KERR BLACK HOLE UNIQUENESS

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Abstract. We prove the black hole uniqueness conjecture in the axially symmetric, stationary, vacuum setting: there is no regular asymptotically flat configuration with more than one horizon component when every degenerate component has nonzero angular momentum. More precisely, it is shown that in any such multi-horizon configuration, the logarithmic angle defect along every finite axis rod is strictly negative, and hence the interaction force along such axis rods is always attractive. The proof is based on a refined asymptotic analysis of the associated singular harmonic maps, in part from the companion paper [16], together with a global bound for the Weyl conformal factor arising from a certain scalar curvature related differential inequality. Furthermore, as a corollary of the extremal version of this uniqueness result, we obtain the conjectured mass-angular momentum inequality for multiple dynamical black holes using the main theorem of [17].

1. Introduction

The black hole uniqueness problem asks whether a regular stationary black hole is determined by the conserved quantities measured at infinity. In four-dimensional vacuum general relativity, the classical uniqueness theory identifies the Kerr family as the unique solution when the event horizon is connected, subject to the usual global regularity hypotheses. The nondegenerate theory is developed in [8, 9, 12, 24, 27], while the connected degenerate case is treated in [10, 15]; see [13] for a survey. A central feature of this theory is the harmonic map structure of the axisymmetric stationary vacuum Einstein equations. In Weyl–Papapetrou coordinates these equations reduce to a singular harmonic map from Euclidean three-space minus the rotation axis into the hyperbolic plane, while the remaining metric coefficient is recovered by quadrature from the harmonic map stress-energy tensor.

The outstanding balance problem concerns a disconnected event horizon. The classical Bach–Weyl analysis [5] already showed in the static two-body setting that a conical singularity, interpreted as a strut, occurs on the axis between the bodies. The static uniqueness theorem of Bunting and Masood-ul-Alam [7], together with the exclusion of degenerate static horizon components by Chruściel–Reall–Tod [11], rules out regular static vacuum multi-black hole configurations. In the rotating setting, Weinstein's harmonic map construction [28, 29] produces axisymmetric stationary vacuum metrics for arbitrary finite nondegenerate rod structures, which are smooth away from possible conical singularities on the axis. An early nonexistence result under an additional symmetry hypothesis was obtained by Li and Tian [21]; the axis and pole regularity theory and the associated force analysis were developed in [22, 23, 28, 30]. For two horizon components, nonexistence of equilibrium was proven by Neugebauer–Hennig in

Q. Han acknowledges the support of NSF Grant DMS-2305038. M. Khuri acknowledges the support of NSF Grant DMS-2405045. J. Xiong acknowledges the partial support of NSFC Grants 12325104. G. Weinstein acknowledges the support of ISF Grant 1119/26.

the subextreme case [25], and extended to the degenerate case in [18]; see also the two-Kerr singularity analysis of Chruściel–Eckstein–Nguyen–Szybka [14]. The corresponding problem for an arbitrary finite number of components, with any mixture of nondegenerate and degenerate horizons, has until now remained open.

We now describe the setting to which the main theorem applies. Let $\Gamma = \{\rho = 0\} \subset \mathbb{R}^3$ denote the rotation z-axis in cylindrical coordinates (ρ, z, ϕ), with ϕ of period 2π. On the domain of outer communications away from the orbit space boundary, the spacetimes under consideration have topology $\mathbb{R} \times (\mathbb{R}^3 \setminus \Gamma)$, and their axisymmetric stationary vacuum metrics take the Weyl–Papapetrou form

$$\mathbf{g} = -e^{2U} d\tau^2 + \rho^2 e^{-2U}(d\phi + w d\tau)^2 + e^{-2U+2\alpha}(d\rho^2 + dz^2). \tag{1.1}$$

Here ∂_τ and ∂_ϕ are respectively the stationary and rotational Killing fields. Setting

$$u = U - \ln \rho,$$

the vacuum Einstein equations reduce to the axisymmetric harmonic map system

$$\Delta u = 2e^{4u}|\nabla v|^2, \qquad \Delta v + 4\nabla u \cdot \nabla v = 0, \tag{1.2}$$

for

$$\Phi = (u, v) : \mathbb{R}^3 \setminus \Gamma \longrightarrow \mathbb{H}^2, \qquad ds^2_{\mathbb{H}^2} = du^2 + e^{4u} dv^2,$$

where v is the twist potential and Δ is the Euclidean Laplacian on $\mathbb{R}^3$. Conversely, once Φ is fixed, the remaining metric coefficients are recovered, up to additive constants, from

$$\begin{aligned} w_\rho &= 2\rho e^{4u} v_z, & w_z &= -2\rho e^{4u} v_\rho, \\ \alpha_\rho &= \rho \left[U_\rho^2 - U_z^2 + e^{4u}(v_\rho^2 - v_z^2) \right], & \alpha_z &= 2\rho \left[U_\rho U_z + e^{4u} v_\rho v_z \right]. \end{aligned} \tag{1.3}$$

The integrability conditions for these quadratures are consequences of (1.2). The asymptotically flat normalization fixes the additive constants by requiring $w \to 0$ and $\alpha \to 0$ at infinity. Thus the rod structure, angular momenta, logarithmic angle defects, and interaction forces appearing below are all determined by the singular harmonic map and its reconstruction [9, 28].

The boundary $\{\rho = 0\}$ of the orbit half-plane decomposes into alternating horizon and axis intervals. Let

$$\mathcal{H}_i = [z_i^-, z_i^+], \qquad z_i^- \le z_i^+ < z_{i+1}^-, \qquad 1 \le i \le N,$$

be the ordered horizon components. If $z_i^- < z_i^+$, the interior of $\mathcal{H}_i$ is a nondegenerate horizon rod. The Killing field

$$\partial_\tau + \Omega_i \partial_\phi, \qquad \Omega_i = -w|_{\mathrm{Int}\mathcal{H}_i},$$

is null there, with the constant Ω_i denoting angular velocity, and the endpoints $(0, 0, z_i^-)$ and $(0, 0, z_i^+)$ are the lower and upper poles of the horizon. If $z_i^- = z_i^+ = z_i$, the horizon is degenerate and is represented in the orbit space by the puncture $p_i = (0, 0, z_i) \in \mathbb{R}^3$. The complementary open intervals

$$\Gamma_1 = (-\infty, z_1^-), \qquad \Gamma_{j+1} = (z_j^+, z_{j+1}^-), \qquad \Gamma_{N+1} = (z_N^+, \infty), \qquad 1 \le j \le N-1,$$

are the axis rods. On each Γ_j the rotational Killing field vanishes and the twist potential has a constant trace, referred to as a potential constant, say

$$v = c_j \quad \text{on } \Gamma_j.$$

With the orientation convention used in [16, 17], the Komar angular momentum of the ith horizon component is then

$$\mathcal{J}_i = \frac{c_{i+1} - c_i}{4}.$$

The *logarithmic angle defects* $\mathbf{b}_j$ also arise directly from the reconstructed harmonic map data. More precisely, standard open axis regularity and (1.3) imply that α has a constant trace on each axis rod. For $z_0 \in \Gamma_j$, the appropriate radius/circumference ratio computed from (1.1) gives

$$e^{\mathbf{b}_j} := \lim_{\rho \to 0} \frac{\int_0^\rho e^{\alpha(s,z_0) - U(s,z_0)}\, ds}{\rho e^{-U(\rho,z_0)}} = e^{\alpha(0,z_0)} \qquad \Rightarrow \qquad \mathbf{b}_j = \alpha|_{\Gamma_j}.$$

The corresponding interaction force [28] is expressed as

$$\mathcal{F}_j = \frac{1}{4}\left(e^{-\mathbf{b}_j} - 1\right),$$

so that $\mathbf{b}_j < 0$ is equivalent to a strictly positive, or attractive, force on the axis rod. Since $\alpha \to 0$ at infinity, the two semi-infinite rods satisfy

$$\mathbf{b}_1 = \mathbf{b}_{N+1} = 0. \tag{1.4}$$

We shall refer to a map Φ satisfying (1.2), with the rod data just described, as a *stationary vacuum singular harmonic map* with black hole rod data when it has: the asymptotics at ordinary axis points and at the poles of nondegenerate horizons as in Li–Tian [21, 22, 23] and Weinstein [28, 29, 31], the standard regular horizon behavior on the interiors of nondegenerate horizons, and the asymptotics at punctures (degenerate horizons) and infinity as in Han–Khuri–Weinstein–Xiong [16]. The precise estimates used in the proof are given in Proposition 2.1 of the next section. The metric reconstructed from such a map is vacuum away from the orbit space boundary and has the prescribed horizon structure, but it is allowed a priori to possess conical singularities on the bounded axis rods. Determining whether those singularities can all be removed is exactly the balance problem addressed here.

