MAT 127, MIDTERM 2
PRACTICE PROBLEMS

The midterm covers chapters 7.1-7.3 and 8.8 in the textbook. The actual exam will contain 5 problems (some multipart), so it will be shorter than this practice exam.

1. Calculate the second degree Taylor polynomial $T_2(x)$ about a for the following functions.

(a) $\sin(x^2)$ where $a = \sqrt{\pi}$.

Answer: We have: $\frac{d}{dx}\sin(x^2) = 2x\cos(x^2)$ and $\frac{d^2}{dx^2}\sin(x^2) = 2\cos(x^2) - 4x^2\sin(x^2)$. So:

$$T_2(x) = 0 + 2\sqrt{\pi}.(-1)(x - \sqrt{\pi} + \frac{1}{2!}(2.(-1) - 2\sqrt{\pi}^2.0)(x - \sqrt{\pi})^2.$$

$$T_2(x) = -2\sqrt{\pi}(x - \sqrt{\pi}) - (x - \sqrt{\pi})^2.$$

(b) $\arccos(x)$ where $a = 1/2$.

Answer:
We have: $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$ and $\frac{d^2}{dx^2}\arccos(x) = -\frac{x}{(1-x^2)^{\frac{3}{2}}}$.
So:

$$T_2(x) = \frac{\pi}{3} - \frac{1}{\sqrt{1 - (\frac{1}{2})^2}}(x - \frac{1}{2}) - \frac{1}{2!}\frac{\frac{1}{2}}{(1 - (\frac{1}{2})^2)^{\frac{3}{2}}}(x - \frac{1}{2})^2.$$

$$T_2(x) = \frac{\pi}{3} - \frac{2}{\sqrt{3}}(x - \frac{1}{2}) - \frac{2}{3\sqrt{3}}(x - \frac{1}{2})^2.$$

(c) x^x around $x = 1$.

Answer:
We have that $x^x = e^{x\ln(x)}$. So $\frac{d}{dx}(x^x) = (\ln(x) + 1)e^{x\ln(x)} = (\ln(x) + 1)x^x$ and $\frac{d^2}{dx^2}(x^x) = (\frac{1}{x} + (\ln(x) + 1)^2)x^x$.
Hence

$$T_2(x) = 1^1 + (\ln(1) + 1)1^1(x - 1) + \frac{1}{2!}(\frac{1}{1} + (\ln(1) + 1)^2)1^1(x - 1)^2.$$

$$T_2(x) = 1 + (x - 1) + (x - 1)^2.$$

2. Using Taylors inequality, how well does $T_2(x)$ (calculated above) approximate $\sin(x^2)$ in the interval $[0, 2\sqrt{\pi}]$?

Answer: Taylors inequality is $|T_2(x) - \sin(x^2)| \leq \frac{M}{3!}|x - \sqrt{\pi}|^3$ where M is greater than or equal to the maximum of $|\frac{d^3}{dx^3}(\sin(x^2))| = |-12x\sin(x^2) - 8x^3\cos(x^2)|$ on the interval $[0, 2\sqrt{\pi}]$. Because $|\sin(x^2)| \leq 1$ and $|\cos(x^2)| \leq 1$, and $|x - \sqrt{\pi}| \leq \sqrt{\pi} < 2$, we have $M \leq 12 \cdot 2 + 8 \cdot 8 = 88$. So $|T_2(x) - \sin(x^2)| \leq \frac{44}{3}|x - \sqrt{\pi}|^3$.

3. Estimate $\cos(0.1)$ to within 2 decimal places. (You may assume that the Maclaurin series for $\sin(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.)

Answer:

First of all we need to find out how many terms we need to calculate using Taylors inequality. We have that: $|T_n(x) - \cos(x)| \leq \frac{M}{(n+1)!}|x|^{n+1}|$ in the interval $-0.1 \leq x \leq 0.1$. Here M is the maximum of $|\frac{d^{n+1}}{dx^{n+1}} \cos(x)|$ on the interval $-0.1 \leq x \leq 0.1$. Since $\frac{d^{n+1}}{dx^{n+1}} \cos(x)$ is equal to one of $\sin(x), \cos(x), -\sin(x), -\cos(x)$, we can assume that $M = 1$. So $|T_n(x) - \cos(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$. So $|T_n(0.1) - \cos(0.1)| \leq \frac{1}{(n+1)!}0.1^{n+1}$. We want to find n large enough so that $|T_n(0.1) - \cos(0.1)| \leq 0.01$. So it is sufficient find n so that: $\frac{1}{(n+1)!}0.1^{n+1} \leq 0.01$. We have: $\frac{1}{1!}0.1 = 0.1 > 0.01$, $\frac{1}{2!}0.1^2 = \frac{1}{2}0.01 < 0.01$. So $n = 2$ will do. So $T_2(0.1) = 1 - 0.1^2 = 0.99$.

