Correspondences for hyperkähler varieties with large Picard numbers

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ABSTRACT. In this note, we explore the connection between hyperkähler manifolds with large Picard numbers and abelian varieties. In particular, we are interested in Morrison's solution to the (modified) Oda's conjecture: every K3 surface whose Picard group is large enough (in a certain precise sense) must be related via an algebraic correspondence to an abelian surface. We generalize this theorem to the case of known examples of hyperkähler manifolds such as pointed Hilbert schemes on K3 surfaces.

1. INTRODUCTION

An important aspect of the study of K3 surfaces is the relationship to abelian surfaces. Namely, given an abelian surface A, the resolution of the quotient $A/\{\pm 1\}$ is a K3 surface, *a Kummer* K3 *surface*. Kummer K3 surfaces play an essential role in the study of all K3 surfaces, for instance they are dense in the moduli space of K3 surfaces, and one can use this fact to prove Torelli-type theorems for K3 surfaces.

A Kummer K3 surface has $\mathrm{rank}(\mathsf{NS}(X)) \geqslant 17$, or equivalently, $\mathrm{rank}(\mathsf{T}(X)) \leqslant 5$. A remarkable theorem of Morrison [Mor84] shows that, in fact, every K3 surface with a large Picard number (e.g., $\rho(X) \geqslant 19$) is closely related to a Kummer surface. More precisely, suppose that there is a primitive embedding $\phi : \mathsf{T}(X) \hookrightarrow U^3$ of lattices, where U is the hyperbolic lattice of rank 2. Equivalently, suppose that there is an embedding $E_8(-1)^2 \hookrightarrow \mathsf{NS}(X)$. Then there exists a Nikulin involution on X with quotient birational to $\mathrm{Kum}(A)$, such that the quotient maps from X and A induce a Hodge isometry $\mathsf{T}(S) \cong \mathsf{T}(A)$. Such a structure on X is labelled a Shioda-Inose correspondence with the abelian surface A. [1]

In this note, we are interested in similar questions for hyperkähler manifolds, which are natural higher dimensional generalizations of K3 surfaces. There are two guiding principles here: (1) a hyperkähler manifold X is "essentially" determined by its transcendental lattice $\mathsf{T}(X)$, or rather its Hodge structure along with the embedding $\mathsf{T}(X) \hookrightarrow H^2(X, \mathbb{Z})$; see Proposition 2.3 below for a more precise formulation, and (2) $\mathsf{T}(X)$ can be "small" enough (for example, when it has rank 2) to embed into the second cohomology of an abelian variety, and thus it can be assumed $\mathsf{T}(X) \cong \mathsf{T}(A)$ for some[2] abelian variety A. By the Hodge conjecture, X and A are related by a correspondence inducing $\mathsf{T}(X)_{\mathbb{Q}} \cong \mathsf{T}(A)_{\mathbb{Q}}$. The purpose of this paper is to explore this correspondence more explicitly in the case of the known hyperkähler manifolds with large Picard number in the spirit of Morrison's results for K3 surfaces.

All the known hyperkähler manifolds ($K3^{[n]}$, Kum_n, OG6 and OG10) arise from moduli of vector bundles/sheaves on K3 surfaces and abelian surfaces. Thus, restricting to certain loci in moduli, they will be naturally in correspondence with symplectic surfaces: either K3 or abelian surfaces. Recently, there has been renewed interest in hyperkähler manifolds with large Picard number and their relation to K3 and abelian surfaces – see esp. [PRO25] and [PM24] (also [Laz21] for a different related question). Here, we sharpen those results a bit, and focus on understanding the connection to abelian surfaces in a uniform way.

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[1] The abelian surface is not necessarily unique. However, there are only finitely many choices; see Theorem A.10 in the appendix.

[2] Once again, not necessarily unique, but determined up to a finite choice.

