MAT 303: MIDTERM REVIEW AND PRACTICE PROBLEMS

SPRING 2024

Midterm rules

Midterm will be on Mon, Feb 19, 2024 in class. It will be a closed book exam: no books, notes, or calculators. There will be no more than 4 problems (but some of them might be multi-part).

Please write full solutions (not just answers!). Solutions should be written so that they are easy to read, understand and follow. Anything that is not part of your final solution (e.g. preliminary computations you later abandoned) should be crossed out.

Sections covered

Exam covers the following textbook sections:
1.1 –1.6, 2.1 – 2.4, 3.1
with the following exceptions:

(1) In 2.2, we haven't discussed section about bifurcation
(2) In 2.3, we haven't discussed case when resistance is proportional to square of air velocity.
(3) In 2.4, we haven't talked about estimating the error of Euler method

Practice problems

Warning: this is just one possible selection of problems — there is no promise that the actual exam will contain the exact same types of problems!

This exam set of practice problems is longer than the actual exam, to give you more opportunities to practice.

Problem 1. Solve each of the following initial value problems. Include **all steps** in your solution. No credit is given for only stating the solution. [15 pts]

(i) $y' + 2xy = x, \quad y(0) = -2.$ [5 pts]

(ii) $t(t+y)y' = y(t-y), \quad y(1) = 1.$ [5 pts]

(iii) $3xy^2y' = 3x^4 + y^3, \quad y(1) = 1.$ [5 pts]

Problem 2. Let $x(t)$ denote the population of fish **in hundreds** in a lake at time t (in months). Suppose that the limiting population of the lake is 1000 fish and that the birth rate is 100 fish per 100 fish per month. [20 pts]

(i) Assuming that population is logistic, write a differential equation for $x(t)$. [5 pts]

(ii) Determine the equilibrium points of the model and classify each of them as stable or unstable. Draw several solutions of the differential equation corresponding to different initial populations. [5 pts]

(iii) Suppose that fish is harvested at a rate of h fish per month. Determine the dependence of the number of critical points on the parameter h. [5 pts]

(iv) What is the maximum value of h so that harvesting is viable for this lake? With that rate of harvesting, what will the population of the lake approximately be after a very long period of time? [5 pts]

Problem 3. In this problem, we consider the motion of a projectile launched vertically from the surface of Earth. We ignore air resistance, only taking into account gravitational force $F = -\frac{GMm}{r^2}$, where M is the mass of Earth, m is the mass of projectile, G is the universal gravitational constant, and r is the distance from the projectile to the center of Earth. We will also denote by R the radius of Earth. [10 pts]

Assume that initial velocity v_0 of the projectile is half of the escape velocity.

(i) Write the formula for velocity as function of r. [5 pts]

(ii) What is the maximal height above earth surface that this projectile will reach? [The answer shoudl be written as fraction of R.] [5 pts]

Problem 4. Consider the initial value problem [10 pts]

$$y' = x + y, \quad y(0) = 1.$$

(i) Sketch a slope field for the differential equation. On the slope field draw the solution of the initial value problem. [5 pts]

(ii) Use Euler's method with step size 0.5 to approximate $y(1.5)$. [5 pts]

Problem 5. Consider the differential equation [15 pts]

$$y'' + 4y - 5y = 0.$$

[5 pts]	(i)	Find two linearly independent solutions of this equation.
[5 pts]	(ii)	Compute the Wronskian of these two solutions and verify that it is non-zero.
[5 pts]	(iii)	FInd a solution of this equation that satisfies initial conditions $y(0) = 5, y'(0) = 7$.