4. For which constants b, c is $\sin(bx)e^{cx}$ a solution of

(a)

$$y'' + 4y = 0$$

$y' = b\cos(bx)e^{cx} + c\sin(bx)e^{cx}$. $y'' = -b^2\sin(bx)e^{cx} + bc\cos(bx)e^{cx} + c^2\sin(bx)e^{cx}$.

Answer:

So $y'' + 4y = (-b^2 + c^2 - 4)\sin(bx)e^{cx} + (bc)\cos(bx)e^{cx} = 0$. So $-b^2 + c^2 + 4 = 0$ and $bc = 0$. If $b = 0$ then $c^2 + 4 = 0$ which has no solution. Hence $c = 0$ and $-b^2 + 4 = 0$. Hence $b = \pm 2$.

Therefore $b = \pm 2$ and $c = 0$. Hence $y = \sin(\pm 2x)e^{0x}$. Hence $y = \sin(2x)$ and $y = \sin(-2x)$ are the only solutions of the form $\sin(bx)e^{cx}$.

(b)

$$y'' + 2y' + 4y = 0,$$

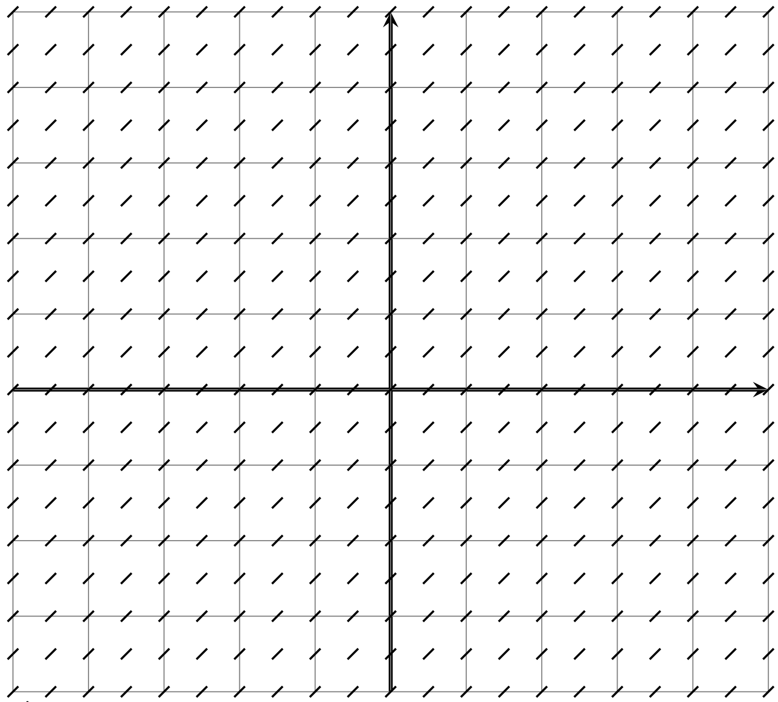
Answer:

Then $y'' + 2y' + 4 = (c^2 - b^2 + 2c + 4)\sin(bx)e^{cx} + (bc + b)\cos(bx)e^{cx} = 0$. Hence $b(c + 1) = 0$ and so $b = 0$ or $c = -1$. If $b = 0$ then $c^2 + 2c + 4 = 0$ which is impossible as this quadratic equation in c has no roots. Hence $c = -1$ and so $1 - b^2 - 2 + 4 = 0$ and so $b^2 = 3$ and so $b = \pm\sqrt{3}$. Hence $c = -1$ and $b = \pm\sqrt{3}$. I.e. $\sin(\pm\sqrt{3}x)e^{-x}$ is a solution.