Theorem 1.1. *Let $N > 1$. Consider an ordered collection $\mathcal{H}_1, \ldots, \mathcal{H}_N$ of horizon rods, and let*

$$\Phi = (u, v) : \mathbb{R}^3 \setminus \Gamma \longrightarrow \mathbb{H}^2$$

be a stationary vacuum singular harmonic map with this black hole rod data in the sense described above. Set $U = u + \ln\rho$, and let w, α, and the Weyl–Papapetrou metric $\mathbf{g}$ be reconstructed from Φ by (1.1)–(1.3) with the asymptotically flat normalization. Assume that

$$\mathcal{J}_i \neq 0$$

for every degenerate horizon component; no hypothesis is imposed on the angular momenta of nondegenerate horizon rods. Then every bounded axis rod has strictly negative logarithmic angle defect:

$$\mathbf{b}_j < 0, \qquad 2 \le j \le N. \tag{1.5}$$

In particular, there is no regular asymptotically flat, axisymmetric, stationary vacuum multi-black hole configuration with more than one horizon component in this class.

Under the standard analyticity, asymptotic flatness, and I^+-regularity hypotheses, the rigidity theorem implies that a stationary vacuum black hole with at least one rotating horizon component is axisymmetric, even when the event horizon is disconnected; see [12, Theorem 4.14]. Theorem 1.1 then forces the horizon to be connected, and the connected-horizon uniqueness theorem [10, Theorem 1.1] implies that the domain of outer communications is that of a Kerr spacetime. Thus, in the rotating analytic class, the Kerr family exhausts the regular asymptotically flat stationary vacuum black holes without any a priori assumption on horizon connectedness. Progress has been made by Alexakis–Ionescu–Klainerman to obtain axisymmetry without the analyticity assumption [1], using instead either a closeness to Kerr hypothesis [2], or a small angular momentum hypothesis [3].

The special case of Theorem 1.1 in which all horizons are degenerate, confirms [17, Conjecture 3.3]. As a consequence, [17, Theorem 1.1] yields the unconditional mass-angular momentum inequality for multiple dynamical black holes, which we state here. This inequality, like the Penrose inequality, was conjectured based on Penrose's heuristic physical arguments [26], and is closely tied with the standard picture of gravitational collapse.

Corollary 1.2. *Let (M, g, k) be a complete, simply connected, axially symmetric, maximal initial data set for the Einstein equations with one designated asymptotically flat end and finitely many other ends, each asymptotically flat or asymptotically cylindrical. Assume that the energy density is nonnegative, $\mu \geq 0$, and that the momentum density vanishes in the rotational direction, $J(\eta) = 0$. If m and $\mathcal{J}$ are respectively the ADM mass and ADM angular momentum of the designated end, then*

$$m \geq \sqrt{|\mathcal{J}|}.$$

Equality holds if and only if the data arise from a constant time slice of an extreme Kerr spacetime.

We now outline the proof of the main theorem. The first observation concerns a global upper bound for the Weyl conformal factor α in terms of the logarithmic angle defects. This arises from a new algebraic identity that produces the following nonlinear differential inequality

$$\Delta_{\rho,z}\alpha + \rho^{-1}|\nabla\alpha| \geq 2e^{4u}|\nabla v|^2 \geq 0. \tag{1.6}$$

By applying the maximum principle on a sequence of exhaustion domains separated from the axis in the orbit space half-plane, and using the asymptotic profile at nondegenerate horizon poles [22] and degenerate horizon punctures [16], we obtain the desired global upper bound

$$\alpha \leq \max_{1\leq j\leq N+1} \mathbf{b}_j. \tag{1.7}$$

The second primary observation is a strict local inequality for α along a bounded axis rod. To obtain it we first show that, starting only with the known $C^{3,\gamma}$ regularity across an open axis rod, the equations force the factorization

$$U(\rho, z) = F(\rho^2, z), \qquad v(\rho, z) = c + \rho^4 G(\rho^2, z), \tag{1.8}$$

where F and G are smooth. This is achieved by lifting $(v - c)/\rho^4$ radially to six transverse variables, where the resulting axis has codimension six and zero H^1 capacity, so the lifted equation extends weakly across it; elliptic bootstrapping then gives smoothness. Next note that the trace $U(0, z)$ tends to $-\infty$ at the boundary of axis rods, and thus it must have an interior maximum there. At an interior maximum of the rod trace of U, using (1.8), the local expansion shows that $\alpha > \mathbf{b}_j$ nearby unless the leading U and twist coefficients both vanish. If v is constant, the conclusion follows directly from harmonicity of U. Otherwise, strong unique continuation and the uniqueness of rotationally invariant homogeneous harmonic modes in dimensions three and seven identify the first nonzero terms. Comparing the free and source-generated parts of U then again yields points arbitrarily close to the rod for which

$$\alpha > \mathbf{b}_j. \tag{1.9}$$

Thus a bounded axis rod cannot have the global maximum of the logarithmic angle defects. Since the two semi-infinite defects vanish, (1.5) follows.

The paper is organized as follows. Section 2 recalls the relevant asymptotic analysis of the harmonic map equations. Section 3 is dedicated to the proof of the differential inequality (1.6), and global upper bound (1.7). The regular axis factorization (1.8) is obtained in Section 4. In Section 5 we establish the strict local inequality (1.9) near bounded axis rods, and Section 6 contains the proof of the main theorem.

Acknowledgements. The authors would like to thank Demetrios Christodoulou who introduced the third author to this problem more than four decades ago. The authors also thank Nan Wu for discussions, and the American Institute of Mathematics for its hospitality.

2. The Singular Harmonic Map

In this section, we collect known results concerning the relevant asymptotics for the singular harmonic maps at the interior of axis rods, at poles and punctures, and at infinity. Recall for later use that if $h = h(\rho, z)$ is an axisymmetric function then

$$\Delta h = h_{\rho\rho} + \frac{1}{\rho} h_\rho + h_{zz}, \qquad \Delta_{\rho,z} h = h_{\rho\rho} + h_{zz}, \qquad |\nabla h|^2 = h_\rho^2 + h_z^2,$$

where Δ is the Euclidean Laplacian on $\mathbb{R}^3$ and $\Delta_{\rho,z}$ is the ordinary Laplacian on the orbit half-plane. In particular, since $\Delta \ln \rho = 0$ for $\rho > 0$, the harmonic map equations (1.2) may be expressed as

$$U_{\rho\rho} + \frac{1}{\rho} U_\rho + U_{zz} = 2e^{4u}(v_\rho^2 + v_z^2), \tag{2.1}$$

$$v_{\rho\rho} + \frac{1}{\rho} v_\rho + v_{zz} = -4\big((U_\rho - \rho^{-1})v_\rho + U_z v_z\big). \tag{2.2}$$

2.1. **Set up and hypotheses.** Let $\Phi = (u, v) \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma, \mathbb{H}^2)$ be an axisymmetric map satisfying the harmonic map equations (1.2) weakly. Assume that Φ is locally minimizing for the renormalized energy in the class of maps having the same prescribed rod traces and singularity data, where the energy and renormalized energy densities are, respectively,

$$e(u, v) = |\nabla u|^2 + e^{4u}|\nabla v|^2, \qquad e'(U, v) = |\nabla U|^2 + e^{4u}|\nabla v|^2.$$

Suppose that the z-axis Γ has the finite mixed black hole rod structure described in Section 1, with axis rods $\Gamma_1, \dots, \Gamma_{N+1}$ and horizon components $\mathcal{H}_1, \dots, \mathcal{H}_N$. Let $\mathcal{I}_{\mathrm{deg}}$ and $\mathcal{I}_{\mathrm{nd}}$ denote, respectively, the sets of degenerate and nondegenerate horizon components. Consider the following three assumptions.

(A1) The appropriate local renormalizations are bounded and have finite renormalized energy at ordinary axis points, at interiors of nondegenerate horizon rods, and at their poles. More precisely, the following conditions hold.

(a) For every compact interval $I \subset \Gamma_j$ contained in an axis rod, there is a neighborhood $\mathcal{O}_I \subset \mathbb{R}^3$ of I such that

$$U, v \in L^\infty(\mathcal{O}_I \setminus \Gamma), \qquad e'(U,v) \in L^1(\mathcal{O}_I),$$

and

$$v = c_j \qquad \text{on } I$$

in the trace sense.