Returning to K3 surfaces, in the situation of Morrison's theorem, there is a Shioda-Inose correpondence between a K3 surface X such that $T(X) \hookrightarrow U^3$, and an abelian surface A, through the common quotient $\mathrm{Kum}(A)$. More generally, one can ask under which conditions there is a correspondence between X and an abelian surface A. This is described by the following conjecture of Oda, modified and proved by Morrison (see the addendum at the end of [Mor84], using foundational results of Mukai on the moduli of sheaves on K3 surfaces [Muk87]).

Theorem 1.1. *([Mor84, Muk87]) Let X be an algebraic K3 surface, and suppose that there is an embedding $\phi : T(X) \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q}$ of $\mathbb{Q}$-lattices. Then there exists an abelian surface A and a correspondence between X and A which induces a Hodge isometry $T(X) \otimes \mathbb{Q} \cong T(A) \otimes \mathbb{Q}$.*

Note that the necessary condition is obvious (since $T(A) \hookrightarrow H^2(A, \mathbb{Q}) \cong U^3$), while the theorem is implied by the Hodge conjecture. Morrison's (unconditional) proof of the theorem is as follows: suppose ϕ is the embedding as in the hypothesis, and let $T = U^3 \cap \phi(T(X) \otimes \mathbb{Q})$. Then T is a lattice with a primitive embedding in U^3, and the right signature. Therefore, there is a K3 surface Y with $T_Y \cong T$, and such that Y is related to an abelian surface A by a Shioda-Inose correspondence. Since $T(X) \otimes \mathbb{Q} \cong T_Y \otimes \mathbb{Q}$ is a Hodge-isometry, and $T(X)$ and T_Y have rank $\leqslant 5$, the K3 surfaces have Picard number $\geqslant 17$. In particular, they satisfy the hypothesis of Mukai's theorem (which only requires Picard number $\geqslant 11$), implying X and Y are related by a correspondence, and therefore so are X and A.

We remark that Buskin [Bus19] generalized Mukai's theorem (removing the Picard number $\geqslant 11$ condition) to show that if two K3 surfaces X and Y are such that there is a Hodge isometry $H^2(X, \mathbb{Q}) \cong H^2(Y, \mathbb{Q})$, then there exists an algebraic correspondence between X and Y inducing the Hodge isometry.

Remark 1.2. *We note in passing that the condition $T(X) \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q}$ is automatically satisfied when $\mathrm{rank}(T(X)) \in \{2, 3\}$; in fact then $T(X) \hookrightarrow U^3$ (see [Mor84, Cor 2.5]). However, when $\mathrm{rank}(T(X)) \in \{4, 5\}$, there are non-trivial arithmetic conditions on $T(X)$ for it to be rationally embeddable in three copies of the hyperbolic plane. For instance, when T has rank 4 (signature $(2, 2)$), it must represent 0 (i.e., have an isotropic vector) in order for $T \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q}$; on the other hand, any even T of signature $(2, 2)$ is $T(X)$ for some K3 surface X. Similarly, when T has rank 5 (signature $(2, 3)$), it can be shown that $T(X) \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q}$ iff $T(X)$ has an isotropic sublattice of rank 2, i.e., two vectors u, v such that $u^2 = v^2 = u \cdot v = 0$.*[3]

The question that we are addressing here is that of finding a correspondence between known hyperkähler manifolds X of large Picard ranks and abelian surfaces A.

Our generalization of Morrison's solution to Oda's conjecture to hyperkähler varieties of $K3^{[n]}$ type is the following.

Theorem 1.3. *Let X be a projective hyperkähler manifold of $K3^{[n]}$ type. If there is an embedding $\phi : T(X) \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q}$ of $\mathbb{Q}$-lattices, then there exists an abelian variety A and a correspondence between X and A which induces a Hodge isometry $T(X) \otimes \mathbb{Q} \cong T(A) \otimes \mathbb{Q}$.*

Remark 1.4. *One can show, similar to the remark above, that the hypothesis of the theorem above is automatically satisfied if $\rho \in \{20, 21\}$, whereas for $\rho = 19$, it is satisfied iff $T(X)$ has an isotropic vector, and for $\rho = 18$, iff $T(X)$ has an isotropic 2-dimensional sublattice.*

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[3]For a proof of these criteria see Lemma 3.2 below.

