THE HYPERBOLOIDAL AND SPACETIME POSITIVE MASS THEOREM IN ALL DIMENSIONS

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Abstract. Using the recent work of Brendle–Wang on the Riemannian positive mass theorem, we prove the spacetime positive mass theorem for asymptotically flat and asymptotically hyperboloidal initial data sets in arbitrary dimensions.

1. Introduction

The spacetime positive mass theorem $E \geq |P|$ for asymptotically flat (AF) and asymptotically hyperboloidal (AH) initial data sets has a long history. In the AF setting, it was originally established in dimension 3 by Schoen–Yau and Witten [11, 40, 41, 45], and in the AH setting for spin manifolds by Wang, Chruściel–Herzlich, and Zhang [14, 44, 46]. These results were subsequently generalized by various authors, and alternative arguments have been found; see for instance [1, 2, 7, 8, 12, 13, 20, 21, 27, 35, 37–39]. The rigidity statements in both settings also have a long history, with additional important contributions by [3, 16, 25, 26, 28, 29, 31–33].

The results above typically rely on either spinors, minimal surfaces, or suitable variants of these methods. As a consequence, for a long time the positive mass inequality in both the AF and AH settings was restricted to either spin manifolds or manifolds of dimension at most 7, which is the threshold below which minimal hypersurfaces are smooth. It has therefore been an important open problem to determine whether the positive mass theorem holds in full generality. Through a series of refinements of the minimal surface method [4, 9, 10, 24, 36, 43], culminating in the recent breakthrough of Brendle–Wang [5], the AF Riemannian ($k = g$) positive mass inequality has now been established in all dimensions and without topological restrictions.

Using the result of Brendle–Wang [5] together with [20, 25, 26, 31, 37], we prove the spacetime positive mass theorem for asymptotically flat and asymptotically hyperboloidal initial data sets satisfying the dominant energy condition, together with corresponding rigidity statements.

We also note the subsequent paper of Brendle–Wang [6], which uses the capillary regularized Jang equation to prove the spacetime positive energy theorem for asymptotically flat initial data sets in all dimensions n, under an additional positivity assumption on the dominant energy scalar $\mu - |J|_g$.

Our first main result is the following:

Theorem 1.1. *Let* (M^n, g, k), $n \geq 3$ *be a complete asymptotically hyperboloidal*[1] *initial data set satisfying the dominant energy condition*

$$\mu \geq |J|_g.$$

Then its total energy–momentum vector (E, P) *satisfies*

$$E \geq |P|.$$

Moreover, if M^n *is spin or* $k = g$*, then equality holds if and only if* (M^n, g, k) *admits an isometric embedding as a spacelike hypersurface into Minkowski space.*

We point out that the rigidity statement remains open in the non-spin case when $k \neq g$, and it would be interesting to close this gap; see also [25, 31–33]. We also note that there is an asymptotically AdS positive mass theorem [17, 30], which is so far completely unknown in the non-spin setting.

Theorem 1.2. *Let* (M^n, g, k), $n \geq 3$ *be a complete asymptotically flat initial data set satisfying the dominant energy condition*

$$\mu \geq |J|_g.$$

Then its ADM energy–momentum vector $(E_{\mathrm{ADM}}, P_{\mathrm{ADM}})$ *satisfies*

$$E_{\mathrm{ADM}} \geq |P_{\mathrm{ADM}}|.$$

Moreover, equality holds if and only if (M^n, g, k) *admits an isometric embedding as a spacelike hypersurface into a pp-wave spacetime* $(\mathbf{M}^{n+1}, \mathbf{g})$*, where* $\mathbf{M}^{n+1} = \mathbb{R}^{n+1}$ *and*

$$\mathbf{g} = -2\,dt\,du + F\,du^2 + dx_1^2 + \cdots + dx_{n-1}^2,$$

for a t*-independent function* F *satisfying*

$$\Delta_{\mathbb{R}^{n-1}} F(\cdot, u) \leq 0 \qquad \text{for every } u \in \mathbb{R}.$$

Remark 1.3. Planar waves with parallel rays, or pp-waves for short, are Lorentzian manifolds that model gravitational waves. The superharmonicity of F corresponds to the dominant energy condition $\mu \geq |J|_g$. In dimensions $n = 3$ and $n = 4$, there are no nontrivial asymptotically flat pp-waves; that is, in this case one has $F = F(u)$ and $(\mathbf{M}^{n+1}, \mathbf{g})$ is Minkowski space. More generally, (M^n, g, k) embeds in Minkowski space whenever (M^n, g, k) is $C^{\ell,\alpha}_{-q}$-asymptotically flat with $q > n - 1 - \ell - \alpha$. For a detailed overview, we refer to [29] and the references therein. On the other hand, as shown in [26], there are no asymptotically hyperboloidal analogues of pp-waves in any dimension.

[1] Several results cited in this paper, such as [5, 25, 33, 37], require slightly stronger decay than is usually assumed. To improve readability, we absorb this into our notion of asymptotically hyperboloidal and asymptotically flat initial data sets; cf. Remark 2.7.

To prove Theorems 1.1 and 1.2, we solve the Jang equation [20, 37] and show that the results of [5] can be applied to the Jang graph. To this end, we establish a new regularity result for singular Jang graphs.

Theorem 1.4. *Let (M^n, g, k), $n \geq 3$ be either an asymptotically hyperboloidal or asymptotically flat initial data set, and let $\Sigma \subset M^n \times \mathbb{R}$ be a geometric Jang graph obtained as a subsequential limit of the capillarity-regularized Jang graphs. Then Σ is a locally C-almost minimizing boundary in $M^n \times \mathbb{R}$. In particular, there is a closed singular set* $\operatorname{Sing}(\Sigma) \subset \Sigma$ *such that* $\Sigma \setminus \operatorname{Sing}(\Sigma)$ *is smooth and*[2]

$$\dim_{\mathcal{M}} \operatorname{Sing}(\Sigma) \leq n - 7.$$

Another difficulty arises from the possibility that the Jang graph blows up at a possibly singular MOTS. This raises the possibility that singularities accumulate along the cylindrical end, so that the resulting singular set is no longer bounded. We overcome this difficulty by constructing the conformal blow-up piece by piece along the singular set.

Finally, using Theorem 1.2 together with gluing methods, we give an alternative and shorter proof of Theorem 1.1 under additional assumptions.

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2. Preliminaries

2.1. Asymptotically hyperboloidal initial data sets.

Let

$$b = \frac{dr^2}{1+r^2} + r^2 g_{S^{n-1}} \tag{2.1}$$

be the hyperbolic metric on $\mathbb{H}^n$, where $g_{S^{n-1}}$ denotes the standard round metric on S^{n-1}.

Definition 2.1 (Asymptotically hyperboloidal initial data sets)**.** For $n \geq 3$, let (M^n, g, k) be a connected, complete initial data set without boundary. Fix $\ell \geq 6$, $\alpha \in (0,1)$, $\tau \in \left(\frac{n}{2}, n\right)$, and $\tau_0 > 0$. We say that (M^n, g, k) is an *asymptotically hyperboloidal initial data set* of type $(\ell, \alpha, \tau, \tau_0)$ if there exist a compact set $\mathcal{C} \subset M^n$ and a diffeomorphism

$$\phi : M^n \setminus \mathcal{C} \to \mathbb{H}^n \setminus \overline{B} \tag{2.2}$$

such that

$$(\phi_* g - b, \phi_*(k - g)) \in C^{\ell,\alpha}_{-\tau}(\mathbb{H}^n \setminus \overline{B}) \times C^{\ell-1,\alpha}_{-\tau}(\mathbb{H}^n \setminus \overline{B}), \tag{2.3}$$

[2] We point out that, in order to prove Theorems 1.1 and 1.2, the weaker estimate $\dim_{\mathcal{M}} \operatorname{Sing}(\Sigma) < n - 4$ would already suffice.

and

$$\phi_*\mu,\ \phi_* J \in C^{\ell-2,\alpha}_{-n-\tau_0}(\mathbb{H}^n \smallsetminus \overline{B}), \tag{2.4}$$

where the energy density μ and momentum density J are given by

$$\mu := \frac{1}{2}\Big(R_g + (\operatorname{tr}_g k)^2 - |k|_g^2\Big), \qquad J := \operatorname{div}_g\big(k - (\operatorname{tr}_g k) g\big). \tag{2.5}$$

For the definitions of weighted Hölder space on hyperbolic spaces and Euclidean space, see [26, 29, Definition 2.1].

Definition 2.2 (Dominant energy condition)**.** We say that (M^n, g, k) satisfies the *dominant energy condition* if

$$\mu \geq |J|_g \tag{2.6}$$

holds pointwise on M^n.