(b) For every $i \in \mathcal{I}_{\mathrm{nd}}$ and every compact interval $I \subset \mathrm{Int}\mathcal{H}_i$ contained in the interior of the horizon rod, there is a neighborhood $\mathcal{O}_I \subset \mathbb{R}^3$ of I such that

$$u, v \in L^\infty(\mathcal{O}_I \setminus \Gamma), \qquad e(u,v) \in L^1(\mathcal{O}_I).$$

(c) If $q_i^\pm = (0,0,z_i^\pm)$ are poles of a nondegenerate horizon rod, write

$$\zeta_i^\pm = z - z_i^\pm, \qquad r_i^\pm = \sqrt{\rho^2 + (\zeta_i^\pm)^2},$$

and define the pole-renormalized functions

$$\widehat{U}_i^- = U - \frac{1}{2}\ln(r_i^- - \zeta_i^-) = u + \frac{1}{2}\ln(r_i^- + \zeta_i^-),$$
$$\widehat{U}_i^+ = U - \frac{1}{2}\ln(r_i^+ + \zeta_i^+) = u + \frac{1}{2}\ln(r_i^+ - \zeta_i^+).$$

In a neighborhood of $q_i^\pm$, the pair $(\widehat{U}_i^\pm, v)$ is locally bounded and satisfies

$$|\nabla \widehat{U}_i^\pm|^2 + e^{4u}|\nabla v|^2 \in L^1.$$

(A2) For every $i \in \mathcal{I}_{\mathrm{deg}}$, let $p_i = (0,0,z_i)$ and introduce polar coordinates (r_i, θ_i, ϕ) centered at p_i. There are constants $\varepsilon_i, \Lambda_i > 0$ such that

$$|u + \ln\sin\theta_i| \le \Lambda_i \quad \text{in } B_{\varepsilon_i}(p_i) \setminus \Gamma, \qquad e'(U,v) \in L^1_{\mathrm{loc}}(B_{\varepsilon_i}(p_i) \setminus \{p_i\}).$$

Moreover,

$$v = c_i \quad \text{on the axis rod below } p_i, \qquad v = c_{i+1} \quad \text{on the axis rod above } p_i$$

in the trace sense, where

$$c_{i+1} - c_i = 4\mathcal{J}_i \neq 0.$$

(A3) There are constants $r_0, \Lambda_\infty > 0$ such that

$$|U| + |v| \le \Lambda_\infty \quad \text{in } \overline{B}_{r_0}^c \setminus \Gamma, \qquad e'(U,v) \in L^1_{\mathrm{loc}}(\overline{B}_{r_0}^c),$$

and v has constant traces c_1 and c_{N+1} on the two semi-infinite axis rods. Moreover, $U \to 0$ as $r \to \infty$.

2.2. **Existence and uniqueness.** The following result is well-known, and is detailed in the survey [32]. Namely, given a rod structure configuration of horizon components $\{\mathcal{H}_i\}_{i=1}^N$, and axis rods $\{\Gamma_j\}_{j=1}^{N+1}$ with potential constants $\{c_j\}_{j=1}^{N+1}$ such that $\mathcal{J}_i \neq 0$ for each $i \in \mathcal{I}_{\mathrm{deg}}$, there exists a smooth axisymmetric model map $\Phi_0 : \mathbb{R}^3 \setminus \Gamma \longrightarrow \mathbb{H}^2$ realizing the prescribed rod structure and potential constants, and satisfying assumptions (A1) – (A3). The model map may be constructed by superposition of appropriate individual Kerr harmonic maps. There is then a unique axisymmetric harmonic map $\Phi : \mathbb{R}^3 \setminus \Gamma \longrightarrow \mathbb{H}^2$ which is asymptotic to Φ_0 in the sense that

$$\sup_{\mathbb{R}^3 \setminus \Gamma} d_{\mathbb{H}^2}(\Phi, \Phi_0) < \infty, \qquad d_{\mathbb{H}^2}(\Phi, \Phi_0) \longrightarrow 0 \quad \text{as } r \longrightarrow \infty.$$

The harmonic map Φ also satisfies (A1) – (A3).

2.3. **Regularity and asymptotics.**

Proposition 2.1. *Let $\Phi = (u, v) : \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2$ be a harmonic map satisfying the assumptions of Section 2.1. The the following statements hold.*

(i) Axis regularity. *On every compact subinterval of an axis rod Γ_j, the pair (U, v) extends across the axis as an axisymmetric $C^{3,\gamma}$ pair for every $\gamma \in (0, 1)$.*

(ii) Degenerate horizon punctures. *Let $i \in \mathcal{I}_{\mathrm{deg}}$. Set $a_i = 2|\mathcal{J}_i|$ and $\epsilon_i = \operatorname{sgn} \mathcal{J}_i$. Then there exists $b_i \in (-1, 1)$ such that*

$$U = \ln r_i + \bar{U}_i(\theta_i) + \widetilde{U}_i, \qquad v = \bar{v}_i(\theta_i) + \widetilde{v}_i, \tag{2.3}$$

where

$$\bar{U}_i(\theta_i) = -\frac{1}{2} \ln \left(\frac{2a_i\sqrt{1 - b_i^2}}{1 + \cos^2\theta_i + 2b_i \cos\theta_i} \right), \tag{2.4}$$

$$\bar{v}_i(\theta_i) = \epsilon_i a_i \left(\frac{b_i + b_i\cos^2\theta_i + 2\cos\theta_i}{1 + \cos^2\theta_i + 2b_i\cos\theta_i} \right) + v_i^0, \qquad v_i^0 = \frac{c_i + c_{i+1}}{2}.$$

Moreover, if

$$\bar{u}_i = \bar{U}_i - \ln \sin\theta_i,$$

then for every fixed $\varsigma \in (0, 1)$, there are $\beta_i \in (0, 1)$ and $C_i > 0$ such that

$$\max_{\mathbb{S}^2} \left(\left| (r_i\partial_{r_i})^\ell \nabla^k_{\mathbb{S}^2} \widetilde{U}_i \right| + e^{(3+\varsigma-k)\bar{u}_i} \left| (r_i\partial_{r_i})^\ell \nabla^k_{\mathbb{S}^2} \widetilde{v}_i \right| \right) \leq C_i r_i^{\beta_i}, \qquad \ell + k \leq 3.$$

The corresponding tangent profile of the Weyl conformal factor is

$$\bar{\alpha}_i(\theta_i) = \frac{\mathbf{b}_i + \mathbf{b}_{i+1}}{2} + \ln \left(\frac{1 + \cos^2\theta_i + 2b_i\cos\theta_i}{2\sqrt{1 - b_i^2}} \right),$$

with

$$|\alpha - \bar{\alpha}_i| + r_i \left|\partial_{r_i}(\alpha - \bar{\alpha}_i)\right| + \left|\partial_{\theta_i}(\alpha - \bar{\alpha}_i)\right| \leq C_i r_i^{\beta_i}$$

uniformly for $0 \leq \theta_i \leq \pi$. Furthermore,

$$\mathbf{b}_{i+1} - \mathbf{b}_i = \ln\left(\frac{1 + b_i}{1 - b_i} \right) = 2 \operatorname{arctanh} b_i. \tag{2.5}$$

(iii) Nondegenerate horizon poles and interiors. *Let $i \in \mathcal{I}_{\mathrm{nd}}$. There are constants $\kappa_i^{\pm} \in \mathbb{R}$ and exponents $\gamma_i^{\pm} \in (0,1)$ such that in a neighborhood of the poles*

$$U = \frac{1}{2}\ln(r_i^- - \zeta_i^-) + \kappa_i^- + R_i^-, \qquad |R_i^-| + r_i^-|\nabla R_i^-| \leq C(r_i^-)^{\gamma_i^-},$$
$$U = \frac{1}{2}\ln(r_i^+ + \zeta_i^+) + \kappa_i^+ + R_i^+, \qquad |R_i^+| + r_i^+|\nabla R_i^+| \leq C(r_i^+)^{\gamma_i^+}, \tag{2.6}$$

while the corresponding Weyl conformal factor expansions are

$$\alpha(r_i^-, \theta_i^-) = \mathbf{b}_i + \ln\sin\frac{\theta_i^-}{2} + O((r_i^-)^{\gamma_i^-}), \tag{2.7}$$

$$\alpha(r_i^+, \theta_i^+) = \mathbf{b}_{i+1} + \ln\cos\frac{\theta_i^+}{2} + O((r_i^+)^{\gamma_i^+}). \tag{2.8}$$

Furthermore, on every compact subinterval $I \subset \mathcal{H}_i$, the functions (u,v) extend across the rod as an axisymmetric smooth pair, and there is a constant $\kappa_i \in \mathbb{R}$, depending only on the horizon rod $\mathcal{H}_i$, such that in a neighborhood of I,

$$\alpha(\rho, z) = \ln\rho + 2u(\rho, z) + \kappa_i + O(\rho^2) \qquad \text{as } \rho \to 0. \tag{2.9}$$

(iv) Infinity expansion. *Set*

$$\mathcal{J} = \frac{c_{N+1} - c_1}{4}, \qquad c_\infty = \frac{c_{N+1} + c_1}{2}.$$

For every fixed $\varsigma \in (0,1)$ there are constants $c_1^\infty \in \mathbb{R}$ and a remainder $\widetilde{v}_\infty$ such that

$$U = \frac{c_1^\infty}{r} + O_1(r^{-2}), \qquad v = \mathcal{J}\cos\theta(3 - \cos^2\theta) + c_\infty + \widetilde{v}_\infty,$$

where

$$|\partial_r \widetilde{v}_\infty| \leq Cr^{-2}\sin^{3+\varsigma}\theta, \qquad |\partial_\theta \widetilde{v}_\infty| \leq Cr^{-1}\sin^{2+\varsigma}\theta.$$

Moreover, the Weyl conformal factor satisfies

$$\alpha = O(r^{-2}), \qquad \partial_r\alpha = O(r^{-3}), \qquad \partial_\theta\alpha = O(r^{-2}). \tag{2.10}$$

Proof. The puncture and infinity statements are from [16, Theorems 2.1–2.3 and equations (7.2)–(7.4)], while the axis and pole statements are based on Li–Tian and Weinstein [22, 23, 28, 29, 31]; see also the summary in [16, Section 2, discussion following Theorem 2.3]. The corresponding statements concerning the Weyl conformal factor arise directly from the harmonic map asymptotics and equations (1.3). □

3. Weyl Conformal Factor Global Upper Bound

The goal of this section is to obtain the global upper bound (1.7) for α. We begin by deriving the nonlinear differential inequality (1.6). Although the equation for $\Delta_{\rho,z}\alpha$ obtained by differentiating the quadratures (1.3) has an indefinite right-hand side, the negative term is controlled exactly by $|\nabla\alpha|/\rho$. Moreover, our proof relates this inequality to the scalar curvature of constant time slices in the stationary vacuum spacetime.