2. Preliminaries

A *hyperkähler manifold* is a simply connected compact Kähler manifold X with a holomorphic symplectic form $\sigma \in H^0(X, \Omega_X^2)$ which is unique up to rescaling. The known classes of hyperkähler examples are: Hilbert schemes of n points on a K3 surface and their deformations (that is, of $K3^{[n]}$ type), generalized Kummer varieties Kum_n in dimension $2n$, and the exceptional examples due to O'Grady OG6 and OG10 in dimensions 6 and 10.

Definition 2.1. Let X be a hyperkähler manifold. The transcendental lattice of X is the lattice $T(X) = NS(X)^{\perp}$, where the orthogonal complement is taken in terms of the Beauville-Bogomolov-Fujiki form q_X. As a lattice, $T(X)$ is endowed with the Beauville-Bogomolov-Fujiki form q_X restricted to $T(X)$ and the weight-two Hodge structure induced by the lattice $(H^2(X, \mathbb{Z}), q_X)$.

Definition 2.2. A pure integral Hodge structure V of weight two is of K3-type if $\dim_{\mathbb{C}}(V_{\mathbb{C}}^{2,0}) = 1$ and $V_{\mathbb{C}}^{p,q} = 0$ if $|p-q| > 2$. Let X be a projective hyperkähler manifold and let (T, q) be a Hodge structure of K3-type. We say that X is induced by T if there exists a Hodge isometry $(T(X), q_X) \cong (T, q)$, where q_X is the Beauville-Bogomolov-Fujiki form of X.

We recall the following proposition from [PRO25, Prop 2.12, Cor 2.13], which makes precise the notion that the transcendental lattice $T(X)$ with its Hodge structure controls the isomorphism class of hyperkähler X, up to finitely many choices:

Proposition 2.3. *Let (T, q) be a fixed Hodge structure of K3 type, and fix a lattice Λ of rank $\geqslant 5$. Then the set*

$$\{X \textit{ hyperkähler with } H^2(X, \mathbb{Z}) \cong \Lambda \textit{ and induced by } (T, q)\}/\cong_{iso}$$

is finite.

In particular, if (T, q) is the transcendental lattice $T(X_0)$ of a fixed HK (say of $K3^{[n]}$-type for a fixed n), along with its Hodge structure, the proposition says that the set of X of type $K3^{[n]}$-type with $T(X)$ Hodge-isometric to $T(X_0)$ is finite, up to isomorphism.

Any hyperkähler manifold of $K3^{[n]}$ type with Picard rank $\rho(X) \geqslant 4$ is isomorphic to a moduli space of twisted stable sheaves on a K3 surface (and this is a sharp bound on the Picard rank) due to the results of Prieto-Montanez [PM24]. If the Picard rank $\rho(X) \geqslant 13$, such a K3 surface is unique. More precisely, we have the following results due to Markman [Mar10], see also Piroddi-Ortiz [PRO25, Theorem 3.7, Corollary 3.9].

Theorem 2.4. *Let X be a projective hyperkähler manifold of $K3^{[n]}$ type and S a projective K3 surface. Then X is induced by the transcendental lattice $T(S)$ if and only if X is birational to the Mukai moduli space $M_v(S, H)$ for some Mukai vector v and a v-generic polarization H.*

Corollary 2.5. *Every hyperkähler manifold of $K3^{[n]}$ type with Picard rank $\rho(X) \geqslant 13$ is induced by a unique K3 surface S.*

In analogy to the results above, in [PRO25, Theorem 3.15, Corollary 3.16], Piroddi-Ortiz also explore the generalized Kummer case.