Definition 2.3 (Wang's asymptotics)**.** Let (M^n, g, k) be asymptotically hyperboloidal of type $(\ell, \alpha, n, \tau_0)$. We say that (M^n, g, k) has *Wang's asymptotics* if, in the chart ϕ,

$$\phi_* g = b + \frac{\mathbf{m}}{r^{n-2}} + O_{\ell,\alpha}(r^{-(n-1)}), \tag{2.7}$$

and

$$\phi_*(k-g)|_{TS_r \times TS_r} = \frac{\mathbf{p}}{r^{n-2}} + O_{\ell-1,\alpha}(r^{-(n-1)}), \tag{2.8}$$

where

$$\mathbf{m}, \mathbf{p} \in C^{\ell,\alpha}\big(S^{n-1}; \operatorname{Sym}^2(T^* S^{n-1})\big). \tag{2.9}$$

Next, let

$$\mathcal{N} := \Big\{ V \in C^\infty(\mathbb{H}^n) : \operatorname{Hess}^b V = V b \Big\}. \tag{2.10}$$

Then

$$\mathcal{N} = \operatorname{span}\{V^{(0)}, V^{(1)}, \ldots, V^{(n)}\}, \tag{2.11}$$

where

$$V^{(0)} = \sqrt{1 + r^2}, \qquad V^{(i)} = x^i, \quad i = 1, \ldots, n. \tag{2.12}$$

Definition 2.4 (Asymptotically hyperboloidal energy–momentum)**.** Set

$$e := \phi_* g - b, \qquad \eta := \phi_*(k - g). \tag{2.13}$$

For each $V \in \mathcal{N}$, define

$$\mathcal{M}(V) := \lim_{r\to\infty} \int_{S_r} \Big(V(\operatorname{div}_b e - d \operatorname{tr}_b e) + \operatorname{tr}_b(e + 2\eta)\, dV - (e + 2\eta)(\nabla^b V, \cdot) \Big)(\nu_r^b)\, dA,$$

where $S_r \subset \mathbb{H}^n$ is a coordinate sphere of radius r having unit outer normal ν_r^b with respect to b, and dA is the induced hypersurface measure. The asymptotically hyperboloidal energy–momentum vector is then defined by

$$E := \frac{\mathcal{M}(V^{(0)})}{2(n-1)\omega_{n-1}}, \qquad P_i := \frac{\mathcal{M}(V^{(i)})}{2(n-1)\omega_{n-1}}, \quad i = 1, \ldots, n, \tag{2.14}$$

where $\omega_{n-1} = |S^{n-1}|_{g_{S^{n-1}}}$. We also write

$$P = (P_1, \dots, P_n). \tag{2.15}$$

In the case of Wang's asymptotics, the charges admit the explicit formulas

$$E = \frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1}} \left(\operatorname{tr}_{g_{S^{n-1}}} \mathbf{p} + \frac{n-2}{2} \operatorname{tr}_{g_{S^{n-1}}} \mathbf{m} \right) dA, \tag{2.16}$$

and

$$P_i = \frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1}} \left(\operatorname{tr}_{g_{S^{n-1}}} \mathbf{p} + \frac{n-2}{2} \operatorname{tr}_{g_{S^{n-1}}} \mathbf{m} \right) x^i \, dA, \tag{2.17}$$

for $i = 1, \dots, n$.

2.2. **Asymptotically flat initial data sets.** We recall the standard definition of asymptotically flat initial data sets and their ADM energy–momentum.

Definition 2.5 (Asymptotically flat initial data sets)**.** Let (M^n, g), $n \geq 3$ be a connected, complete Riemannian manifold without boundary, and let k be a symmetric $(0,2)$-tensor on M^n. Fix $\ell \geq 6$, $\alpha \in (0,1)$, and

$$q \in \left(\frac{n-2}{2}, n-2 \right). \tag{2.18}$$

We say that (M^n, g, k) is an *asymptotically flat initial data set* of class $C^{\ell,\alpha}$ and decay rate q if there exist a compact set $\mathcal{C} \subset M^n$ and a diffeomorphism

$$\phi : M^n \setminus \mathcal{C} \to \mathbb{R}^n \setminus B \tag{2.19}$$

onto the complement of a Euclidean ball such that, in the corresponding asymptotic coordinates,

$$\left(\phi_* g - \delta, \phi_* k \right) \in C^{\ell,\alpha}_{-q}(\mathbb{R}^n \setminus B) \times C^{\ell-1,\alpha}_{-q-1}(\mathbb{R}^n \setminus B), \tag{2.20}$$

where δ denotes the Euclidean metric. Moreover, we assume that

$$\mu, \ J \in L^1(M^n). \tag{2.21}$$

Definition 2.6 (ADM energy and momentum)**.** Let (M^n, g, k) be an asymptotically flat initial data set, and fix asymptotic coordinates $x = (x^1, \dots, x^n)$ on the unique end. Let ν and dA denote respectively the outward Euclidean unit normal and Euclidean hypersurface measure on the coordinate sphere

$$S_r = \{ x \in \mathbb{R}^n : |x| = r \}. \tag{2.22}$$

Then the *ADM energy* and *ADM linear momentum* are defined by

$$E_{\mathrm{ADM}} := \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \to \infty} \int_{S_r} \left(\partial_j g_{ij} - \partial_i g_{jj} \right) \nu^i \, dA, \tag{2.23}$$

and

$$P^i_{\mathrm{ADM}} := \frac{1}{(n-1)\omega_{n-1}} \lim_{r \to \infty} \int_{S_r} \pi^i_j \, \nu^j \, dA, \qquad i = 1, \dots, n, \tag{2.24}$$

where

$$\pi := k - (\operatorname{tr}_g k) g, \tag{2.25}$$

and $\omega_{n-1} = |S^{n-1}|$ is the Euclidean volume of the unit sphere in $\mathbb{R}^n$. We write $P_{\mathrm{ADM}} = (P^1_{\mathrm{ADM}}, \dots, P^n_{\mathrm{ADM}})$, and, if $E_{\mathrm{ADM}} \geq |P_{\mathrm{ADM}}|$, the corresponding ADM mass is defined by

$$m_{\mathrm{ADM}} := \sqrt{E^2_{\mathrm{ADM}} - |P_{\mathrm{ADM}}|^2}. \tag{2.26}$$

Remark 2.7. Typically, one assumes only $\ell \geq 2$ in the definitions of asymptotically hyperboloidal and asymptotically flat initial data sets. By imposing the stronger assumption $\ell \geq 6$, we are able to directly invoke the results in [5, 25, 33, 37]. Apart from this, the condition $\ell \geq 6$ is not used in the present work. In addition, in the AH setting we impose stronger decay on μ and J which is required in [37].

3. Reduction to $E \geq 0$

3.1. **The asymptotically hyperboloidal case.** We show that, in the asymptotically hyperboloidal setting, it suffices to prove nonnegativity of the energy in an arbitrary asymptotic chart. This is based on a classical boost argument, see for instance [30, Section 3.3].

Let $\phi : M^n_{\mathrm{end}} \to \mathbb{H}^n \setminus \overline{B}$ be the chosen asymptotic chart and let

$$(E, P) = (E, P_1, \dots, P_n) \tag{3.1}$$

be the corresponding energy–momentum vector from Definition 2.4. Recall that in hyperboloidal coordinates on $\mathbb{H}^n \subset \mathbb{R}^{1,n}$ we have

$$t = \sqrt{1 + r^2}, \qquad x = (x^1, \dots, x^n), \tag{3.2}$$

and

$$E = \frac{\mathcal{M}(t)}{2(n-1)\omega_{n-1}}, \qquad P_i = \frac{\mathcal{M}(x^i)}{2(n-1)\omega_{n-1}}, \tag{3.3}$$

where $\mathcal{M}$ denotes the mass functional associated with ϕ.

Proposition 3.1. *Let $\Psi \in SO(1,n)$ be a hyperbolic isometry, and define a new asymptotic chart by*

$$\tilde{\phi} := \Psi \circ \phi.$$

Let $\widetilde{\mathcal{M}}$ denote the corresponding mass functional, and let

$$(\widetilde{E}, \widetilde{P})$$

be the energy–momentum vector computed with respect to $\tilde{\phi}$. Then

$$(\widetilde{E}, \widetilde{P}) = \Psi(E, P).$$

In particular, after a spatial rotation we may assume that

$$P_1 = |P|, \qquad P_\alpha = 0, \quad \alpha = 2, \dots, n.$$

If moreover $E > |P|$, then the boost

$$\Psi = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}, \quad \cosh\theta = \frac{E}{\sqrt{E^2 - |P|^2}}, \quad \sinh\theta = -\frac{|P|}{\sqrt{E^2 - |P|^2}},$$

yields

$$\widetilde{E} = \sqrt{E^2 - |P|^2}, \qquad \widetilde{P} = 0.$$

Proof. Since Ψ is an isometry of $(\mathbb{H}^n, b)$, the composition $\tilde{\phi} = \Psi \circ \phi$ is again an admissible asymptotic chart. Moreover, if $V \in \mathcal{N} = \{V \in C^\infty(\mathbb{H}^n) : \mathrm{Hess}^b V = Vb\}$, then the charge computed in the new chart satisfies

$$\widetilde{\mathcal{M}}(V) = \mathcal{M}(V \circ \Psi). \tag{3.4}$$

Indeed, the integrands in the flux formula are preserved by the isometry Ψ, and hence the corresponding limits agree. Now let

$$\tilde{t} := t \circ \Psi, \qquad \tilde{x}^i := x^i \circ \Psi. \tag{3.5}$$

By the definition of the charges in the new chart,

$$\begin{aligned} \widetilde{E} &= \frac{\widetilde{\mathcal{M}}(t)}{2(n-1)\omega_{n-1}} = \frac{\mathcal{M}(\tilde{t})}{2(n-1)\omega_{n-1}}, \\ \widetilde{P}_i &= \frac{\widetilde{\mathcal{M}}(x^i)}{2(n-1)\omega_{n-1}} = \frac{\mathcal{M}(\tilde{x}^i)}{2(n-1)\omega_{n-1}}. \end{aligned} \tag{3.6}$$