3.1. The Weyl conformal factor identity/inequality.

Lemma 3.1. *Away from Γ, the Weyl conformal factor satisfies the identity*

$$\Delta_{\rho,z}\alpha = -|\nabla U|^2 + 3e^{4u}|\nabla v|^2, \tag{3.1}$$

and consequently the differential inequality

$$\Delta_{\rho,z}\alpha + \frac{|\nabla\alpha|}{\rho} \geq 2e^{4u}|\nabla v|^2 \geq 0. \tag{3.2}$$

Proof. Consider a time slice $\{\tau = \text{constant}\}$, of the stationary vacuum spacetime with metric (1.1). The induced metric induced on Σ_τ is given by

$$h = e^{-2U}\bar{q}, \qquad \bar{q} = e^{2\alpha}(d\rho^2 + dz^2) + \rho^2 d\phi^2.$$

A standard Riemannian geometry formula yields the scalar curvature of this slice

$$R_h = e^{2U}\left(\bar{R} + 4\bar{\Delta}U - 2|\bar{\nabla}U|^2_{\bar{q}}\right).$$

Moreover, for the metric $\bar{q}$, a direct calculation yields

$$\bar{R} = -2e^{-2\alpha}\Delta_{\rho,z}\alpha, \qquad \bar{\Delta}U = e^{-2\alpha}\Delta U, \qquad |\bar{\nabla}U|^2_{\bar{q}} = e^{-2\alpha}|\nabla U|^2,$$

and hence

$$R_h = e^{2U-2\alpha}\left(-2\Delta_{\rho,z}\alpha + 4\Delta U - 2|\nabla U|^2\right). \tag{3.3}$$

On the other hand, we may compute the scalar curvature from the vacuum Hamiltonian constraint [20, (2.17)] to find

$$R_h = 2e^{2U-2\alpha}e^{4u}|\nabla v|^2.$$

Furthermore, by the harmonic map equations

$$\Delta U = \Delta u + \Delta \ln\rho = 2e^{4u}|\nabla v|^2, \tag{3.4}$$

since $\ln\rho = 0$ is harmonic. The desired identity (3.1) now follows by combining (3.3)-(3.4).

To obtain (3.2), first observe that the two quadratures (1.3) give

$$\frac{|\nabla\alpha|^2}{\rho^2} = \left[U_\rho^2 - U_z^2 + e^{4u}(v_\rho^2 - v_z^2)\right]^2 + 4\left[U_\rho U_z + e^{4u}v_\rho v_z\right]^2.$$

The elementary identity

$$\left[(X_1^2 - X_2^2) + (Y_1^2 - Y_2^2)\right]^2 + 4(X_1X_2 + Y_1Y_2)^2 = (|X|^2 - |Y|^2)^2 + 4(X\cdot Y)^2$$

with $X = \nabla U$ and $Y = e^{2u}\nabla v$ produces

$$\rho^{-1}|\nabla\alpha| = \left||\nabla U|^2 - e^{4u}|\nabla v|^2\right|.$$

Combining this with (3.1) yields

$$\Delta_{\rho,z}\alpha + \rho^{-1}|\nabla\alpha| \geq -|\nabla U|^2 + 3e^{4u}|\nabla v|^2 + \left||\nabla U|^2 - e^{4u}|\nabla v|^2\right| \geq 2e^{4u}|\nabla v|^2,$$

which proves (3.2). $\square$

3.2. **Puncture, pole, and axis estimates.** We will use the asymptotics obtained in Section 2 to provide estimates for the Weyl conformal factor near sensitive locations, in preparation for the maximal principle argument.

Lemma 3.2. *The Weyl conformal factor satisfies the following estimates.*

(i) Punctures. *Let $i \in \mathcal{I}_{\text{deg}}$, and let (r_i, θ_i, ϕ) be polar coordinates centered at puncture p_i. Then there exists a constant $C > 0$ such that in a neighborhood of this puncture*

$$\sup_{0 \leq \theta_i \leq \pi} \alpha(r_i, \theta_i) \leq \max\{\mathbf{b}_i, \mathbf{b}_{i+1}\} + C r_i^{\beta_i}. \tag{3.5}$$

(ii) Poles. *Let $i \in \mathcal{I}_{\text{nd}}$, and let $(r_i^{\pm}, \theta_i^{\pm}, \phi)$ be polar coordinates centered at poles $q_i^{\pm}$. Then there exists a constant $C > 0$ such that in a neighborhood of these poles*

$$\sup_{0 < \theta_i^- \leq \pi} \alpha(r_i^-, \theta_i^-) \leq \mathbf{b}_i + C(r_i^-)^{\gamma_i^-}, \qquad \sup_{0 \leq \theta_i^+ < \pi} \alpha(r_i^+, \theta_i^+) \leq \mathbf{b}_{i+1} + C(r_i^+)^{\gamma_i^+}.$$

(iii) Axis. *Let I be a compact subinterval of an axis rod Γ_j. Then there exist constants $C_I, \rho_I > 0$ such that*

$$\sup_{z \in I} |\alpha(\rho, z) - \mathbf{b}_j| \leq C_I \rho^2, \qquad 0 \leq \rho \leq \rho_I.$$

Proof. Consider the three regions separately.

(i) *Punctures.* According to Proposition 2.1(ii), in a neighborhood of the puncture

$$\alpha(r_i, \theta_i) = \frac{\mathbf{b}_i + \mathbf{b}_{i+1}}{2} + \ln\left(\frac{1 + \cos^2\theta_i + 2b_i \cos\theta_i}{2\sqrt{1 - b_i^2}}\right) + O(r_i^{\beta_i}). \tag{3.6}$$

Set $x = \cos\theta_i$, and note that the function $x \mapsto 1 + x^2 + 2b_i x$ is convex, so that its maximum on $[-1, 1]$ occurs at an endpoint. Since ln is increasing, the second term on the right-hand side of (3.6) is bounded above by the larger of its two pole values. This, combined with (2.5) yields the desired inequality (3.5).

(ii) *Poles.* Equations (2.7) and (2.8), together with the elementary inequalities $\ln\sin\frac{\theta_i^-}{2} \leq 0$ and $\ln\cos\frac{\theta_i^+}{2} \leq 0$ give the two desired estimates immediately.

(iii) *Axis.* Proposition 4.2 gives, uniformly for $z \in I$,

$$U_\rho = O(\rho), \qquad U_z = O(1), \qquad v_\rho = O(\rho^3), \qquad v_z = O(\rho^4).$$

Since $e^{4u} = \rho^{-4} e^{4U}$, the quadratures (1.3) imply

$$\alpha_\rho = O(\rho), \qquad \alpha_z = O(\rho^2).$$

Thus, since $\alpha(0, z) = \mathbf{b}_j$ we have

$$|\alpha(\rho, z) - \mathbf{b}_j| \leq \int_0^\rho |\alpha_t(t, z)| dt \leq C_I \rho^2,$$

which yields the desired result. $\square$

3.3. **Maximum Principle.** We will now establish the global upper bound for α, by applying the maximum principle on a sequence of exhaustion domains.

Proposition 3.3. *The Weyl conformal factor satisfies the following estimate*

$$\alpha(\rho,z) \le \max_{1\le j\le N+1} \mathbf{b}_j \qquad \text{for all } \rho>0.$$

Proof. Set $M = \max_{1\le j\le N+1} \mathbf{b}_j$, and observe that $M \ge 0$, since according to (1.4) the logarithmic angle defects of the semi-infinite rods vanish. Let

$$\mathcal{Q} = \{p_i \mid i \in \mathcal{I}_{\mathrm{deg}}\} \cup \{q_i^-, q_i^+ \mid i \in \mathcal{I}_{\mathrm{nd}}\}$$

be the finite set of punctures and poles, and for $q=(0,0,z_q)\in\mathcal{Q}$, put

$$r_q = \sqrt{\rho^2 + (z-z_q)^2}.$$

Choose $r_* > 0$ so that all puncture and pole estimates from Lemma 3.2 hold for $r_q \le 2r_*$. Since $\mathcal{Q}$ is finite, there is a common exponent

$$\kappa_* = \min\big(\{\beta_i \mid i\in\mathcal{I}_{\mathrm{deg}}\} \cup \{\gamma_i^-, \gamma_i^+ \mid i \in \mathcal{I}_{\mathrm{nd}}\}\big) > 0.$$

Fix $\sigma>0$ smaller than r_* and smaller than a quarter of the distance between any two distinct points of $\mathcal{Q}$. Choose R so large that all points of $\mathcal{Q}$ lie in the half-disk of radius $R/2$, and take $0<\eta<\sigma/2$. Following [17, Section 4] define

$$\Omega_{\eta,\sigma,R} = \Big\{(\rho,z) \mid \rho>\eta,\ \rho^2+z^2<R^2,\ r_q>\sigma \text{ for every } q\in\mathcal{Q}\Big\},$$

and observe that this domain is bounded with piecewise smooth boundary.