Theorem 2.6. *Let X be a hyperkähler manifold of Kum^n-type. Then X is induced by $T(A)$ for an abelian surface A if and only if X is birational to the generalized Kummer variety $K_v(A, H) = \mathrm{Alb}^{-1}(0)$ associated with A, or to $K_v(A^\vee, H)$ associated with $A^\vee$, for some v-generic polarization H.*

Corollary 2.7. *Every hyperkähler manifold of Kum^n-type with Picard rank $\rho(X) \geqslant 4$ is induced by a unique abelian surface A or its dual $A^\vee$.*

3. Main result

The generalization of Morrison's solution to Oda's conjecture to hyperkähler varieties of $K3^{[n]}$ type is as follows:

Theorem 3.1. *Let X be a projective hyperkähler manifold of $K3^{[n]}$ type. If there is an embedding $\phi : T(X) \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q}$ of $\mathbb{Q}$-lattices, then there exists an abelian variety A and a correspondence between X and A which induces a Hodge isometry $T(X) \otimes \mathbb{Q} \cong T(A) \otimes \mathbb{Q}$.*

Proof. Since $\mathrm{rank}(T(X)) = \dim_{\mathbb{Q}} T(X) \otimes \mathbb{Q} \leqslant 6$, we have that $T(X)$ has signature $(2, k)$, for $0 \leqslant k \leqslant 4$. By Nikulin's theorem A.8, we have a (unique) primitive embedding $i : T(X) \hookrightarrow L = \Lambda_{K3} = E_8(-1)^2 \oplus U^3$. According to Morrison's Corollary 1.9, there is a K3 surface S with $T(S) \cong T(X)$. In order to show that we can make choices such that $T(S) \cong T(X)$ is in fact a Hodge isometry, we adapt Morrison's proof (and notation) of Corollary 1.9, as follows.

The lattice $T(X)$ comes equipped with a Hodge structure $T(X) \otimes C \cong T^{2,0} \oplus T^{1,1} \oplus T^{0,2}$, where $T^{2,0} = \mathbb{C}\omega$ is 1-dimensional. Let $S = i(T(X))^{\perp}$ in L. Define $L^{2,0} = \mathbb{C}i(\omega)$ and $L^{0,2} = \mathbb{C}i(\bar{\omega})$, and $L^{1,1}$ the orthogonal complement of $L^{2,0} \oplus L^{0,2}$ in $L \otimes \mathbb{C}$. Let $\Sigma = L^{1,1} \cap (L \otimes \mathbb{R})$, and $N = \Sigma \cap L$. It is easy to see that N and $i(T(X))$ are orthogonal complements in L, and the quadratic form q on L, extended to $L \otimes \mathbb{R}$, has signature $(1, 19)$ on the subspace Σ. Also, $L \otimes \mathbb{C} = L^{2,0} \oplus L^{1,1} \oplus L^{0,2}$, is a Hodge decomposition. Choosing a component of the light cone $\{x \in L^{1,1} \cap (L \otimes R) : q(x) > 0\}$ makes it a signed Hodge structure. By the surjectivity of the period map (see for example, [Mor84, Theorem 1.7]), there is a K3 surface S and a signed Hodge isometry $\phi : H^2(S, \mathbb{Z}) \to L$. It follows that $\phi|_{NS(S)}$ gives an isometry of $NS(S)$ with $L^{1,1} \cap L = \Sigma \cap L = N$, which has signature $(1, \rho - 1)$ for some $\rho \leqslant 20$, and therefore S is algebraic. Finally, $\phi|_{T(S)}$ gives a Hodge isometry $T(S) \cong i(T(X))$.

By [PRO25, Theorem 3.7] (essentially proved by Markman in [Mar10]), since X is induced by $T(S)$, we have that X is birational to a moduli space $M_v(S, H)$ for some primitive[4] vector $v \in \tilde{H}(S)$ and a v-generic polarization H. (Their remark 3.8 further shows that X is a moduli space of semistable sheaves for some Bridgeland stability condition.)