Since $\mathcal{M}$ is linear on $\mathcal{N}$, this shows that the energy–momentum vector transforms by the Lorentz transformation Ψ. For the explicit boost, we compute

$$\tilde{t} = t \cosh\theta + x^1 \sinh\theta = t \frac{E}{\sqrt{E^2 - |P|^2}} - x^1 \frac{|P|}{\sqrt{E^2 - |P|^2}}, \tag{3.7}$$

$$\tilde{x}^1 = x^1 \cosh\theta + t \sinh\theta = x^1 \frac{E}{\sqrt{E^2 - |P|^2}} - t \frac{|P|}{\sqrt{E^2 - |P|^2}}, \tag{3.8}$$

and

$$\tilde{x}^\alpha = x^\alpha, \qquad \alpha = 2, \dots, n. \tag{3.9}$$

Therefore,

$$\widetilde{E} = \frac{\mathcal{M}(\tilde{t})}{2(n-1)\omega_{n-1}} = \frac{E^2 - |P|^2}{\sqrt{E^2 - |P|^2}} = \sqrt{E^2 - |P|^2}, \tag{3.10}$$

while

$$\widetilde{P}_1 = \frac{\mathcal{M}(\tilde{x}^1)}{2(n-1)\omega_{n-1}} = \frac{E|P| - E|P|}{\sqrt{E^2 - |P|^2}} = 0, \tag{3.11}$$

and clearly $\widetilde{P}_\alpha = 0$ for $\alpha = 2, \dots, n$. □

As an immediate consequence, to prove Theorem 1.1 it is enough to establish nonnegativity of the energy in every admissible asymptotic chart.

Corollary 3.2. *Assume that for every asymptotically hyperboloidal initial data set satisfying the dominant energy condition, and for every admissible asymptotic chart, the corresponding energy is nonnegative. Then the full energy–momentum inequality*

$$E \geq |P|$$

holds.

Proof. Suppose, for contradiction, that (E,P) is not future causal, i.e.

$$E < |P|. \tag{3.12}$$

Then there exists a future unit timelike vector

$$a = (a_0, a_1, \dots, a_n) \in \mathbb{R}^{1,n}, \qquad a_0 > 0, \qquad a_0^2 - \sum_{i=1}^n a_i^2 = 1, \tag{3.13}$$

such that

$$a_0 E + \sum_{i=1}^n a_i P_i < 0. \tag{3.14}$$

Choose $\Psi \in SO(1,n)$ whose first row is a, and let $\tilde{\phi} = \Psi \circ \phi$. By Proposition 3.1, the corresponding energy satisfies

$$\widetilde{E} = a_0 E + \sum_{i=1}^n a_i P_i < 0, \tag{3.15}$$

contradicting the assumed nonnegativity of the energy in every chart. Hence $E \geq |P|$. □

We will make use of the following density result from [18].

Theorem 3.3. *Let (M^n, g, k) be an asymptotically hyperboloidal initial data set of type $(\ell, \alpha, \tau, \tau_0)$ satisfying the dominant energy condition. Then for every $\varepsilon > 0$ there exists an initial data set $(M^n, \check{g}, \check{k})$ of type $(\ell - 1, \alpha, n, \tau_0')$ with Wang's asymptotics such that $\check{\mu} > |\check{J}|$, and its energy-momentum vector satisfies*

$$|\check{E} - E| + |\check{P} - P| < \varepsilon.$$

Moreover, there exists $\lambda > 0$ such that

$$\check{\mu} - |\check{J}|_{\check{g}} \geq \lambda r^{-n-1}.$$

Proof. The proof follows closely the arguments in [18, Theorem 3.1 and 5.2]. Choose a bounded positive function f such that $f = r^{-n-1}$. There exists $(\check{g}, \check{k})$ such that, by [18, Equations (23) and (24)],

$$(1 + tv)^\kappa \check{\mu} > \mu + \frac{t}{3} f, \quad (1 + tv)^\kappa |\check{J}|_{\check{g}} < |J|_g + \frac{t}{4} f, \tag{3.16}$$

where $v \in C^{\ell-1,\alpha}_{-n}$, $\kappa = \frac{4}{n-2}$, and t is sufficiently small. Moreover, according to [18, Theorem 5.2], there exists a coordinate chart such that $(\check{g}, \check{k})$ has Wang's asymptotics. □

3.2. The asymptotically flat case.

Theorem 3.4. *Suppose that there exists an asymptotically flat initial data set (M, g, k) satisfying the dominant energy condition*

$$\mu \geq |J|$$

and

$$E_{\mathrm{ADM}} < |P_{\mathrm{ADM}}|.$$

Then there exists another asymptotically flat initial data set, which by abuse of notation we still denote by (M, g, k), such that

$$E_{\mathrm{ADM}} < 0, \qquad \mu > |J|$$

everywhere on M, and such that the data has harmonic asymptotics. *More precisely, if*

$$\pi := k - (\mathrm{tr}_g\, k)\, g,$$

then outside a compact set there are asymptotically flat coordinates $x = (x^1, \dots, x^n)$ in which

$$g_{ij} = u^{\frac{4}{n-2}} \delta_{ij}, \qquad \pi_{ij} = u^{\frac{2}{n-2}} \Big((L_\delta Y)_{ij} - (\mathrm{div}_\delta\, Y) \delta_{ij} \Big),$$

for some function u and vector field Y satisfying

$$u(x) = 1 + a|x|^{2-n} + O_{2,\alpha}(|x|^{1-n}), \qquad Y_i(x) = b_i |x|^{2-n} + O_{2,\alpha}(|x|^{1-n}).$$

Proof. Assume that there exists an asymptotically flat initial data set (M, g, k) satisfying $\mu \geq |J|$ and $E_{\mathrm{ADM}} < |P_{\mathrm{ADM}}|$. We first use the density theorem of Eichmair–Huang–Lee–Schoen [21, p.119], together with the remark following its proof, to perturb the data slightly so that the dominant energy condition is preserved, the inequality $E_{\mathrm{ADM}} < |P_{\mathrm{ADM}}|$ still holds, and

$$\mu = 0, \qquad J = 0 \tag{3.17}$$

outside a large compact set. In particular, the data are smooth near infinity. After this perturbation, we continue to denote the resulting initial data set by (M, g, k).

If already $E_{\mathrm{ADM}} < 0$, there is nothing further to prove at this stage. Otherwise we have

$$0 \leq E_{\mathrm{ADM}} < |P_{\mathrm{ADM}}|. \tag{3.18}$$

Choose coordinates so that P_{ADM} points in the x^n-direction. Since the energy–momentum vector is spacelike, we may choose a boost parameter $\theta \in (0, 1)$ with $\theta > E_{\mathrm{ADM}}/|P_{\mathrm{ADM}}|$. The Lorentz-transformed energy then satisfies

$$E^{\theta}_{\mathrm{ADM}} = \frac{E_{\mathrm{ADM}} - \theta |P_{\mathrm{ADM}}|}{\sqrt{1 - \theta^2}} < 0. \tag{3.19}$$

By the boost theorem of Christodoulou–Ó Murchadha [11], the vacuum end of the spacetime development may be replaced by a boosted asymptotically flat slice with energy $E^{\theta}_{\mathrm{ADM}}$. Thus we obtain a new asymptotically flat initial data set, again denoted by (M, g, k), satisfying

$$E_{\mathrm{ADM}} < 0 \tag{3.20}$$

and still obeying the dominant energy condition.

Finally, we apply the density theorem of Eichmair–Huang–Lee–Schoen once more [21, Theorem 18]. Choosing the perturbation sufficiently small, we preserve the inequality $E_{\mathrm{ADM}} < 0$, while obtaining an initial data set with harmonic asymptotics and with strict dominant energy condition

$$\mu > |J|. \tag{3.21}$$

Relabeling this final perturbation by (M, g, k) completes the proof. □

4. Existence and Regularity Theory of Singular Jang Graphs

In 1978 P. S. Jang introduced a quasilinear elliptic equation [34], which Schoen and Yau [41] successfully employed in dimension $n = 3$ to reduce the positive mass theorem for general initial data to the case of time symmetry. This was later extended to dimensions $3 \leq n \leq 7$ by Eichmair [20], who introduced a key advancement in the modern Jang equation approach with the use of geometric measure theory techniques to analyze weak solutions. Analogs of these results in the asymptotically hyperboloidal setting were obtained by Sakovich [39] for dimension $n = 3$, and more recently by Lundberg [37] for dimensions $3 \leq n \leq 7$.