On $\Omega_{\eta,\sigma,R}$ set

$$B(\rho,z) = \begin{cases} \dfrac{\nabla\alpha}{\rho|\nabla\alpha|}, & \nabla\alpha\neq 0,\\ 0, & \nabla\alpha = 0,\end{cases}$$

and notice that B is bounded and measurable, with $|B|\le\eta^{-1}$. Furthermore, by Lemma 3.1 we have

$$\Delta_{\rho,z}\alpha + B\cdot\nabla\alpha \ge 0.$$

The standard weak maximum principle then yields

$$\sup_{\Omega_{\eta,\sigma,R}} \alpha \le \sup_{\partial\Omega_{\eta,\sigma,R}} \alpha. \tag{3.7}$$

We will now estimate the boundary pieces. On a half-circle $r_q=\sigma$, Lemma 3.2 (i) and (ii) apply when q is a puncture or pole. Since there are finitely punctures or poles, we have

$$\alpha \le M + C\sigma^{\kappa_*}.$$

Next consider the boundary portions with $\rho=\eta$. The condition $r_q>\sigma$ and the inequality $\eta<\sigma/2$ imply

$$|z-z_q| > \sqrt{\sigma^2-\eta^2} > \frac{\sqrt{3}}{2}\sigma.$$

For fixed σ and R, the projections of these portions onto Γ lie in a finite union of compact subintervals, each contained either in an axis rod or in the interior of a nondegenerate horizon rod. On the axis pieces, Lemma 3.2 (iii) gives

$$\alpha \leq M + C_{\sigma,R}\eta^2,$$

while on the horizon pieces, (2.9) gives

$$\alpha \leq \ln \eta + C_{\sigma,R}.$$

Lastly, on the outer half-circle (2.10) shows that

$$|\alpha| \leq CR^{-2}.$$

Combining these estimates together with (3.7) yields

$$\sup_{\Omega_{\eta,\sigma,R}} \alpha \leq \max\left\{M + C\sigma^{\kappa_*},\, M + C_{\sigma,R}\eta^2,\, \ln \eta + C_{\sigma,R},\, CR^{-2}\right\}. \tag{3.8}$$

Fix (ρ_0, z_0) arbitrarily with $\rho_0 > 0$. For all sufficiently large R and sufficiently small σ, η, this point belongs to $\Omega_{\eta,\sigma,R}$. In (3.8), first let $\eta \to 0$ with σ, R fixed. Then let $\sigma \to 0$, and finally let $R \to \infty$. Since $M \geq 0$, the last term causes no difficulty. It follows that $\alpha(\rho_0, z_0) \leq M$. □

4. Regular Axis Factorization

The strict local argument of the next section requires more than the $C^{3,\gamma}$ regularity supplied by Proposition 2.1 (i), at the interior of axis rods. In this section, the factorization (1.8) will be established. We first present two elementary facts about radial functions.

Lemma 4.1. *Let $I \subset \mathbb{R}$ be an interval and let $m, \ell \geq 2$.*

(i) *Suppose that $q = q(\rho, z)$ is defined for $\rho \geq 0$, and that*

$$h_m(x, z) = q(|x|, z), \qquad (x, z) \in \mathbb{R}^m \times I,$$

is $C^{k,\gamma}$ and $\mathrm{SO}(m)$-invariant, where $k \geq 0$ and $0 < \gamma < 1$. Then the same radial representative defines an $\mathrm{SO}(\ell)$-invariant function

$$h_\ell(X, z) = q(|X|, z)$$

of class $C^{k,\gamma}$ on $\mathbb{R}^\ell \times I$.

(ii) *If h_m is smooth, then there is a function $H = H(s, z)$, which is smooth for $s \geq 0$ up to $s = 0$, such that*

$$h_m(x, z) = H(|x|^2, z). \tag{4.1}$$

(iii) *If $q(|x|, z)$ is $C^{k+2,\gamma}$ on $\mathbb{R}^m \times I$, then q_ρ/ρ, initially defined for $\rho > 0$, extends as a radial $C^{k,\gamma}$ function across $x = 0$.*

Proof. We may work locally on a compact subinterval $I' \subset I$. Let $e_1 \in \mathbb{R}^m$ be a unit vector. Restriction to the line defined by e_1 gives an even extension

$$Q(\rho, z) := h_m(\rho e_1, z), \qquad (\rho, z) \in (-\rho_0, \rho_0) \times I',$$

of class $C^{k,\gamma}$. At a point of the z-axis, every transverse Taylor polynomial of an $\mathrm{SO}(m)$-invariant function is an invariant polynomial, and hence is a linear combination of the powers $|x|^{2j}$.

Applying Taylor's theorem in the transverse variables x, also after n differentiations in z with $n \leq k$, gives

$$\partial_z^n Q(\rho, z) = \sum_{2j \leq k-n} a_{j,n}(z)\rho^{2j} + R_{k-n,n}(\rho, z), \tag{4.2}$$

where, for $0 \leq \ell \leq k - n$,

$$\left|\partial_\rho^\ell R_{k-n,n}(\rho, z)\right| \leq C|\rho|^{k-n+\gamma-\ell}, \tag{4.3}$$

and the corresponding Hölder difference estimates hold uniformly for $z \in I'$. In particular, all odd radial derivates permitted by the regularity vanish at $\rho = 0$.

For $X \neq 0$, repeated differentiation of a radial representative has the form

$$\partial_X^\beta [Q(|X|, z)] = \sum_{\ell=1}^{|\beta|} \partial_\rho^\ell Q(|X|, z)|X|^{\ell-|\beta|} P_{\beta,\ell}\left(\frac{X}{|X|}\right), \tag{4.4}$$

where the $P_{\beta,\ell}$ are smooth functions on the unit sphere depending only on β and ℓ. By substituting (4.2) into (4.4), we find that the polynomial part is a polynomial in X, while (4.3) controls every apparent negative power of $|X|$ and gives the required $C^{k-n,\gamma}$ extension at $X = 0$. Applying this for each $n \leq k$ proves that $Q(|X|, z)$ is $C^{k,\gamma}$, jointly in (X, z) . This does not depend on the transverse dimension, and thus proves (i).

For (ii), observe that the evenness of Q and repeated Taylor expansion give, for every L,

$$Q(\rho, z) = \sum_{j=0}^{L} a_j(z)\rho^{2j} + \rho^{2L+2} Q_{L+1}(\rho, z),$$

where Q_{L+1} smooth and even. Hence, $H(s, z) := Q(\sqrt{s}, z)$ for $s \geq 0$ has derivatives of every order extending continuously to $s = 0$. Rotational invariance of h_m then yields (4.1).

Lastly, evenness implies $q_\rho(0, z) = 0$, and therefore

$$\frac{q_\rho(\rho, z)}{\rho} = \int_0^1 q_{\rho\rho}(t\rho, z)dt.$$

Moreover, since $Q(\rho, z) = q(|\rho|, z)$ is even and $C^{k+2,\gamma}$, its second radial derivative $Q_{\rho\rho}$ is even and $C^{k,\gamma}$. The finite regularity argument from part (i), applied to $Q_{\rho\rho}$, shows that the radial function $(x, z) \mapsto q_{\rho\rho}(|x|, z)$ is $C^{k,\gamma}$. Therefore, $|x|^{-1} q_\rho(|x|, z)$ extends as a $C^{k,\gamma}$ function across $x = 0$. This proves (iii). □

The next result gives the desired smooth factorization of the harmonic map functions at the interior of axis rods.