Next, by [PRO25, Theorem 3.5] (due to Mukai, O'Grady, and Yoshioka), the moduli space $M_v(S, H)$ is deformation equivalent to $S^{[n]}$, where $2n = \langle v, v \rangle + 2$. By [Mar24, Theorem 1.1], it is enough to construct a correspondence between $Y = S^{[n]}$ and S. Now we have the maps $Y \xrightarrow{HC} \mathrm{Sym}^n(S)$, and $Z = S^n \xrightarrow{\pi} \mathrm{Sym}^n(S)$. On $Z \times S = S^{n+1}$ we have the obvious correspondence $\Psi = \sum_i \pi^*_{i,n+1}(\Delta)$ where $\Delta \subset S \times S$ is the diagonal. It induces a diagonal map on H^2. So we can push it forward by π and pull it back by HC, the Hilbert-Chow morphism, to get the correspondence from Y to S.

Note that by Oda-Morrison's theorem for K3 surfaces (1.1), since $T(S) \otimes \mathbb{Q} \cong T(X) \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q}$, we already know there is a correspondence between S and A inducing a Hodge isometry $T(S) \otimes \mathbb{Q} \cong T(A) \otimes \mathbb{Q}$, and composing with the previous correspondence $X \sim S$, we prove Oda-Morrison's conjecture for the hyperkähler manifold $S^{[n]}$. By Markman's [Mar24, Theorem 1.1], since we are in the projective case, there exists an algebraic correspondence between X of $K3^{[n]}$ type and $Y = S^{[n]}$, and therefore between X and A, inducing a Hodge isometry $T(X) \otimes \mathbb{Q} \cong T(A) \otimes \mathbb{Q}$.

□

Lemma 3.2. *Let t be a quadratic space over $\mathbb{Q}$ of signature $(2, k)$, and let u be the hyperbolic plane over $\mathbb{Q}$. Then*

(1) If $\dim t \leqslant 3$, (i.e., if $k \leqslant 1$), then $t \hookrightarrow u^3$.
(2) If $k = 2$, then $t \hookrightarrow u^3$ iff t has an isotropic vector.
(3) If $k = 3$, then $t \hookrightarrow u^3$ iff t has a 2-dimensional isotropic subspace.

Remark 3.3. *In the lemma, $k = 4$ is impossible due to signature reasons. Also, the proof of the lemma is analogous to that of* [Mor84, Cor 2.6]

Proof. The first part follows from the fact that u represents every rational number. In the case when $\dim t = 3$, we can diagonalize $t = \langle a, b, c \rangle$, we can find vectors α, β, γ in each of the three orthogonal copies of u with $\langle \alpha, \alpha \rangle = a$ etc. The case of $\dim t = 2$ is even simpler; one can embed t in u^2, in fact.

For the second part, first note that if t has an isotropic vector α, then we can find $\beta \in t$ such that $\langle \alpha, \beta \rangle = 1$; then $\mathbb{Q}\alpha + \mathbb{Q}\beta$ is a hyperbolic plane. Therefore, $t = u \oplus t'$ for some t' of signature $(1, 1)$, which

[4]The theorem as stated does not mention that v is primitive; however, the proof constructs such a primitive vector v.

embeds into u^2 by the proof of part (1). Therefore $t \hookrightarrow u^3$. Conversely, if $t \hookrightarrow u^3$, then $t^\perp$ has signature $(1,1)$ and embeds in u^2 by part (1). Then $u^3 = t \oplus t^\perp \hookrightarrow t \oplus u^2$, and so $u \hookrightarrow t$ by Witt cancellation.

Finally, if t has dimension 5, and t has an isotropic subspace of dimension 2 generated by α, α', then as in the previous paragraph, we can find β, β' such that these four vectors span u^2. Therefore, $t = u^2 \oplus t'$ for some 1-dimensional t', which embeds in u. Therefore $t \hookrightarrow u^3$. Conversely, if $t \hookrightarrow u^3$, then $t^\perp$ is 1-dimensional and embeds in u. Therefore, $u^3 = t \oplus t' \hookrightarrow t \oplus u$, and then $u^2 \hookrightarrow t$ by Witt cancellation.