Consider the regularized Jang equation

$$H(f_\tau) - \mathrm{tr}_g(k)(f_\tau) = \tau f_\tau \qquad \text{on } M^n. \tag{4.1}$$

For each $\tau > 0$ a solution $f_\tau \in C^{3,\alpha}_{loc}(M^n)$, $\alpha \in (0,1)$ may be obtained through an exhaustion procedure, which entails solving an appropriate Dirichlet problem on finite exhausting domains and using barriers in the asymptotic ends to control decay. The resulting solutions [20, Proposition 5], [37, Proposition 4.4] satisfy

$$|f_\tau| \leq C\tau^{-1} \qquad \text{on all of } M^n \tag{4.2}$$

via a maximum principle, where C is a constant depending only on the initial data, which in the asymptotically flat setting may be taken to be $C = 1 + n \sup_{M^n} |k|_g$. Furthermore, in the asymptotically flat case the regularized solutions satisfy

$$|f_\tau(x)| \leq c_\beta |x|^{2-\beta} \qquad \text{for all } |x| \geq \Lambda_\beta, \tag{4.3}$$

where $\beta = 1 + q \in (2, n)$ and the constants $c_\beta > 0$ and $\Lambda_\beta \geq 1$ are independent of τ, while in the asymptotically hyperboloidal case [37, Proposition 4.4]

$$f_-(r) \leq f_\tau(x) \leq f_+(r) \qquad \text{for all } r \geq \Lambda, \tag{4.4}$$

where $\Lambda \geq 1$ and the radial barriers $f_\pm$ are independent of τ with

$$f_\pm(r) = \sqrt{1 + r^2} + O(r^{3-n}) \tag{4.5}$$

when $n \geq 4$.

Theorem 4.1 (Higher-Dimensional AF Geometric Jang Limit)**.** *Let* (M^n, g, k), $n \geq 4$ *be a complete asymptotically flat initial data set. Then there exists a sequence* $\tau_j \downarrow 0$, *a Caccioppoli set* $\mathbf{E} \subset M \times \mathbb{R}$, *and a closed set*

$$\bar{\Sigma} := \operatorname{spt}(\partial \mathbf{E}) \subset M^n \times \mathbb{R}$$

such that the following hold.

(i) $\partial \mathbf{E}$ *is a* $2C$*-almost minimizing boundary in* $M^n \times \mathbb{R}$.

(ii) *If*

$$\operatorname{Reg}(\bar{\Sigma}) := \bar{\Sigma} \setminus \operatorname{Sing}(\bar{\Sigma}),$$

then $\operatorname{Reg}(\bar{\Sigma})$ *is a* $C^{3,\alpha}_{\mathrm{loc}}$ *embedded hypersurface, and*

$$\dim_{\mathcal{H}}\big(\operatorname{Sing}(\bar{\Sigma})\big) \leq n - 7.$$

(iii) *Every connected component* Σ *of* $\operatorname{Reg}(\bar{\Sigma})$ *is either*

(a) *a vertical cylinder*

$$\Sigma = \Sigma_0^{\mathrm{Reg}} \times \mathbb{R},$$

where $\Sigma_0^{\mathrm{Reg}} \subset M^n$ *is a smooth embedded marginally trapped hypersurface, whose closure is a* $2C$*-almost minimizing boundary in* M^n *with singular set of Hausdorff dimension at most* $n - 8$, *or*

(b) *a graph*

$$\Sigma = \operatorname{graph}(f_\Sigma, U_\Sigma)$$

over an open set $U_\Sigma \subset M^n$, *where* $f_\Sigma \in C^{3,\alpha}_{\mathrm{loc}}(U_\Sigma)$ *solves the Jang equation*

$$H(f_\Sigma) - \operatorname{tr}_g(k)(f_\Sigma) = 0 \qquad \text{on } U_\Sigma.$$

(iv) *There exists at least one graphical component*

$$\Sigma_\infty = \operatorname{graph}(f_\infty, U_\infty)$$

such that, for some $\Lambda_\beta \geq 1$ *and* $c_\beta > 0$, *we have* $\{x \in M : |x| > \Lambda_\beta\} \subset U_\infty$ *and*

$$|\nabla^l f_\infty(x)| \leq c_\beta |x|^{2-\beta-l} \quad \text{for } |x| > \Lambda_\beta,$$

with $\beta = 1 + q \in (2, n)$ *and* $l = 0, \ldots, 5$. *In particular, the Jang graph is smooth near infinity.*

(v) *For every graphical component* $\operatorname{graph}(f_\Sigma, U_\Sigma)$, *the frontier* ∂U_Σ *is carried by cylindrical components of* $\bar{\Sigma}$. *Along the regular part of the corresponding cross-sections one has*

$$f_\Sigma(x) \to \pm\infty,$$

and vertical translates of the graph converge in the sense of currents to the corresponding cylinders.

Theorem 4.2 (Higher-Dimensional AH Geometric Jang Limit)**.** *Let (M^n, g, k), $n \geq 4$, be a complete asymptotically hyperboloidal initial data set with Wang asymptotics. Then the conclusions* (i)*,* (ii)*,* (iii)*, and* (v) *of Theorem 4.1 hold unchanged. Moreover, the conclusion* (iv) *is replaced by the following statement.*

(iv) *There exists at least one graphical component*

$$\Sigma_\infty = \mathrm{graph}(f_\infty, U_\infty)$$

which contains the chosen asymptotic end. Moreover, for any $\epsilon > 0$ there exists $\Lambda \geq 1$ such that $\{r > \Lambda\} \subset U_\infty$ and

$$f_\infty(r,\theta) = \sqrt{1+r^2} + \frac{\boldsymbol{\alpha}(\theta)}{r^{n-3}} + O_5(r^{2-n+\epsilon}),$$

where $\boldsymbol{\alpha} \in C^2(S^{n-1})$ is the unique solution of

$$\Delta_{g_{S^{n-1}}}\boldsymbol{\alpha} - (n-3)\boldsymbol{\alpha} = \frac{n-2}{2}\,\mathrm{tr}_{g_{S^{n-1}}}\mathbf{m} + \mathrm{tr}_{g_{S^{n-1}}}\mathbf{p}.$$

In particular, the Jang graph is smooth near infinity.

Proof of Theorems 4.1 and 4.2. These theorems are a combination of existing results in the literature. Here we will summarize the main points and provide the appropriate references. For each $\tau > 0$, [20, Proposition 6] shows that $\Gamma_\tau := \mathrm{graph}(f_\tau) \subset M \times \mathbb{R}$ is a $2C$-almost minimizing boundary. Choose a sequence $\tau_j \downarrow 0$. Since the Γ_{τ_j} are uniformly $2C$-almost minimizing, the compactness theorem for $2C$-almost minimizing boundaries yields, after passing to a subsequence, a Caccioppoli set $\mathbf{E} \subset M \times \mathbb{R}$ such that $\Gamma_{\tau_j} \to \partial\mathbf{E}$ as currents and as varifolds. It follows that $\bar\Sigma := \mathrm{spt}(\partial E)$ is an n-dimensional almost minimizing hypersurface in the ambient $(n+1)$-manifold $M^n \times \mathbb{R}$. Therefore, the regularity theorem for almost minimizing boundaries applies to yield $\dim_{\mathcal{H}}\big(\mathrm{Sing}(\bar\Sigma)\big) \leq n-7$. On $\mathrm{Reg}(\bar\Sigma)$, Allard regularity yields local $C^{1,\alpha}$ convergence of a subsequence of the Γ_{τ_j}, and the equation (4.1) upgrades this to local $C^{3,\alpha}$ convergence by standard quasilinear elliptic estimates. This proves (ii).

Let Σ be a connected component of $\mathrm{Reg}(\bar\Sigma)$. Since the approximating graphs satisfy

$$H(f_{\tau_j}) - \mathrm{tr}_g(k)(f_{\tau_j}) = \tau_j f_{\tau_j}, \tag{4.6}$$

and since the convergence on compact subsets of Σ is smooth while $\tau_j f_{\tau_j} \to 0$ locally along the convergent sheets, one obtains

$$H_\Sigma - \mathrm{tr}_\Sigma(k) = 0. \tag{4.7}$$

Let $v_\tau = \sqrt{1+|\nabla f_\tau|_g^2}$, and note that this quantity [22, Lemma A.1] (see also [19, Lemma 2.3]) satisfies

$$\Delta_{\Sigma_\tau} v_\tau^{-1} = -(|h_\tau|^2 + \nu_\tau(H_\tau) + \mathrm{Ric}(\nu_\tau,\nu_\tau))v_\tau^{-1} \leq \sigma v_\tau^{-1} + \langle Y, \nabla_{\Sigma_\tau} v_\tau^{-1}\rangle, \tag{4.8}$$

for some locally bounded function σ and vector field Y whose bounds depend only on the initial data, where Ric is the ambient Ricci curvature of $M^n \times \mathbb{R}$, h_τ is the second fundamental form of the graph, and $\nu_\tau = v_\tau^{-1}(f_\tau^i \partial_i - \partial_t)$ is the unit normal to the graphs. In particular, the Harnack principle for limits of graphs implies that every connected component of the regular set of such a geometric limit is either a vertical cylinder or a graph over an open subset of M^n. Hence each connected component of $\mathrm{Reg}(\bar{\Sigma})$ is of one of the two types in (iii), and in the graphical case the defining function solves the unregularized Jang equation. See Eichmair [19, Section 3] for more details.