Proposition 4.2. *Let I be a compact subinterval of an axis rod Γ_j, and write $v = c$ on that rod. There exist functions F and G, smooth on $\mathbb{R}^2_+$ up to the boundary, such that on a sufficiently small neighborhood of I in $\mathbb{R}^3$,*

$$U(\rho, z) = F(\rho^2, z), \qquad v(\rho, z) = c + \rho^4 G(\rho^2, z).$$

Proof. Set $y = v - c$. Since $u = U - \ln \rho$, equation (2.2) becomes

$$y_{\rho\rho} - \frac{3}{\rho} y_\rho + y_{zz} + 4U_\rho y_\rho + 4U_z y_z = 0. \tag{4.5}$$

By Proposition 2.1 (i), y is the restriction of an SO(2)-invariant Cartesian $C^{3,\gamma}$ function in (x_1, x_2, z), and $y(0,z) = 0$. Rotational invariance eliminates all odd transverse Taylor tensors, while the transverse Hessian is a scalar multiple of the identity. Uniformly on compact subintervals of the rod,

$$\begin{aligned} y(\rho, z) &= a(z)\rho^2 + O(\rho^{3+\gamma}), \\ y_\rho(\rho, z) &= 2a(z)\rho + O(\rho^{2+\gamma}), \\ y_{\rho\rho}(\rho, z) &= 2a(z) + O(\rho^{1+\gamma}), \end{aligned} \tag{4.6}$$

with the analogous estimates involving the z-differentiations allowed by $C^{3,\gamma}$ regularity. Since $y(0,z)$ vanishes identically, $y_{zz}(0,z) = 0$. Moreover,

$$U_\rho = O(\rho), \qquad U_z = O(1), \qquad y_z = O(\rho^2).$$

Letting $\rho \to 0$ in (4.5) and using (4.6) produces

$$0 = 2a(z) - 6a(z) + o(1) = -4a(z) + o(1).$$

Thus $a(z) \equiv 0$ on the rod interval, and we have

$$|y| \le C\rho^{3+\gamma}, \qquad |y_\rho| + |y_z| \le C\rho^{2+\gamma}. \tag{4.7}$$

For $\rho > 0$, define

$$h(\rho, z) = \rho^{-4} y(\rho, z).$$

A direct substitution into (4.5) gives

$$h_{\rho\rho} + \frac{5}{\rho} h_\rho + h_{zz} + 4U_\rho h_\rho + 4U_z h_z + 16\frac{U_\rho}{\rho} h = 0. \tag{4.8}$$

The estimates (4.7) imply

$$|h| \le C\rho^{-1+\gamma}, \qquad |\nabla h| \le C\rho^{-2+\gamma}. \tag{4.9}$$

Now lift the radial representatives to six transverse variables:

$$\widehat{U}(X, z) = U(|X|, z), \qquad \widehat{G}(X, z) = h(|X|, z), \qquad X \in \mathbb{R}^6.$$

By Lemma 4.1 (i), $\widehat{U}$ is $C^{3,\gamma}$, and Lemma 4.1 (iii) shows that U_ρ/ρ extends at least as a $C^{1,\gamma}$ function across $X = 0$. Moreover from (4.9), in a bounded z-cylinder, we have

$$\int_0^\varepsilon |h|^2 \rho^5 d\rho = O(\varepsilon^{4+2\gamma}), \qquad \int_0^\varepsilon |\nabla h|^2 \rho^5 d\rho = O(\varepsilon^{2+2\gamma}). \tag{4.10}$$

Thus $\widehat{G} \in H^1_{\mathrm{loc}}(\mathbb{R}^7)$.

We now remove the lifted axis $\{X = 0\}$. Choose a radial cut-off $\chi_\varepsilon(X)$ equal to one for $|X| < \varepsilon$, zero for $|X| > 2\varepsilon$, and satisfying $|\nabla \chi_\varepsilon| \le C\varepsilon^{-1}$. Then

$$\int_{\mathbb{R}^6} |\nabla \chi_\varepsilon|^2 dX \le C\varepsilon^{-2} \int_\varepsilon^{2\varepsilon} \rho^5 d\rho = O(\varepsilon^4). \tag{4.11}$$

Use $(1 - \chi_\varepsilon)\varphi$ as a test function in (4.8), written in lifted divergence form, where $\varphi \in C_c^\infty(\mathbb{R}^7)$. The cut-off term involving the principal part is bounded by

$$\left(\int_{\{\varepsilon < |X| < 2\varepsilon\}} |\nabla \widehat{G}|^2 \right)^{1/2} \left(\int_{\{\varepsilon < |X| < 2\varepsilon\}} |\nabla \chi_\varepsilon|^2 \right)^{1/2} \|\varphi\|_{L^\infty} = O(\varepsilon^{1+\gamma}) \cdot O(\varepsilon^2) = o(1),$$

by (4.10) and (4.11). Since the square root of the tube volume is $O(\varepsilon^3)$, the remaining omitted error terms satisfy

$$O(\varepsilon^{1+\gamma})\cdot O(\varepsilon^3), \qquad O(\varepsilon^{4+\gamma}), \qquad O(\varepsilon^{5+\gamma}),$$

for the principal term containing $\nabla\varphi$, the first-order term, and the zeroth-order term, respectively. Here we used (4.10), and boundedness of the coefficients. It follows that the lifted equation holds weakly across $\{X=0\}$:

$$\Delta_{\mathbb{R}^7}\widehat{G} + 4\nabla\widehat{U}\cdot\nabla\widehat{G} + 16\widehat{\left(\frac{U_\rho}{\rho}\right)}\widehat{G} = 0. \tag{4.12}$$

We may now bootstrap. The lower-order coefficients in (4.12) are bounded, and $\widehat{G}\in H^1$, so the right-hand side of the corresponding Poisson equation initially belongs to L^2. Interior $W^{2,p}$ estimates and Sobolev embedding successively raise the integrability of $\nabla\widehat{G}$ according to $2\to\frac{14}{5}\to\frac{14}{3}\to 14$, since in dimension seven the critical Sobolev exponent is $7p/(7-p)$. After the last step, $\widehat{G}\in W^{2,14}_{\text{loc}}$, and Morrey's inequality yields $\widehat{G}\in C^{1,1/2}_{\text{loc}}$. If $0<\gamma_0<\min\{\gamma,\frac{1}{2}\}$, then Schauder estimates applied to (4.12) produce $\widehat{G}\in C^{3,\gamma_0}$.

Next observe that the U equation (2.1), expressed in terms of h, is given by

$$U_{\rho\rho} + \frac{1}{\rho}U_\rho + U_{zz} = 2e^{4U}\left[\rho^2(4h+\rho h_\rho)^2 + \rho^4 h_z^2\right]. \tag{4.13}$$

Lemma 4.1 allows us to move the improved radial representative of $\widehat{G}$ back to the original two transverse variables. Since the right-hand side of (4.13) is then a Cartesian C^{2,γ_0} function across the original axis, the three-dimensional Schauder estimates show that $U\in C^{4,\gamma_0}$.

We now proceed inductively. Suppose that

$$U\in C^{m+3,\gamma_0}, \qquad \widehat{G}\in C^{m+2,\gamma_0}.$$

By Lemma 4.1 (iii), it holds that $U_\rho/\rho\in C^{m+1,\gamma_0}$ after the radial dimension transfer. Equation (4.12) then yields $\widehat{G}\in C^{m+3,\gamma_0}$, and thus (4.13) gives $U\in C^{m+4,\gamma_0}$. Continuing this iteration shows that both U and the lifted $\widehat{G}$ are smooth across the axis. We may now apply Lemma 4.1 (ii), first in two transverse dimensions for U and then in six transverse dimensions for $\widehat{G}$, to find smooth functions F and G with

$$U(\rho,z)=F(\rho^2,z), \qquad h(\rho,z)=G(\rho^2,z).$$

Although this holds only in a neighborhood of the interval I, the functions F and G maybe extended smoothly to all of $\mathbb{R}^2_+$. Lastly, since $v-c=\rho^4 h$, the desired result follows. □

Remark 4.3. *For later use, we write the relevant equations in terms of the functions F and G, and with respect to the variable $s=\rho^2$. On an axis rod of potential constant value c, write*

$$U=F(s,z), \qquad v=c+s^2G(s,z), \qquad A=\alpha-\mathbf{b}_j.$$

A direct substitution into the harmonic map equations (2.1)-(2.2), and the radial quadrature equation (1.3) produces

$$4(sF_{ss}+F_s)+F_{zz} = 2e^{4F}\left[4s(2G+sG_s)^2+s^2G_z^2\right], \tag{4.14}$$

$$4sG_{ss}+12G_s+G_{zz} = -16sF_sG_s - 4F_zG_z - 32F_sG,$$

$$2A_s = 4sF_s^2 - F_z^2 + 4se^{4F}(2G+sG_s)^2 - s^2e^{4F}G_z^2, \qquad A(0,z)=0. \tag{4.15}$$

5. Weyl Conformal Factor Strict Local Lower Bound

The purpose of this section is to prove a strict local lower bound for α near any bounded axis rod. The proof must cover arbitrarily degenerate maxima of the axis trace of U; this is the reason for the homogeneous-mode analysis below.

Proposition 5.1. *Let $\Gamma_i = (z_{i-1}^+, z_i^-)$, $2 \le i \le N$, be a bounded open axis rod on which $v = c_i$ and $\alpha = \mathbf{b}_i$. If $q \in \Gamma_i$ is a maximum point of $z \mapsto U(0,z)$, then every neighborhood of $(0,q)$ in the half-plane $\rho > 0$ contains a point where*

$$\alpha > \mathbf{b}_i. \tag{5.1}$$

Proof. After translating q to the origin, we may assume that $q = 0$. Near the rod, by Proposition 4.2, we can write

$$U = F(s,z), \qquad v = c_i + s^2 G(s,z), \qquad A = \alpha - \mathbf{b}_i,$$

where $s = \rho^2$ and F, G are smooth. Set

$$f(z) = F(0,z), \qquad g(z) = G(0,z).$$

Then $A(0,z) = 0$, $f'(0) = 0$, and equations (4.14)–(4.15) hold.