□

We obtain the following corollary by combining Theorem 3.1 and Lemma 3.2.

Corollary 3.4. *Let X by a projective hyperkähler manifold of $K3^{[n]}$ type. Then if any of the following conditions are met, there is an abelian variety A and a correspondence between X and A inducing a Hodge isometry $T(X) \otimes \mathbb{Q} \cong T(A) \otimes \mathbb{Q}$.*

- $\rho(X) \in \{20, 21\}$.
- $\rho(X) = 19$ *and* $T(X)$ *has an isotropic vector.*
- $\rho(X) = 18$ *and* $T(X)$ *has a 2-dimensional isotropic sublattice (i.e.,* $\mathbb{Z}x + \mathbb{Z}y$ *such that* $x^2 = y^2 = x \cdot y = 0$*).*

Remark 3.5. *To summarize, now we know how to handle the $K3^{[n]}$ type case: for large Picard numbers, X is precisely the Mukai moduli space $M_v(S)$ for a unique $K3$ surface S with a large Picard number. By Morrison's results, there is a correspondence between S and an abelian surface, which in turn corresponds to an abelian variety. The other known cases of of hyperkähler manifolds can be treated similarly. The generalized Kummer varieties Kum_n already arise from a Kummer-type construction due to Beauville's classical construction* [Bea83]. *The exceptional O'Grady examples OG10 and OG6 are related (up to an index 2) to a $K3^{[5]}$-type and a $K3^{[3]}$-type hyperkähler manifold, respectively. The example OG6 is a quotient of $K3^{[3]}$ by a birational involution (*[MRS18]*). For the exceptional example OG10, if the transcendental lattice $T(X)$ contains a copy of the hyperbolic lattice U, then X is of LSV-type (*[LSV17]*), and then it is in correspondence with a cubic fourfold Z, and in turn, for large $NS(Z)$ one obtains a correspondence to a $K3$ surface (as noticed in Laza's paper* [Laz21]*).*

Appendix A. Some lattice theory

We briefly recall some background on lattices.

Definition A.1. A *lattice* will denote a finitely generated free abelian group Λ equipped with a symmetric bilinear form $B : \Lambda \times \Lambda \to \mathbb{Z}$.

We abbreviate the data (Λ, B) to Λ sometimes, when the form is understood, and we interchangeably write $u \cdot v = \langle u, v\rangle = B(u, v)$ and $u^2 = \langle u, u\rangle = B(u, u)$ for $u \in \Lambda$.

The *signature* of the lattice is the real signature of the form B, written (r_+, r_-, r_0) where r_+, r_- and r_0 are the number of positive, negative and zero eigenvalues of B, counted with multiplicity. We say that the lattice is *non-degenerate* if the form B has zero kernel, i.e. $r_0 = 0$. In that case, the signature is abbreviated to (r_+, r_-). We say Λ is *even* if $x^2 \in 2\mathbb{Z}$ for all $x \in \Lambda$.

Let Λ be a non-degenerate lattice. The *discriminant* of the lattice is $\mathrm{disc}(\Lambda) := |\det(B)|$. The lattice is said to be *unimodular* if its discriminant is 1. From now on, we restrict attention to non-degenerate lattices. We may think of Λ as embedded in the real or p-adic space $\Lambda \otimes \mathbb{R}$ or $\Lambda \otimes \mathbb{Q}_p$, and extend the bilinear form to the ambient space.

Definition A.2. The *dual lattice* of Λ is $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z})$. We can identify Λ^* with a subgroup of $\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$, using the bilinear form B to identify $\Lambda_{\mathbb{Q}}$ with its $\mathbb{Q}$-dual. It is easy to show that $\mathrm{disc}(\Lambda) = [\Lambda^* : \Lambda]$.

Definition A.3. The finite abelian group Λ^*/Λ is called the *discriminant group* A_Λ.