Lastly, (iv) in both the asymptotically flat and asymptotically hyperboloidal cases follows from the interior gradient estimate [19, Lemma 2.1] ([37, Lemma 6.1]) together with a standard boot-strap, and the barrier bounds (4.3) and (4.4). Moreover, the boundary behavior in (v) is the standard blow-up mechanism for geometric limits of regularized Jang graphs: near the frontier of a graphical domain, vertical translates of the graph subconverge to cylindrical components, and along the regular part of the corresponding cross-sections the defining function tends uniformly to $+\infty$ or $-\infty$. This is exactly the same cylinder-versus-graph alternative and translation argument used in the smooth case, and it is local on the regular set, so it carries over unchanged here. □

A particularly advantageous feature of Jang graphs is the weak positivity property enjoyed by their scalar curvature when the dominant energy condition is satisfied. This is recorded in the next result [41, (2.25)].

Proposition 4.3 (Schoen–Yau Scalar Curvature Identity)**.** *Let $\Sigma \subset \mathrm{Reg}(\bar{\Sigma})$ be a graphical connected component with induced Jang metric $\bar{g} = g + df^2$. Then the scalar curvature of the Jang metric takes the form*

$$\bar{R} = 2(\mu - J(w)) + |h-k|_{\bar{g}}^2 + 2|X|_{\bar{g}}^2 - 2\,\mathrm{div}_{\bar{g}}\, X,$$

where h is the second fundamental form of Σ and w, X are 1-forms given by

$$w_i = \frac{f_i}{1+|\nabla f|_g^2}, \qquad h_{ij} = \frac{\nabla_{ij} f}{1+|\nabla f|_g^2}, \qquad X_i = \bar{g}^{j\ell}(h_{ij} - k_{ij})w_\ell,$$

with ∇_{ij} denoting covariant differentiation with respect to g.

Proof of Theorem 1.4. We have already proven existence of the Jang equation above. Thus, it remains only to justify the Minkowski dimension estimate for the singular set. Our argument is purely local and does not distinguish between the AH and the AF case.

By Eichmair's compactness theory for Jang graphs, Σ is locally the boundary of a Caccioppoli set and is locally C-almost minimizing in the sense of currents. More precisely, if $W \subseteq M \times \mathbb{R}$ is open and X is an integral $(n+1)$-current supported in W, then

$$\mathbf{M}_W(\partial E) \le \mathbf{M}_W(\partial E + \partial X) + C\,\mathbf{M}_W(X), \tag{4.9}$$

where $\Sigma = \partial E$ in W and C depends only on the local mean curvature bound coming from the regularized Jang equation.

Fix now a relatively compact coordinate ball

$$U \subseteq M \times \mathbb{R}, \tag{4.10}$$

and identify it with a bounded open subset of $\mathbb{R}^{n+1}$ by a smooth chart. Since Minkowski dimension is invariant under bi-Lipschitz changes of coordinates, it is enough to work in this Euclidean chart.

Let $E \subset U$ be the set of finite perimeter bounded by $\Sigma \cap U$. We claim that E satisfies the almost minimizing hypothesis of [23, (6.8)–(6.9)]. Indeed, let

$$E \triangle F \subseteq B_r(x) \subseteq U. \tag{4.11}$$

Choose the filling current $X := [\![F]\!] - [\![E]\!]$. Then $\partial X = \partial F - \partial E$ in $B_r(x)$ and

$$\mathbf{M}(X) = |E \triangle F|. \tag{4.12}$$

Applying the C-almost minimizing property gives

$$\operatorname{Per}(E; B_r(x)) \leq \operatorname{Per}(F; B_r(x)) + C\,|E \triangle F|. \tag{4.13}$$

Since $E \triangle F \subset B_r(x) \subset \mathbb{R}^{n+1}$, we have

$$|E \triangle F| \leq \omega_{n+1} r^{n+1}, \tag{4.14}$$

and therefore

$$\operatorname{Per}(E; B_r(x)) \leq \operatorname{Per}(F; B_r(x)) + C\omega_{n+1} r^{n+1}. \tag{4.15}$$

Thus E is a perimeter almost minimizer in the sense of [23], with

$$\alpha(r) := C\omega_{n+1} r, \tag{4.16}$$

so that

$$\operatorname{Per}(E; B_r(x)) \leq \operatorname{Per}(F; B_r(x)) + \alpha(r)\, r^n. \tag{4.17}$$

Moreover, α is nondecreasing, $\alpha(r) \to 0$ as $r \downarrow 0$, $t \mapsto \alpha(t)/t$ is constant, and

$$\int_0^T \frac{\alpha(t)^{1/2}}{t}\, dt = \sqrt{C\omega_{n+1}} \int_0^T t^{-1/2}\, dt < \infty. \tag{4.18}$$

Hence all assumptions of [23, Theorem 6.7] are satisfied.

Applying [23, Theorem 6.7] in the chart, with ambient dimension $n+1$, yields

$$\dim_{\mathcal{M}} \operatorname{Sing}(\Sigma \cap U) \leq (n+1) - 8 = n - 7. \tag{4.19}$$

Since $U \subseteq M \times \mathbb{R}$ was arbitrary, the same estimate holds locally on all of Σ, and this proves

$$\dim_{\mathcal{M}} \operatorname{Sing}(\Sigma) \leq n - 7. \tag{4.20}$$

□

5. Desingularizing the Jang Graph and Proof of Theorem 1.2

Lemma 5.1. *Let (M^n, g, k), $n \geq 4$ be either asymptotically flat, or asymptotically hyperboloidal with Wang asymptotics, satisfying a strict dominant energy condition with $\mu - |J|_g \geq \lambda r^{-n-1}$ in the asymptotic end for some constant $\lambda > 0$. Consider a regular Jang graphical component*

$$\Sigma_\infty = \mathrm{graph}(f_\infty, U_\infty) \subset \mathrm{Reg}(\bar{\Sigma})$$

obtained from Theorem 4.1 or 4.2, and let $\bar{g}$ be its induced metric. Then there exist smooth positive functions ρ and Q on Σ_∞ and $c, C_l \in \mathbb{R}$ such that

$$|\nabla^l \left(\rho - (1 + cr^{2-n})\right)|_\delta \leq C_l r^{1-n-l}, \qquad |\nabla^l Q|_\delta \leq C_l r^{-n-1-l}, \tag{5.1}$$

and the following inequality holds for all $\phi \in C^\infty(\Sigma_\infty)$ with the property that ϕ vanishes in a neighborhood of the singular set, is constant in the asymptotically flat end, and has bounded supported outside this same end:

$$\lim_{r\to\infty} \int_{\Sigma_\infty^r} \rho|\nabla\phi|_{\bar{g}}^2 + \frac{1}{2}\rho\left(R_{\bar{g}} - 2\Delta_{\bar{g}}\log\rho - \frac{n+1}{n+2}|\nabla\log\rho|_{\bar{g}}^2\right)\phi^2 \geq \int_{\Sigma_\infty} \rho Q \phi^2, \tag{5.2}$$

where Σ_∞^r denotes the region contained within coordinate sphere S_r.

Proof. We will treat the asymptotically hyperboloidal case first, and will assume for simplicity that the initial data possesses a single end. Let ρ be a smooth positive function to be chosen. Utilizing the Jang scalar curvature formula Proposition (4.3) while integrating $\mathrm{div}_{\bar{g}} X$ and $\Delta_{\bar{g}} \log\rho$ by parts produces

$$\begin{aligned} &\int_{\Sigma_\infty^r} \rho|\nabla\phi|_{\bar{g}}^2 + \frac{\rho}{2}\left(R_{\bar{g}} - 2\Delta_{\bar{g}}\log\rho - \frac{n+1}{n+2}|\nabla\log\rho|_{\bar{g}}^2\right)\phi^2 \\ \geq &\int_{\Sigma_\infty^r} \rho|\nabla\phi|_{\bar{g}}^2 + (\mu - |J|_g)\rho\phi^2 + \rho\phi^2|X|_{\bar{g}}^2 + \left(1 - \frac{n+1}{2(n+2)}\right)\phi^2\frac{|\nabla\rho|_{\bar{g}}^2}{\rho} \\ &+ \int_{\Sigma_\infty^r} 2\phi\langle\nabla\phi, \nabla\rho\rangle + \phi^2\langle\nabla\rho, X\rangle + 2\rho\phi\langle\nabla\phi, X\rangle \\ &- \int_{S_r} \left(\rho\phi^2\langle X, \nu\rangle + \phi^2\nu(\rho)\right), \end{aligned} \tag{5.3}$$

where S_r denote coordinate spheres with unit outer normal ν in the asymptotically flat end of Σ_∞. Note that the Jang scalar curvature is not necessarily integrable in the asymptotically flat end, due to the expansion [37, Lemma B.3]

$$R_{\bar{g}} = 2(n-2)\frac{\Delta_{S^{n-1}}\boldsymbol{\alpha}}{r^n} + O(r^{-n-1+\epsilon}) \tag{5.4}$$

for any $\epsilon > 0$ and some $\boldsymbol{\alpha} \in C^\infty(S^{n-1})$. However, the integral on the left-hand side of (5.3) is finite even in the limit as $r \to \infty$ since the leading term of this expansion integrates to zero on coordinate spheres. It follows that

$$\begin{aligned}
&\int_{\Sigma_\infty^r} \rho|\nabla\phi|_{\bar g}^2 + \frac{\rho}{2}\left(R_{\bar g} - 2\Delta_{\bar g}\log\rho - \frac{n+1}{n+2}|\nabla\log\rho|_{\bar g}^2\right)\phi^2 \\
\geq &\int_{\Sigma_\infty^r} (\mu - |J|_g)\rho\phi^2 + \rho|\nabla\phi + \phi(X + \nabla\log\rho)|_{\bar g}^2 \\
&- \int_{\Sigma_\infty^r} \left(\left(1 - \frac{n+3}{2(n+2)}\right)\phi^2\frac{|\nabla\rho|_{\bar g}^2}{\rho} + \phi^2\langle\nabla\rho, X\rangle\right) \\
&- \int_{S_r} \rho\phi^2\left(\langle X,\nu\rangle + \nu(\log\rho)\right).
\end{aligned} \tag{5.5}$$

