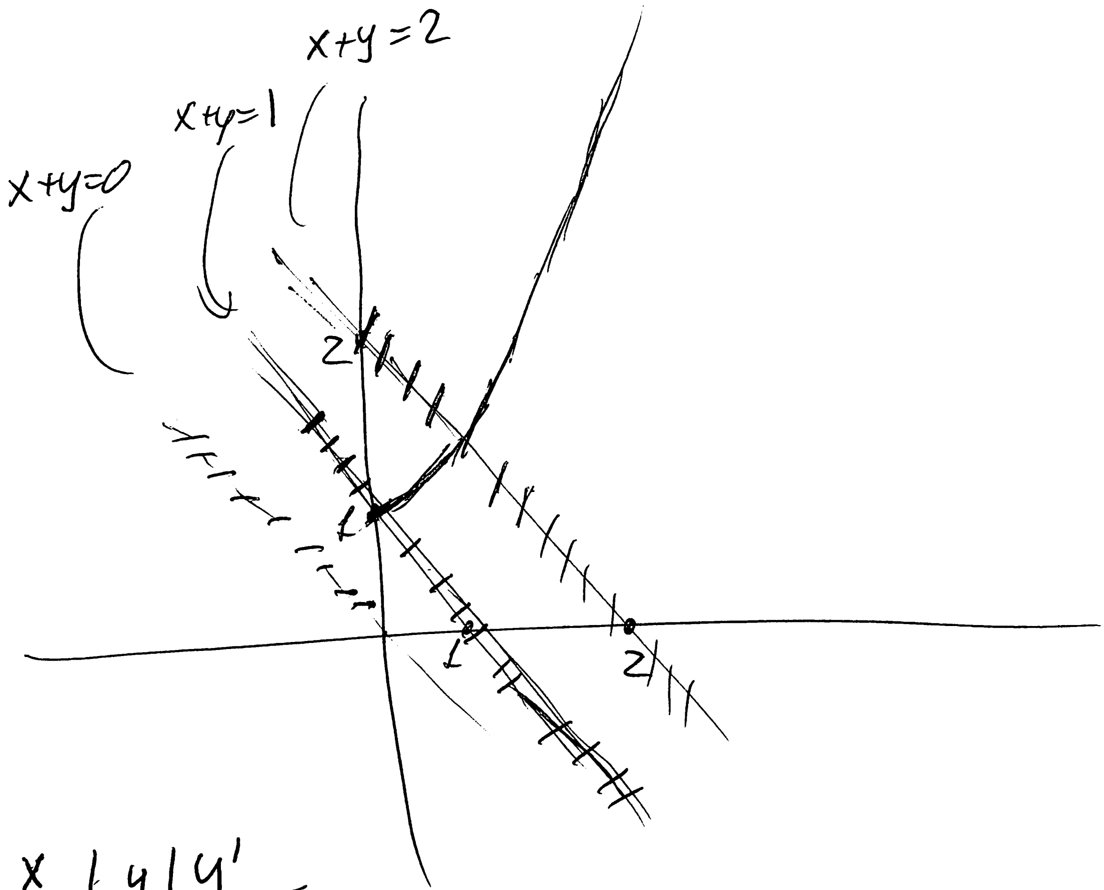
$$\frac{1}{r}=\frac{3}{4R}$$

$$r=\frac{4}{3}R$$

~~dis~~ height $= r-R=\frac{1}{3}R$

Problem 4

(i)



(ii)

x	y	y'
0	1	1
0.5	1.5	2
1.0	2.5	3.5
1.5	$2.5 + 0.5 \cdot 3.5 = 4.25$	

Problem 5

$$y'' + 4y' - 5y = 0$$

(i) Characteristic equation

$$r^2 + 4r - 5 = 0$$

~~roots~~ $(r+5)(r-1) = 0$

roots: $-5, 1$

Corresponding solutions: $y_1 = e^{-5x}, \quad y_2 = e^x$

(ii)

$$W = \begin{vmatrix} e^{-5x} & e^x \\ -5e^{-5x} & e^x \end{vmatrix} = e^{-5x} \cdot e^x - (-5e^{-5x}) \cdot e^x$$

$$= e^{-4x} + 5e^{-4x}$$

$$= 6 \cdot e^{-4x} \neq 0.$$

(iii) General solution:

$$y = c_1 \cdot e^{-5x} + c_2 \cdot e^x$$

$$y(0) = 5 \Rightarrow c_1 + c_2 = 5$$

$$y'(0) = 7 \Rightarrow -5c_1 + c_2 = 7$$

Solving this system gives

$$c_1 = -\frac{1}{3} \qquad c_2 = \frac{16}{3}$$

$$y = -\frac{1}{3} e^{-5x} + \frac{16}{3} e^x$$