Let Λ be an even non-degenerate lattice.

Definition A.4. A stronger invariant of the lattice is its *discriminant form* q_Λ, which is defined as follows. The form B on Λ induces a $\mathbb{Q}$-valued bilinear symmetric form on Λ^*, and consequently also induces a quadratic form on A_Λ as follows. The induced form $\Lambda^* \times \Lambda^* \to \mathbb{Q}$ takes $\Lambda^* \times \Lambda$ into $\mathbb{Z}$, and the diagonal of $\Lambda \times \Lambda$ to $2\mathbb{Z}$. Therefore we get an induced symmetric form $b : A_\Lambda \times A_\Lambda \to \mathbb{Q}/\mathbb{Z}$ and a quadratic form $q : A_\Lambda \to \mathbb{Q}/2\mathbb{Z}$ such that for all $n \in \mathbb{Z}$ and all $a, b \in \Lambda$, we have

$$q(na) = n^2 q(a)$$

$$q(a + a') - q(a) - q(a') \equiv 2b(a, a') \mod 2\mathbb{Z}$$

This data (A_Λ, b, q) will be abbreviated to q_Λ.

Note that the discriminant of the lattice is just the size of the discriminant group. We shall let $l(A)$ denote the minimum number of generators of an abelian group A. Note that $l(A_\Lambda) \leqslant \operatorname{rank}(\Lambda^*) = \operatorname{rank}(\Lambda)$. For a unimodular lattice Λ, we have $l(A_\Lambda) = 0$.

The discriminant form of a unimodular lattice is trivial, and if $M \subset L$ is a *primitive* embedding of non-degenerate even lattices (that is, L/M is a free abelian group), with L unimodular, then we have

$$q_{M^\perp} = -q_M$$

For a lattice Λ and a real number α, we denote by $\Lambda(\alpha)$ the lattice which has the same underlying group but with the bilinear form scaled by α. The lattice $\mathbb{Z}\langle\alpha\rangle$ of rank one with a generator of norm α will be abbreviated $\langle\alpha\rangle$.

A *root* of a positive definite lattice, we will mean an element x such that $x^2 = 2$, whereas for a negative-definite or indefinite lattice, we will mean an element x such that $x^2 = -2$. A *root lattice* is a lattice that is spanned by its roots.

The familiar root lattices (positive definite by convention) are listed below:

- A_n : This is the n-dimensional simplex lattice $\{x \in \mathbb{Z}^{n+1} \,|\, \sum x_i = 0\}$ inside the sum-zero hyperplane in R^{n+1}. It has $n(n+1)$ roots, and discriminant $n+1$.
- D_n : The n-dimensional checkerboard lattice is $\{x \in \mathbb{Z}^n \,|\, \sum_{i=1}^{n} x_i \equiv 0 \mod 2\}$. It has $2n(n-1)$ roots, and discriminant 4.
- E_8 : This is an "exceptional root lattice"; one realization of E_8 is as the span of D_8 and the all-halves vector $(1/2, \ldots, 1/2)$. It has 240 roots, and is unimodular.
- E_7 : taking the orthogonal complement of any root in E_8 gives us the root lattice E_7. It has 126 roots, and discriminant 2.
- E_6 : taking the orthogonal complement of any A_2 sublattice of E_8 (i.e., spanned by two roots e_1, e_2 such that $e_1 \cdot e_2 = -1$), gives us E_6. It has 72 roots, and discriminant 3.

The ADE classification theorem of root lattices states that any positive definite root lattice can be decomposed into an orthogonal direct sum of the root lattices $A_n, D_n, E_{\{6,7,8\}}$ listed above.

Let U be the *hyperbolic plane*, i.e. the indefinite rank 2 lattice whose matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that $U(-1) \cong U$.

Positive definite even unimodular lattices only exist in dimensions which are multiples of n, and their number grows at least doubly exponentially in the dimension. (More generally, the Smith-Minkowski-Siegel mass formula can be used to enumerate or estimate the number of lattices of positive definite lattices of a given dimension with specified local character at all primes, i.e., in a genus.)