According to [37, (B.15)] the flux density is

$$\langle X,\nu\rangle = (n-2)(n-3)\frac{\boldsymbol{\alpha}}{r^{n-1}} + O(r^{-n+\epsilon}). \tag{5.6}$$

This motivates the choice

$$\rho = 1 + \psi\cdot\frac{(n-3)\alpha_0}{r^{n-2}}, \qquad \alpha_0 = \frac{1}{\omega_{n-1}}\int_{S^{n-1}}\boldsymbol{\alpha}, \tag{5.7}$$

where ψ is a smooth nonnegative cut-off function which vanishes inside S_{r_0} and is 1 outside S_{2r_0} in the asymptotic end, and satisfies $|\nabla\psi|_{\bar g} \leq 2r_0^{-1}$. By [37, (B.15) and (B.18)] we have $|X|_{\bar g} = O(r^{1-n})$, and thus the strict DEC assumption $\mu - |J|_g \geq \lambda r^{-n-1}$ implies that

$$\mu - |J|_g \geq \left(1 - \frac{n+3}{2(n+2)}\right)|\nabla\log\rho|^2 + \langle\nabla\log\rho, X\rangle \tag{5.8}$$

if r_0 is chosen sufficiently large. Moreover, with this choice of ρ we find that for r large enough

$$\langle X,\nu\rangle + \nu(\log\rho) = (n-2)(n-3)\frac{\boldsymbol{\alpha} - \alpha_0}{r^{1-n}} + O(r^{-n+\epsilon}). \tag{5.9}$$

Hence, the flux integral in (5.5) converges to zero, and we obtain

$$\begin{aligned}
&\lim_{r\to\infty}\int_{\Sigma_\infty^r} \rho\,|\nabla\phi|_{\bar g}^2 + \frac{\rho}{2}\left(R_{\bar g} - 2\Delta_{\bar g}\log\rho - \frac{n+1}{n+2}|\nabla\log\rho|_{\bar g}^2\right)\phi^2 \\
&\qquad\geq \int_{\Sigma_\infty} \rho\, Q\,\phi^2 + \rho|\nabla\phi + \phi(X + \nabla\log\rho)|_{\bar g}^2,
\end{aligned} \tag{5.10}$$

for some positive smooth function Q satisfying (5.1) and $Q \geq \frac{\lambda}{2}r^{-n-1}$ for large r.

In the case of asymptotically flat initial data, an analogous argument holds with many simplifications. In particular, we may choose $\rho = 1$, the Jang scalar curvature is integrable in the asymptotically flat end, and there are no flux terms arising from the vector field X. □

Lemma 5.1 is the analogue, in the present Jang-graph setting, of [5, Corollary 3.31]. It provides the coercive estimate needed for the conformal blow-up argument.

We follow [5, Section 3.7] to blow up the singular set in order to obtain a complete manifold without singularities. However, in our setting the singular set may not be compact, which differs from the situation in [5]. Therefore, we modify the proof accordingly.

Theorem 5.2. *Let (M, g, k) be an asymptotically flat initial data set satisfying $\mu > |J|$. Let Σ_∞ be the Jang graph defined in Lemma 5.1. In what follows, ∇ and Δ denote the gradient and Laplacian on Σ_∞.*

(i) *Let $\overline{M} = M \times \mathbb{R}$ be equipped with the product metric $\overline{g}$, $\overline{\Sigma}_\infty$ is the closure of Σ_∞ in $\overline{M}$, and let $\mathcal{S}$ denote the singular set in $\overline{\Sigma}_\infty$. Then $\mathcal{S}$ can be decomposed into a union of compact sets $\mathcal{S}_i$ such that*

$$d_{(\overline{M},\overline{g})}(\mathcal{S}_i, \mathcal{S}_j) \geq |i-j| - 1.$$

(ii) *There exists a function $\Psi_i \in C^2(\overline{M} \setminus \mathcal{S}_i)$ supported on a small neighborhood of $\mathcal{S}_i$ such that for $x \in \Sigma_\infty$ the following statements hold.*
  (a) There exists a constant $C_i > 1$,

$$\Delta \Psi_i + \frac{n-2}{n+2}\langle \nabla \log \rho, \nabla \Psi_i \rangle \leq C_i.$$

  (b) If $d_{(\overline{M},\overline{g})}(x, \mathcal{S}_i)$ is sufficiently small, then

$$\Delta \Psi_i + \frac{n-2}{n+2}\langle \nabla \log \rho, \nabla \Psi_i \rangle < 0,$$
$$\Psi_i(x) \geq \frac{1}{2(n-2)} 3^{2-n} d_{(\overline{M},\overline{g})}(x, \mathcal{S}_i)^{-2}.$$

(iii) *Let $\overline{\Psi} = \sum_{i=1}^\infty C_i^{-1}\Psi_i$. Then there exists ε_0 such that on Σ_∞*

$$\varepsilon_0 \left(\Delta \Psi + \frac{n-2}{n+2}\langle \nabla \log \rho, \nabla \Psi \rangle \right) \leq \frac{n-2}{4(n+2)} Q.$$

(iv) *Let $w = 1 + \varepsilon_0 \overline{\Psi}$ and $\widetilde{g} = w^{\frac{n+2}{n-2}} g_{\Sigma_\infty}$, then $(\Sigma_\infty, \widetilde{g})$ is complete.*

Proof. (i) Note that the singular set can only concentrate along the cylindrical ends and the asymptotically flat end is regular by the interior estimates [37, Proposition 6.2]. Set

$$\Omega_i := \overline{\{t_{i-1} \leq |f_\infty| \leq t_i\}}, \quad \mathcal{S}_i := \mathcal{S} \cap \Omega_i, \tag{5.11}$$

with $t_0 = 0$ and t_i $(i > 0)$ chosen large enough so that $d_{(\overline{M},\overline{g})}(\Omega_i, \Omega_j) \geq |i-j| - 1$. Therefore, each $\mathcal{S}_i$ is compact and $d_{(\overline{M},\overline{g})}(\mathcal{S}_i, \mathcal{S}_j) \geq |i-j| - 1$.

(ii) Following [5, Page 31], choose $t_* \in (0,1)$ such that $\sqrt{t_*} \leq \frac{1}{2}\operatorname{inj}_{(\overline{M},\overline{g})}(p)$ for all $p \in \mathcal{S}$. Construct Ψ_i as in [5, Page 32]. Properties (a) and (b) then follow from [5, Proposition 3.34, 3.35 & 3.36].

(iii) Since Ψ_i is supported on a small neighborhood of $\mathcal{S}_i$, it follows that $\Psi_i \Psi_j = 0$ whenever $|i-j| > 1$. Note that as shown in Lemma 5.1 and Proposition A.2, we may choose $Q = \frac{1}{4}(\mu - |J|)$ which is uniformly positive on cylindrical ends, therefore, there exists a suitable $\varepsilon_0 > 0$ as required.

(iv) Let $y_0, y_1 \in \Sigma_\infty$ and let $\sigma : [0,1] \to \Sigma_\infty$ be a smooth path connecting them. Define

$$I := \{i | \Omega_i \cap \sigma([0,1]) \neq \varnothing\}. \tag{5.12}$$

Let c_i and ε_i be positive constants such that

$$w(x) \geq \varepsilon_i d_{(\overline{M},\overline{g})}(x,\mathcal{S})^{-2} \quad \text{for } d_{(\overline{M},\overline{g})}(x,\mathcal{S}_i) \in (0, c_i]. \tag{5.13}$$

Thus, on the region $\cup_{i\in I}\Omega_i$, we have

$$w(x) \geq \min_{i\in I}\{\varepsilon_i\} d_{(\overline{M},\overline{g})}(x,\mathcal{S})^{-2} \quad \text{for } d_{(\overline{M},\overline{g})}(x,\mathcal{S}_i) \in (0, \min_{i\in I}\{c_i\}]. \tag{5.14}$$