5. Draw direction fields for the following differential equations.

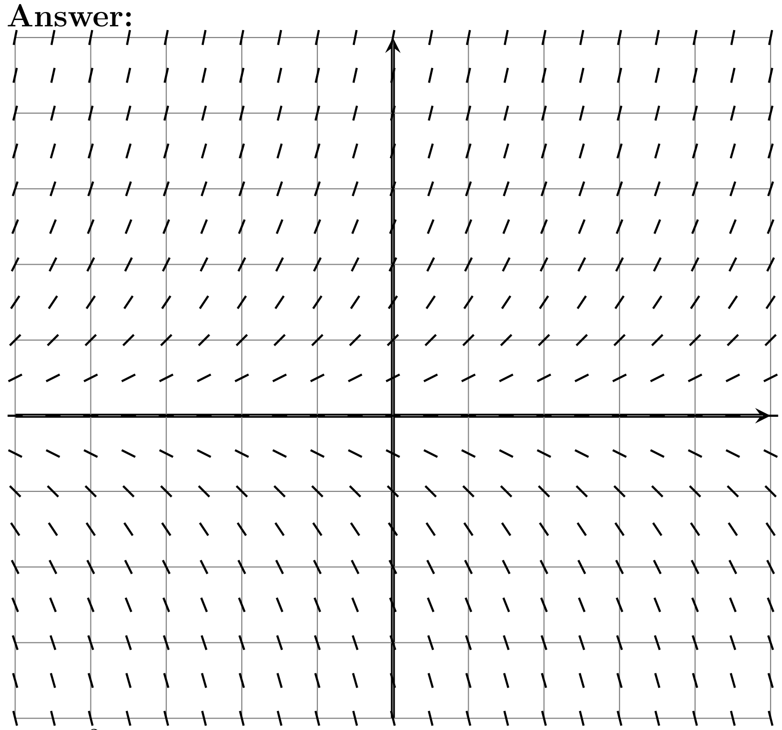
(a) $y' = 1$

Answer:

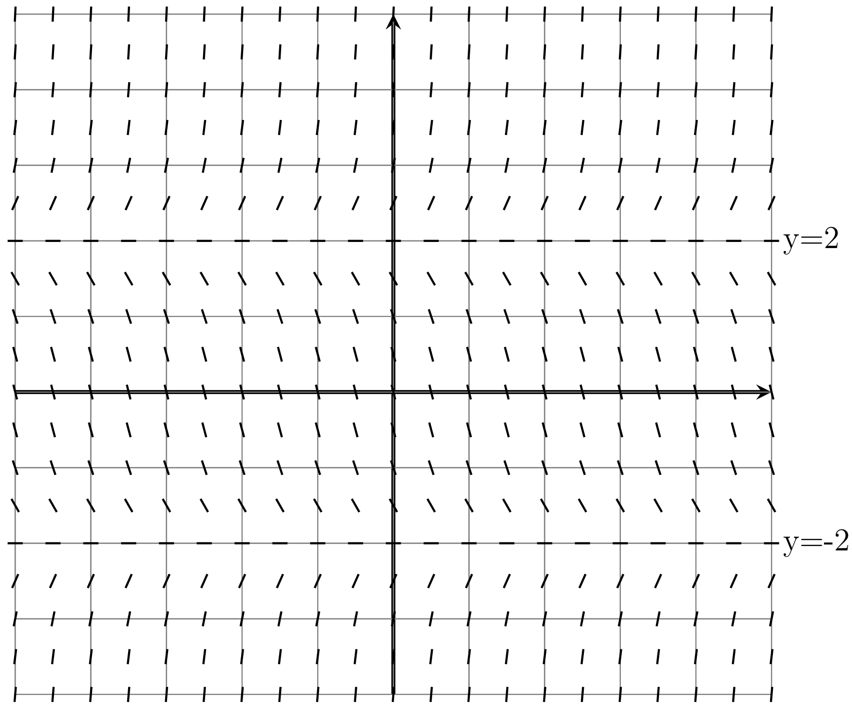


(b) $y' = y$

Answer:

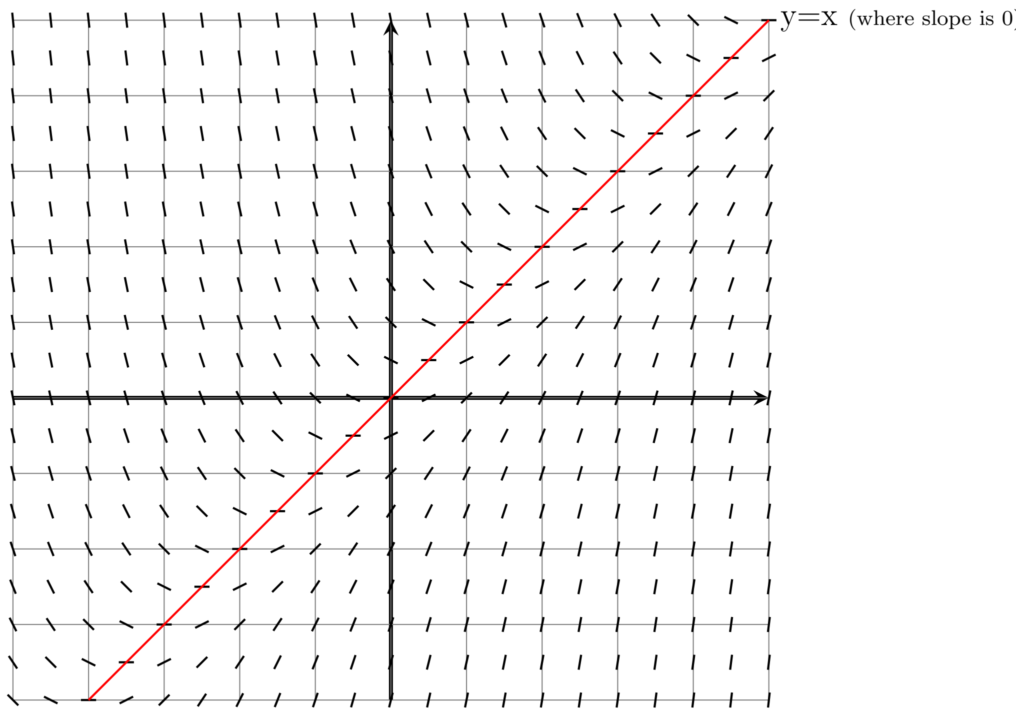


(c) $y' = y^2 - 4$. **Answer:**



(d) $y' = x - y$.

Answer:



6. Use Eulers Method with step size 0.01 to estimate $y(0.02)$ where y satisfies:
(a) $y' = y, \quad y(0) = 1$.

Answer:

$x_0 = 0, x_1 = 0.01, x_2 = 0.02$. So $y_0 = 1, y_1 = 1 + 1 \times 0.01 = 1.01, y_2 = 1.01 + 1.01 \times 0.01 = 1.01 + 0.0101 = 1.0201$.

(b) $y' = xy, \quad y(0) = 3$.

Answer:
$x_0 = 0, x_1 = 0.01, x_2 = 0.02$. So $y_0 = 3, y_1 = 3 + 0 \times 3 \times 0.01 = 3, y_2 = 3 + 0.01 \times 3 \times 0.01 = 3.0003$.

7. Solve the following differential equations:

(a) $y' = y^2, \quad y(0) = 1$.

Answer:
Solve using separation of variables. So $\frac{1}{y^2}y' = 1$ and hence $\int \frac{1}{y^2}dy = \int 1dx$. Hence $-\frac{1}{y} = x + C$. Therefore $y = \frac{1}{C-x}$.
We also have $y(0) = 1$. Hence $\frac{1}{C-0} = 1$ which implies that $C = 1$. Hence $y = \frac{1}{1-x}$.

(b) $y' = 1 + y^2, y(0) = 0$.

Answer: Solve using separation of variables. $\int \frac{1}{1+y^2}dy = \int 1dx = x + C$.
We have that $\int \frac{1}{1+y^2}dy = \arctan(y)$. Hence $\arctan(y) = x + C$. Hence $y = \tan(x + C)$.
We have $y(0) = \tan(0 + C) = 0$ and so $C = 0$. Hence $y = \tan(x)$.

(c) $y' = x - y, \quad y(0) = 1$ (by substituting $u = x - y$).

Answer:
We have $y' = u$. Hence $y' = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}(1 - y') = \frac{dy}{du}(1 - u) = u$. Therefore $\frac{dy}{du} = \frac{u}{1-u}$.
Hence $y = \int \frac{u}{1-u}du$. Substitute $v = 1 - u$ then $dv = -du$.
Hence $y = -\int \frac{1-v}{v}dv = -\int \frac{1}{v} + 1dv = -\ln|v| + v + C = -\ln|1-u| + 1 - u + C = -\ln|1 - x + y| + 1 - x + y + C$.
Hence $y = -\ln|1-x+y| + 1 - x + y + C$. Therefore $0 = -\ln|1-x+y| + 1 - x + C$.
Hence $\ln|1-x+y| = 1-x+C$. Hence $|1-x+y| = e^{1-x+C}$. Hence $1-x+y = Ae^{1-x}$ for some constant A. Therefore $y = Ae^{1-x} + x - 1$.
Now $y(0) = 1$ and hence $1 = Ae + 0 - 1$. Hence $Ae = 2$, so $A = 2e^{-1}$. Hence $y = 2e^{-1}e^{1-x} + x - 1 = 2e^{-1}e^{1}e^{-x} + x - 1 = 2e^{-x} + x - 1$.
Therefore $y = 2e^{-x} + x - 1$ is our solution.