Step 1: Setting $s = 0$ in (4.14) gives

$$F_s(0,0) = -\frac{1}{4} f''(0).$$

Along $z = 0$, smoothness and $f'(0) = 0$ yield

$$F_s(s,0) = F_s(0,0) + O(s), \qquad F_z(s,0) = sF_{sz}(0,0) + O(s^2),$$
$$G(s,0) = g(0) + O(s), \qquad G_z(s,0) = g'(0) + O(s).$$

Substitution into (4.15), followed by integration in s, yields

$$A(s,0) = s^2 \left[\frac{f''(0)^2}{16} + 4e^{4f(0)} g(0)^2 \right] + O(s^3).$$

Hence (5.1) follows immediately unless

$$f''(0) = 0, \qquad g(0) = 0. \tag{5.2}$$

Thus, we may now assume (5.2) for the remainder of the proof.

Step 2: For $n \ge 3$ and $m \ge 0$, let $\mathcal{Z}_m^{(n)}$ be the space of zonal homogeneous polynomials P of degree m on

$$\mathbb{R}_x^{n-1} \times \mathbb{R}_z$$

which are invariant under $\mathrm{SO}(n-1)$ in the x variables and harmonic in $\mathbb{R}^n$. Every such polynomial has the form

$$P(x,z) = \sum_{j=0}^{\lfloor m/2 \rfloor} a_j \rho^{2j} z^{m-2j},$$

where $\rho = |x|$. Since

$$\Delta_x \rho^{2(j+1)} = 2(j+1)(2j+n-1)\rho^{2j},$$

comparison of the coefficients in $\Delta_{\mathbb{R}^n} P = 0$ produces

$$2(j+1)(2j+n-1)a_{j+1} + (m-2j)(m-2j-1)a_j = 0. \tag{5.3}$$

Thus a_0 determines every coefficient. Conversely, the recurrence produces a harmonic polynomial for every choice of a_0; at the final index the remaining power of z is zero or one, and no additional condition occurs. Therefore restriction to the z-axis is an isomorphism

$$\mathcal{Z}_m^{(n)} \longrightarrow \operatorname{span}\{z^m\}, \qquad P \longmapsto P(0,z). \tag{5.4}$$

In particular, $\mathcal{Z}_m^{(n)}$ is one-dimensional. Let $H_m^{(n)}$ denote its unique element normalized by

$$H_m^{(n)}(0,z) = z^m. \tag{5.5}$$

Writing $r = (\rho^2 + z^2)^{1/2}$, the two cases used below are

$$H_m^{(3)}(\rho,z) = r^m P_m(z/r), \qquad H_m^{(7)}(\rho,z) = r^m \frac{C_m^{5/2}(z/r)}{C_m^{5/2}(1)},$$

where P_m and $C_m^{5/2}$ are the Legendre and Gegenbauer polynomials. The explicit formulas are not essential; what matters is (5.4). We shall also use the consequence of the recurrence (5.3) that, for even m,

$$H_m^{(n)}(\rho,0) = \lambda_m^{(n)} \rho^m \quad \text{with } \lambda_m^{(n)} \neq 0, \qquad \partial_z H_m^{(n)}(\rho,0) = 0. \tag{5.6}$$

Step 3: Suppose first that v is constant. Then U is an axisymmetric harmonic function on $\mathbb{R}^3$. This is the static case treated in [5, 6] for two black holes, and generalized to any number of black holes in [19]. The explicit formula for α in this case shows (5.1). For the remainder of the proof assume that v is not constant.

Step 4: Let $\widehat{G}(X,z)$ be the smooth seven-dimensional lift of $G(\rho^2,z)$ constructed in the proof of Proposition 4.2. We first show that $g(z) = \widehat{G}(0,z)$ cannot be flat at the origin. Suppose otherwise:

$$g^{(j)}(0) = 0 \qquad \text{for every } j \geq 0. \tag{5.7}$$

If the full Taylor expansion of $\widehat{G}$ has a first nonzero homogeneous term Q_m, then Q_m is SO(6)-invariant. Moreover, Q_m is harmonic. Indeed, this is immediate for $m = 0, 1$, and for $m \geq 2$, the contributions generated by Q_m in (4.12) have lowest possible homogeneous degrees $m-2$, $m-1$, and m, from the Laplacian, first-order term, and zeroth-order term, respectively, while all lower Taylor terms of $\widehat{G}$ vanish by the choice of Q_m. It follows that $Q_m \in \mathcal{Z}_m^{(7)}$, and its restriction to the axis is

$$Q_m(0,z) = \frac{g^{(m)}(0)}{m!} z^m = 0$$

by (5.7). The injectivity in (5.4) forces $Q_m = 0$, a contradiction. Therefore every Taylor coefficient of $\widehat{G}$ vanishes at the origin.

Taylor's theorem then gives $|\widehat{G}(X,z)| = O(r^\ell)$ for every ℓ. In particular, $\widehat{G}$ vanishes to infinite order. Equation (4.12) has smooth lower-order coefficients and implies locally

$$|\Delta_{\mathbb{R}^7} \widehat{G}| \leq C\big(|\nabla \widehat{G}| + |\widehat{G}|\big).$$

Standard unique-continuation [4] yields $\widehat{G} \equiv 0$ in a neighborhood of the origin. Hence $v - c_i = \rho^4 G$ vanishes on a nonempty open subset of $\{\rho > 0\}$. There, $v - c$ satisfies the linear uniformly elliptic equation

$$\Delta(v - c_i) + 4\nabla u \cdot \nabla(v - c_i) = 0.$$

Unique continuation then forces $v - c_i \equiv 0$, so v is globally constant, contrary to the hypothesis. We conclude that g is not flat.

It follows that the integer

$$d = \min\{j \geq 1 \mid g^{(j)}(0) \neq 0\}$$

exists. Set

$$\sigma_d = \frac{g^{(d)}(0)}{d!} \neq 0.$$

No full Taylor term of $\widehat{G}$, of degree less than d, can be nonzero: the first such term would again be a zonal harmonic polynomial with zero axial restriction. The degree-d term is a zonal harmonic with axial restriction $\sigma_d z^d$, and hence, by (5.5), equals $\sigma_d H_d^{(7)}$. Taylor's theorem, with one differentiated remainder, yields

$$\left| D^\ell (h - \sigma_d H_d^{(7)}) \right| \leq C r^{d+1-\ell}, \qquad \ell = 0, 1, \tag{5.8}$$

where derivatives are understood in the lifted variables and then restricted to the orbit half-plane.

Step 5: We write the equation (4.13) as

$$\Delta_{\mathbb{R}^3} U = \mathcal{S},$$

where

$$\mathcal{S} = 2e^{4U} \left[\rho^2 (4h + \rho h_\rho)^2 + \rho^4 h_z^2 \right].$$

For any integer $M \geq 1$, consider the homogeneous Taylor decompositions in the original three-dimensional variables,

$$U = f(0) + \sum_{n=1}^{M} U_n + O(r^{M+1}), \qquad \mathcal{S} = \sum_{n=0}^{M-2} \mathcal{S}_n + O(r^{M-1}),$$

where U_n and $\mathcal{S}_n$ are homogeneous of degrees n. A simple comparison gives

$$\Delta_{\mathbb{R}^3} U_n = \mathcal{S}_{n-2},$$

with $\mathcal{S}_n = 0$ for $n < 0$.

Write

$$U_n(0, z) = c_n z^n$$

and define the normalized particular component

$$\widetilde{U}_n = U_n - c_n H_n^{(3)}. \tag{5.9}$$

Then

$$U_n = \widetilde{U}_n + c_n H_n^{(3)}, \qquad \Delta_{\mathbb{R}^3} \widetilde{U}_n = \mathcal{S}_{n-2}, \qquad \widetilde{U}_n(0, z) = 0. \tag{5.10}$$

The zero-axis normalization makes the particular component unique, because the difference of two such choices would be a zonal harmonic polynomial with zero axial restriction. The second term in (5.9) is the free harmonic component.

Let k be the first nonzero Taylor order of $f(z)-f(0)$, with $k=\infty$ if f vanishes up to infinite order at $z=0$. If $k<\infty$, the local maximum at 0 implies

$$f(z)=f(0)-\kappa z^k+O(z^{k+1}),\quad \kappa>0,\quad k \text{ even.} \tag{5.11}$$

Thus $c_n=0$ for $1\le n<k$, and $c_k=-\kappa$.