Following [5, Proposition 3.41], this implies

$$\begin{aligned} d_{(\overline{M},\overline{g})}(y_1,\mathcal{S}_l)^{-\frac{4}{n-2}} \leq & \max\left\{d_{(\overline{M},\overline{g})}(y_0,\mathcal{S}_l)^{-\frac{4}{n-2}}, \max_{i\in I}\{c_i^{-\frac{4}{n-2}}\}\right\} \\ & + \frac{4}{n-2}\max_{i\in I}\left\{\varepsilon_i^{-\frac{n+2}{2(n-2)}}\right\}\int_0^1 |\sigma'(t)|_{\widetilde{g}} dt, \end{aligned} \tag{5.15}$$

where l satisfies $y_1 \in \Omega_l$. On the other hand, since $d_{(\overline{M},\overline{g})}(\Omega_i,\Omega_j) \geq |i-j| - 1$, we have $|\sigma|_{\widetilde{g}} \geq |I| - 1$. Therefore, $d_{(\Sigma_\infty,\widetilde{g})}(y_0,y_1) \to \infty$ when $d_{(\overline{M},\overline{g})}(y_1,\mathcal{S}_l) \to \infty$. Hence, $\widetilde{g}$ is complete. □

Proof of Theorem 1.1. By Corollary 3.2, to prove $E \geq |P|$, it suffices to show that $E \geq 0$. By Proposition 3.3, it suffices to study Wang asymptotics which allows us to apply Theorem 4.2 and construct a Jang graph. Proposition A.1 and A.2 imply that the Jang graph $(\Sigma_\infty, \tilde{g})$ becomes an n-dataset in the Brendle-Wang terminology after performing the deformation in Appendix A. By [5, Theorem 1.5], the following mass quantity is nonnegative

$$(n-1)c + 2(n-3)\alpha_0 \geq 0, \tag{5.16}$$

where α_0 appears in the expansion (5.7), and $\tilde{g} = (1 + cr^{2-n})\delta$ in the asymptotic end as defined in Proposition A.1. Note that $E_{ADM} = \frac{n-2}{2}c$, and by [37, (3.4) and (C.1)] we find $\alpha_0 = -\frac{1}{n-3}E_{ADM}$. It follows that

$$0 \leq (n-1)c + 2(n-3)\alpha_0 = \frac{2}{n-2}E_{ADM}. \tag{5.17}$$

Then by [37, (A.3) and (C.1)], $E = \frac{E_{ADM}}{n-1}$ is nonnegative. Finally, the equality case follows by [30, 31]. □

Proof of Theorem 1.2. We first prove $E_{\mathrm{ADM}} \geq |P_{\mathrm{ADM}}|$. By Theorem 3.4 and Theorem 5.2, it suffices to show that the initial data set (M, g, k) satisfies $E_{\mathrm{ADM}} \geq 0$. We argue by contradiction and assume $E_{\mathrm{ADM}} < 0$. Applying the perturbation in Appendix A, we construct an asymptotically flat Riemannian manifold $(\Sigma_\infty, \tilde{g})$ with a Schwarzschild end, negative energy, and scalar curvature $R_{\tilde{g}}$ satisfying Equation (A.2). Note that we can choose $\rho = 1$; therefore, [5, Theorem 1.5] implies $E_{\mathrm{ADM}} \geq 0$.

Next, we analyze the case $E_{\mathrm{ADM}} = |P_{\mathrm{ADM}}|$. We apply [25] to deduce that (M, g, k) embeds into a spacetime $(\overline{M}, \mathbf{g})$ containing a null parallel vector field in case $(g,k) \in C^5_{\mathrm{loc}}(M) \times C^4_{\mathrm{loc}}(M)$. In this case, we may apply [29] to find that the ambient metric may be written in Brinkmann coordinates as

$$\mathbf{g} = -2\,dt\,du + F(x,u)\,du^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2, \tag{5.18}$$

where F is independent of t and satisfies

$$\Delta_{\mathbb{R}^{n-1}} F(\cdot,u) \leq 0 \qquad \text{for every } u \in \mathbb{R}. \tag{5.19}$$

This proves the claim. □

6. Asymptotically Hyperboloidal Positive Mass Theorem Via the Asymptotically Flat Positive Mass Theorem

In this section we explain that Theorem 1.2 implies Theorem 1.1 under stronger asymptotic assumptions at infinity.

6.1. **Proof for Wang asymptotics.** In the Riemannian ($k = g$) case, the desired inequality for asymptotically hyperboloidal manifolds with Wang asymptotics follows directly from the recent work of Chruściel and Delay [12]. More precisely, they show that the causal-future-directed character of the asymptotically hyperboloidal energy–momentum vector can be reduced to the spacetime positive mass theorem for asymptotically Euclidean initial data sets.

Therefore, combining their reduction with Theorem 1.2, we obtain the hyperbolic positive mass theorem for asymptotically hyperboloidal Riemannian manifolds with Wang asymptotics. In particular, if (M^n, g) is an asymptotically hyperboloidal manifold with Wang asymptotics and scalar curvature

$$R_g \geq -n(n-1), \tag{6.1}$$

then its energy–momentum vector (E, P) satisfies

$$E \geq |P|. \tag{6.2}$$

6.2. **Proof for exact AdS–Schwarzschild asymptotics.** We now explain how Theorem 1.2 yields the hyperbolic positive mass theorem for initial data sets which are exactly AdS–Schwarzschild near infinity.

Assume that (M^n, g, k) is an asymptotically hyperboloidal initial data set satisfying the dominant energy condition, and that for some compact set $K \subset M^n$ the exterior region $M^n \setminus K$ is realized as a spacelike hypersurface in the Schwarzschild spacetime of mass m, with induced initial data equal to the corresponding exact AdS–Schwarzschild data.

Since both the asymptotically hyperboloidal AdS–Schwarzschild slice and the standard asymptotically flat Schwarzschild slice occur in the same ambient Schwarzschild spacetime, we may bend the given hypersurface in the exterior region so as to replace the hyperboloidal end by an asymptotically flat end, while keeping the data unchanged on a sufficiently large compact set. In this way one obtains an asymptotically flat initial data set

$$(\widetilde{M}, \widetilde{g}, \widetilde{k}) \tag{6.3}$$

which still satisfies the dominant energy condition and whose asymptotically flat end is exactly Schwarzschild of mass m. Applying Theorem 1.2 to $(\widetilde{M}, \widetilde{g}, \widetilde{k})$, we conclude that $m \geq 0$, i.e. that $E \geq |P|$ including rigidity.

Appendix A. Density Lemma

In this appendix, we deform the Jang graph metric to produce a Schwarzschild end, while preserving the weighted inequality for the scalar curvature (cf. Proposition A.2) and ensuring that the total energy changes only slightly.

Proposition A.1. *Let $n \geq 3$, and let $(M^n, \bar{g})$ be asymptotically flat on an end*

$$\mathcal{E} \simeq \mathbb{R}^n \setminus B_{R_0}.$$

Assume that, in the asymptotic coordinates,

$$\bar{g}_{ij} = \delta_{ij} + O_2(r^{2-n}).$$

Let $E_{\mathrm{ADM}}(\bar{g})$ denote the ADM energy of $\bar{g}$. Fix $\Lambda > 0$ and $\epsilon \in (0, n-2)$. Then, for all sufficiently large r_0, there exists a smooth asymptotically flat metric $\tilde{g}$ such that

$$\tilde{g} = \bar{g} \qquad \text{on } \{r \leq r_0\},$$
$$\tilde{g}_{ij} = (1 + cr^{2-n})\delta_{ij} \qquad \text{on } \{r \geq 2r_0\},$$

and

$$E_{\mathrm{ADM}}(\tilde{g}) - E_{\mathrm{ADM}}(\bar{g}) = O(\Lambda r_0^{-\epsilon}).$$

Proof. Let

$$E_0 := E_{\mathrm{ADM}}(\bar{g}). \tag{A.1}$$

Choose the positive energy increment

$$\eta_{r_0} := \frac{\Lambda}{(n-1)\epsilon} r_0^{-\epsilon}. \tag{A.2}$$

Define

$$c = c(r_0) := \frac{2}{n-2}(E_0 + \eta_{r_0}), \tag{A.3}$$

and set

$$g_c := (1 + cr^{2-n})\delta. \tag{A.4}$$

With the ADM energy normalization

$$E_{\mathrm{ADM}}(g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{\rho \to \infty} \int_{S_\rho} (\partial_j g_{ij} - \partial_i g_{jj})\nu^i \, dS, \tag{A.5}$$

we have

$$E_{\mathrm{ADM}}(g_c) = \frac{n-2}{2} c = E_0 + \eta_{r_0}. \tag{A.6}$$

Let $\chi \in C^\infty([0,\infty))$ satisfy

$$\chi(s) = 0 \quad \text{for } s \le 1, \qquad \chi(s) = 1 \quad \text{for } s \ge 2, \qquad 0 \le \chi \le 1. \tag{A.7}$$

Define

$$\chi_{r_0}(r) := \chi(r/r_0). \tag{A.8}$$

Then

$$\chi_{r_0} = 0 \quad \text{for } r \le r_0, \qquad \chi_{r_0} = 1 \quad \text{for } r \ge 2r_0, \tag{A.9}$$

and on the transition annulus $r_0 \le r \le 2r_0$,

$$|\partial^\ell \chi_{r_0}| \le C_\ell r^{-\ell}. \tag{A.10}$$

Define $\tilde{g}$ on the end $\mathcal{E}$ by

$$\tilde{g} := (1-\chi_{r_0})\bar{g} + \chi_{r_0} g_c, \tag{A.11}$$

and set $\tilde{g} = \bar{g}$ away from the chosen end. For r_0 sufficiently large, both $\bar{g}$ and g_c are uniformly close to δ on $\{r \ge r_0\}$, so $\tilde{g}$ is positive definite. By construction,

$$\tilde{g} = \bar{g} \qquad \text{on } \{r \le r_0\}, \tag{A.12}$$

and

$$\tilde{g} = g_c = (1 + cr^{2-n})\delta \qquad \text{on } \{r \ge 2r_0\}. \tag{A.13}$$

Finally,

$$E_{\text{ADM}}(\tilde{g}) - E_{\text{ADM}}(\bar{g}) = \eta_{r_0} = \frac{\Lambda}{(n-1)\epsilon} r_0^{-\epsilon} = O(\Lambda r_0^{-\epsilon}), \tag{A.14}$$

which tends to zero as $r_0 \to \infty$. The proof is complete. □

Next, we show that $R_{\tilde{g}}$ satisfies a weighted inequality.