In contrast, the structurex of indefinite unimodular lattices is quite simple to describe.

Theorem A.5. (Milnor [Mil58]) *Let Λ be an indefinite unimodular lattice. If Λ is even, then $\Lambda \cong E_8(\pm 1)^m \oplus U^n$ for some m and n. If Λ is odd, then $\Lambda \cong \langle 1\rangle^m \oplus \langle -1\rangle^n$ for some m and n.*

Theorem A.6. (Kneser [Kne56], Nikulin [Nik79]) *Let L be an even lattice with signature (s_+, s_-) and discriminant form q_L such that*

(1) $s_+ > 0$
(2) $s_- > 0$
(3) $l(A_L) \leqslant \mathit{rank}(L) - 2$.

Then L *is the unique lattice with that signature and discriminant form, up to isometry.*

Proposition A.7. *Let* M *be a unimodular lattice. Then if* $M \subset L$*, we have an orthogonal decomposition* $L = M \oplus M^{\perp}$.

Theorem A.8. (Nikulin [Nik79] Theorem 1.14.4) *Let* M *be an even lattice with invariants* (t_+, t_-, q_M) *and let* L *be an even unimodular lattice of signature* (s_+, s_-)*. Suppose that*

(1) $t_+ < s_+$.
(2) $t_- < s_-$.
(3) $l(A_M) \leqslant \mathit{rank}(L) - \mathit{rank}(M) - 2$.

Then there exists a unique primitive embedding of M *into* L*, up to automorphisms of* L.

Remark A.9. *Note that* $l(A_M) \leqslant \mathrm{rank}(M)$*, so the third condition in Theorem A.8 may be replaced by the sufficient condition* $\mathrm{rank}(M) \leqslant \mathrm{rank}(L)/2 - 1$.

Theorem A.10. *Fix an abelian surface* A*. There are only finitely many abelian surfaces* B *up to isomorphism such that there is a Hodge isometry* $T(A) \cong T_B$.

It is a theorem of Mukai and Orlov that A and B satisfy this property iff they are Fourier-Mukai partners, i.e., their derived categories of bounded complexes of coherent sheaves are equivalent. [GL04].

Proof. Denote the abstract lattice $T(A) \cong T_B$ by M, and let L be the class of the abstract lattice $H^2(A, \mathbb{Z}) \cong H^2(B, Z) \cong U^3$. Then by [MM86], there are finitely many classes of embeddings $M \hookrightarrow L$, up to isometries of L. If the embeddings $T(A) \hookrightarrow H^2(A, \mathbb{Z})$ and $T_B \hookrightarrow H^2(B, \mathbb{Z})$, identified by $\phi_T : T(A) \cong T_B$, are the same up to isometries of L, then it follows that the Hodge isometry ϕ_T can be extended to a Hodge isometry $\phi : H^2(A, \mathbb{Z}) \to H^2(B, \mathbb{Z})$ (since $NS(A) = T(A)^{\perp}$ lies in the $(1, 1)$ part of the Hodge decomposition and similarly for B) and then $A \cong B$ by Torelli. Therefore, there are only finitely many possibilities for B, up to isomorphism. □

As a corollary, for a given K3 surface X there are only finitely many abelian surfaces A related to X by a Shioda-Inose structure (since any other such B would have $T(A) \cong T(X) \cong T_B$ by Hodge isometries).

We end with a brief mention of the theory of quadratic spaces over $\mathbb{Q}$: it is significantly simpler than that of integral lattices. A good reference is the book "Rational Quadratic Forms" [Cas78] by Cassels. We have used the following key lemma in the proof of Lemma 3.2.

Lemma A.11 (Witt cancellation)**.** *Let* v, w_1, w_2 *be quadratic spaces over* $\mathbb{Q}$*, such that* $v \oplus w_1 \cong v \oplus w_2$*. Then* $w_1 \cong w_2$.

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