Step 6: We now study the equation

$$\alpha_\rho=\rho(U_\rho^2-U_z^2)+e^{4U}\big[\rho^3(4h+\rho h_\rho)^2-\rho^5 h_z^2\big], \tag{5.12}$$

and examine which term on the right-hand side provides the lowest degree when expanded in terms of ρ. We note that the lowest degree of ρ from h is $2d+3$ by (5.8), and the lowest degree of ρ from U is $2k-1$ by (5.11) if k is finite. By comparing $2d+3$ and $2k-1$, we consider three exhaustive cases.

Case I: $k<d+2$. Then $k<2d+4$, and

$$U(\rho,z)=f(0)-\kappa H_k^{(3)}(\rho,z)+O(r^{k+1})$$

with one differentiated remainder. Since k is even, (5.6) produces

$$H_k^{(3)}(\rho,0)=\lambda_k\rho^k,\quad \lambda_k\neq 0,\quad \partial_z H_k^{(3)}(\rho,0)=0.$$

Therefore

$$U_\rho(\rho,0)=-\kappa k\lambda_k\rho^{k-1}+o(\rho^{k-1}),\qquad U_z(\rho,0)=o(\rho^{k-1}).$$

The direct twist contribution to A_ρ is $O(\rho^{2d+3})=o(\rho^{2k-1})$. Thus the radial quadrature gives

$$A_\rho(\rho,0)=\kappa^2k^2\lambda_k^2\rho^{2k-1}+o(\rho^{2k-1}).$$

Since $A(0,0)=0$,

$$A(\rho,0)=\frac{k}{2}\kappa^2\lambda_k^2\rho^{2k}+o(\rho^{2k})>0$$

for all sufficiently small $\rho>0$.

Case II: $k=d+2$. Here d is even. Set

$$\lambda_k=H_k^{(3)}(1,0)\neq 0,\qquad \mu_d=H_d^{(7)}(1,0)\neq 0.$$

At $z=0$, (5.10) with $n=k$ and (5.8) give

$$\begin{aligned} U_\rho&=-\kappa k\lambda_k\rho^{k-1}+o(\rho^{k-1}), & U_z&=o(\rho^{k-1}),\\ h&=\sigma_d\mu_d\rho^d+o(\rho^d), & h_z&=o(\rho^{d-1}),\\ 4h+\rho h_\rho&=\sigma_d(d+4)\mu_d\rho^d+o(\rho^d). \end{aligned}$$

The positive radial squares in (5.12) have the same order, while the negative vertical squares are of higher order. Hence

$$A_\rho(\rho,0)=\big[\kappa^2k^2\lambda_k^2+e^{4f(0)}\sigma_d^2(d+4)^2\mu_d^2\big]\rho^{2d+3}+o(\rho^{2d+3}).$$

Integration yields

$$A(\rho,0)=\Big[\frac{k}{2}\kappa^2\lambda_k^2+\frac{(d+4)^2}{2d+4}e^{4f(0)}\sigma_d^2\mu_d^2\Big]\rho^{2d+4}+o(\rho^{2d+4})>0.$$

Case III: $d+2<k$, including $k=\infty$. Set

$$L=\min\{k,2d+4\}.$$

The free/particular decomposition gives

$$U-f(0)=O(r^L),\qquad |\nabla U|=O(r^{L-1}). \tag{5.13}$$

Since $L>d+2$, for every fixed $\eta>0$,

$$\int_0^{\eta z}\rho|\nabla U|^2d\rho=O(z^{2L})=o(z^{2d+4})\qquad \text{as } z\downarrow 0. \tag{5.14}$$

Define the homogeneous polynomial

$$W_d(\xi,\tau)=\xi^4H_d^{(7)}(\xi,\tau)$$

and, for $\eta>0$,

$$I_d(\eta)=\int_0^\eta \xi^{-3}\left[(\partial_\xi W_d(\xi,1))^2-(\partial_\tau W_d(\xi,1))^2\right]d\xi. \tag{5.15}$$

The normalization $H_d^{(7)}(0,\tau)=\tau^d$ and the recurrence (5.3) imply, as $\xi\downarrow 0$, that

$$H_d^{(7)}(\xi,1)=1+O_d(\xi^2),$$

so

$$\partial_\xi W_d(\xi,1)=4\xi^3+O_d(\xi^5),\quad \partial_\tau W_d(\xi,1)=d\xi^4+O_d(\xi^6).$$

Consequently,

$$I_d(\eta)=4\eta^4+O_d(\eta^6)>0 \tag{5.16}$$

for every sufficiently small fixed $\eta>0$.

It remains to justify that (5.15) is the leading ray integral, including the remainder. Write

$$R=h-\sigma_dH_d^{(7)}.$$

On $0\le\rho\le\eta z$, estimate (5.8) yields

$$|R|\le Cz^{d+1},\qquad |\nabla R|\le Cz^d.$$

Since $v-c=\rho^4(\sigma_dH_d^{(7)}+R)$ and $\rho=\xi z$, homogeneity gives

$$v_\rho(\xi z,z)=\sigma_dz^{d+3}\partial_\xi W_d(\xi,1)+O(z^{d+4}(\xi^3+\xi^4)), \tag{5.17}$$

$$v_z(\xi z,z)=\sigma_dz^{d+3}\partial_\tau W_d(\xi,1)+O(z^{d+4}\xi^4). \tag{5.18}$$

The powers of ξ in these errors make all integrals below convergent at $\xi=0$.

Because $A(0,z)=0$ and $e^{4u}=\rho^{-4}e^{4U}$, the radial quadrature is

$$A(\eta z,z)=\int_0^{\eta z}\left\{\rho(U_\rho^2-U_z^2)+e^{4U}\rho^{-3}(v_\rho^2-v_z^2)\right\}d\rho. \tag{5.19}$$

The first term is $o(z^{2d+4})$ by (5.14). In the second term, the leading/error cross terms from (5.17)–(5.18) are $O(z^{2d+5})$ after integration, and the error squares are $O(z^{2d+6})$. Moreover, (5.13) produces

$$e^{4U}=e^{4f(0)}(1+o(1)),$$

uniformly on the ray segment. Substituting $\rho=\xi z$ into (5.19) therefore yields

$$A(\eta z,z)=e^{4f(0)}\sigma_d^2I_d(\eta)z^{2d+4}+o(z^{2d+4})>0,$$

for all sufficiently small $z > 0$, by (5.16).

Each of the three cases gives points with $A > 0$ arbitrarily close to the origin. Since $A = \alpha - \mathbf{b}_i$, this proves (5.1) and the proposition. $\square$

Remark 5.2. *Only smoothness obtained in Proposition 4.2, and finite Taylor expansions are used. Infinite-order vanishing of the lifted twist mode is treated by strong unique continuation. No real-analytic regularity of the coupled harmonic map system is assumed.*

6. Proof of the Main Theorem

We first verify the endpoint behavior required to apply Proposition 5.1. Define

$$\nu_i = \begin{cases} 1, & i \in \mathcal{I}_{\mathrm{deg}}, \\ \frac{1}{2}, & i \in \mathcal{I}_{\mathrm{nd}}. \end{cases}$$

Lemma 6.1. *For $2 \le j \le N$, let $f_j(z) = U(0,z)$ for $z_{j-1}^+ < z < z_j^-$. Then*

$$\begin{aligned} f_j(z) &= \nu_{j-1} \ln(z - z_{j-1}^+) + O(1) \to -\infty && \textit{as } z \downarrow z_{j-1}^+, \\ f_j(z) &= \nu_j \ln(z_j^- - z) + O(1) \to -\infty && \textit{as } z \uparrow z_j^-. \end{aligned} \tag{6.1}$$

Consequently, f_j attains a maximum at an interior point of Γ_j.

Proof. At the bottom endpoint, first suppose that $\mathcal{H}_{j-1}$ is degenerate. Approach its puncture along the north axis and use (2.3)–(2.4); since the denominator in the ln in (2.4) is $2(1+b_{j-1}) > 0$, this gives

$$U(0,z) = \ln(z - z_{j-1}^+) + O(1).$$

If $\mathcal{H}_{j-1}$ is nondegenerate, approach its upper pole along the adjacent north axis. Formula (2.6) gives instead

$$U(0,z) = \frac{1}{2}\ln(z - z_{j-1}^+) + O(1).$$

This proves (6.1). The top endpoint behavior is obtained similarly. Axis regularity gives continuity in the interior. The two endpoint limits force an interior maximum. $\square$

Proof of Theorem 1.1. Let

$$M = \max_{1 \le j \le N+1} \mathbf{b}_j = \max\{0, \mathbf{b}_2, \dots, \mathbf{b}_N\} \ge 0.$$

Proposition 3.3 gives

$$\alpha \le M \qquad \text{throughout } \{\rho > 0\}.$$

By Lemma 6.1, the trace of U on any bounded axis rod has an interior maximum, hence by Proposition 5.1, α cannot achieve its maximum on any bounded axis rod. Because $\alpha = 0$ on the unbounded axis rods, it follows that $M = 0$. Applying Proposition 5.1 again immediately shows that for any bounded axis rod Γ_j we must have $\mathbf{b}_j < M = 0$. $\square$

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