Proposition A.2. *Let $\tilde{g}$ be the metric constructed in Proposition A.1. Let Σ^s_∞ and Q be the region and function defined in Lemma 5.1. Suppose $\mu - |J|_g \ge \lambda r^{-n-1}$ in the asymptotically flat end. Define*

$$I_s(\tilde{g}) := \int_{\Sigma^s_\infty} \left(\rho|\nabla\phi|^2_{\tilde{g}} + \frac{\rho}{2}\left(R_{\tilde{g}} - 2\Delta_{\tilde{g}} \log\rho - \frac{n+1}{n+2}|\nabla \log \rho|^2_{\tilde{g}} \right)\phi^2 \right) dV_{\tilde{g}},$$

and let $I(\tilde{g}) := \lim_{s\to\infty} I_s(\tilde{g})$. Then

$$I(\tilde{g}) \ge \int_{\Sigma_\infty} \frac{1}{2}\rho f^2 \phi^2 Q \, dV_{\tilde{g}}.$$

Proof. We compare $I_s(\tilde{g})$ and $I_s(\bar{g})$. Observe that

(A.15)
$$
\begin{aligned}
&I_s(\tilde{g}) - I_s(\bar{g}) \\
&= \int_{\Sigma^s_\infty} \left(\rho|\nabla\phi|^2_{\tilde{g}} + \frac{\rho}{2}\left(R_{\tilde{g}} - 2\Delta_{\tilde{g}}\log\rho - \frac{n+1}{n+2}|\nabla\log\rho|^2_{\tilde{g}}\right)\phi^2\right)(\sqrt{\det\tilde{g}} - \sqrt{\det\bar{g}})dx \\
&\quad + \int_{\Sigma^s_\infty}\left(\rho(\tilde{g}^{ij} - \bar{g}^{ij})\phi_i\phi_j + \frac{1}{2}\rho\phi^2(R_{\tilde{g}} - R_{\bar{g}})\right)dV_{\bar{g}} \\
&\quad - \int_{\Sigma^s_\infty}\frac{1}{2}\rho\phi^2\left(2(\Delta_{\tilde{g}} - \Delta_{\bar{g}})\log\rho + \frac{n+1}{n+2}(\tilde{g}^{ij} - \bar{g}^{ij})(\log\rho)_i(\log\rho)_j\right)dV_{\bar{g}}.
\end{aligned}
$$

First, we estimate the scalar curvature integral. Write

$$\bar{h} := \bar{g} - \delta, \qquad \tilde{h} := \tilde{g} - \delta. \tag{A.16}$$

The scalar curvature expansion about the Euclidean metric is

$$R_{\delta+h} = L_\delta h + \mathcal{Q}(h, \partial h, \partial^2 h), \tag{A.17}$$

where $\mathcal{Q}$ is the quadratic term and

$$L_\delta h = \partial_i\partial_j h_{ij} - \Delta_\delta(\operatorname{tr}_\delta h), \tag{A.18}$$

and, for h sufficiently small in C^2,

$$|\mathcal{Q}(h, \partial h, \partial^2 h)| \le C\big(|h||\partial^2 h| + |\partial h|^2\big). \tag{A.19}$$

Since

$$\bar{h} = O_2(r^{2-n}), \qquad g_c - \delta = O_2(r^{2-n}), \tag{A.20}$$

the cutoff construction gives

$$\tilde{h} = O_2(r^{2-n}), \tag{A.21}$$

with constants independent of r_0, for r_0 sufficiently large. Hence

$$\mathcal{Q}(\tilde{h}, \partial\tilde{h}, \partial^2\tilde{h}) = O(r^{2-2n}), \quad \mathcal{Q}(\bar{h}, \partial\bar{h}, \partial^2\bar{h}) = O(r^{2-2n}). \tag{A.22}$$

Using that $\tilde{g} = \bar{g}$ in $\{r \le r_0\}$ and

$$R_{\tilde{g}} - R_{\bar{g}} = L_\delta(\tilde{h} - \bar{h}) + O(r^{2-2n}), \tag{A.23}$$

we may integrate L_δ by parts to obtain

$$
\begin{aligned}
&\int_{\Sigma^r_\infty}\frac{1}{2}\rho\phi^2(R_{\tilde{g}} - R_{\bar{g}})dV_{\bar{g}} \\
&= \int_{\Sigma^r_\infty\setminus\Sigma^{r_0}_\infty} -\rho\phi\partial_i\phi\left(\partial_j(\tilde{h}_{ij} - \bar{h}_{ij}) - \partial_i(\tilde{h}_{jj} - \bar{h}_{jj})\right) + \rho\phi^2 O(r^{2-2n})dV_{\bar{g}}.
\end{aligned} \tag{A.24}
$$

Applying the decay of all relevant terms to (A.15) yields

$$I_r(\tilde{g}) \ge I_r(\bar{g}) + \int_{\Sigma^r_\infty\setminus\Sigma^{r_0}_\infty}\left(O(r^{2-n})\rho|\nabla\phi|^2_{\bar{g}} + O(r^{2-2n})\rho\phi^2 + O(r^{1-n})\rho\phi|\nabla\phi|_{\bar{g}}\right)dV_{\bar{g}}.$$

Now apply equation (5.10) and the inequality

$$r^{1-n}\rho\phi|\nabla\phi|_{\bar{g}} \le \frac{r^{3.1-n}}{2}\rho|\nabla\phi|^2_{\bar{g}} + \frac{r^{-n-1.1}}{2}\rho\phi^2, \tag{A.25}$$

and taking $r \to \infty$, we obtain

$$I(\tilde{g}) \geq \int_{\Sigma_\infty} \left(\frac{4Q}{5} \rho \phi^2 + \rho |\nabla \phi + \phi(X + \nabla \log \rho)|_{\bar{g}}^2 \right) dV_{\bar{g}} - \int_{\Sigma_\infty \setminus \Sigma_\infty^{r_0}} c_0 r^{3.1-n} \rho |\nabla \phi|_{\bar{g}}^2 dV_{\bar{g}} \tag{A.26}$$

for some $c_0 > 0$, where we use $Q \geq \frac{\lambda}{2} r^{-n-1}$ to absorb $O(r^{-n-1.1})\rho\phi^2$. Since

$$\begin{aligned} &|\nabla \phi|_{\bar{g}}^2 \\ =&|\nabla \phi + \phi(X + \nabla \log \rho)|_{\bar{g}}^2 - 2\phi \nabla \phi \cdot (X + \nabla \log \rho) - \phi^2 |X + \nabla \log \rho|_{\bar{g}}^2 \\ \leq&|\nabla \phi + \phi(X + \nabla \log \rho)|_{\bar{g}}^2 + \frac{1}{2}|\nabla \phi|_{\bar{g}}^2 + \phi^2 |X + \nabla \log \rho|_{\bar{g}}^2, \end{aligned} \tag{A.27}$$

it follows that

$$\begin{aligned} |\nabla \phi|_{\bar{g}}^2 \leq& 2|\nabla \phi + \phi(X + \nabla \log \rho)|_{\bar{g}}^2 + 2\phi^2 |X + \nabla \log \rho|_{\bar{g}}^2 \\ \leq& 2|\nabla \phi + \phi(X + \nabla \log \rho)|_{\bar{g}}^2 + c_1 r^{2-2n} \phi^2, \end{aligned} \tag{A.28}$$

for some $c_1 > 0$. Therefore, we have

$$I(\tilde{g}) \geq \int_{\Sigma_\infty} \left(\frac{3Q}{5} \rho \phi^2 + \frac{1}{2} \rho |\nabla \phi + \phi(X + \nabla \log \rho)|_{\bar{g}}^2 \right) dV_{\bar{g}}. \tag{A.29}$$

Hence, if r_0 is chosen large enough such that $\sqrt{\det \bar{g}} \geq \frac{5}{6} \sqrt{\det \tilde{g}}$, we have

$$I(\tilde{g}) \geq \int_{\Sigma_\infty} \frac{3Q}{5} \rho \phi^2 dV_{\bar{g}} \geq \int_{\Sigma_\infty} \frac{1}{2} \rho \phi^2 Q dV_{\tilde{g}}. \tag{A.30}$$

□

Remark A.3. Note that $R_{\tilde{g}} \in L^1$; thus, $I(\tilde{g})$ is invariant for any exhaustions.

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