DEFINITION OF STACKS

1. Introduction

The origin of "stacks" lies in work of Grothendieck, Giraud, et al., on non-Abelian cohomology [REFERENCE, Giraud]. The use of stacks as a generalization of schemes (and algebraic spaces), with extensions to stacks of the main theorems about schemes (e.g., theorems about cohomology of sheaves on schemes) dates to the article of Deligne and Mumford, [REFERENCE, Deligne – Mumford], who used stacks to prove irreducibility of the moduli space of genus-g curves in arbitrary characteristic (following the proof in characteristic zero by Hurwitz [REFERENCE] and partial results in positive characteristic by Fulton [REFERENCE]). Their original notion is what is now called a *Deligne-Mumford stack*. Soon after, Michael Artin [REFERENCE, Versal deformations] introduced the generalized notion of what is now called an *algebraic stack* (or *Artin stack*). The notion of algebraic stacks unifies and generalizes earlier concepts such as V-manifolds and orbifolds.

2. Classes

The language for describing stacks is category theory. Although it is possible to develop category theory directly as a formalism for the foundation of mathematics without first developing set theory, here category theory is formalized within Zermelo-Frankel set theory. The *signature* of Zermelo-Frankel set theory includes the signature of first-order predicate logic, e.g., symbols for countably many variables (e.g., by repetition of a single variable symbol "x", i.e., "x", "xx", "xxx", etc., when preceded / succeeded by non-variable symbols are interpreted as variables "x_1", "x_2", "x_3", etc.), for parentheses, for logical connectives such as "$\neg$" and "$\Rightarrow$" (from which we can derive the other common logical connectives "$\wedge$", "$\vee$", "$\Leftarrow$", and "$\Leftrightarrow$"), for equality,"$=$", and for the quantifier "$\forall$" (from which we can derive the quantifier "$\exists$"). The signature of Zermelo-Fraenkel set theory also includes "$\in$", the set membership symbol. There are many other standard symbols of logic and set theory that are abbreviations for predicates in Zermelo-Fraenkel set theory with the signature above. We use these freely. Altogether, the collection of all symbols in the most common signature of Zermelo-Fraenkel set theory is finite, and the collection of all strings of symbols in this signature is countably infinite. The axioms of Zermelo-Fraenkel set theory are recorded in many places; they include two axiom schemata of countably many axioms (in the first-order theory, or just two axioms in a second-order theory), and axioms that allow for an "arity collapse" via predicates that encode a finite ordered tuple of sets as a single set – the Kuratowski ordered tuple set – and that decode the individual coordinates of a Kuratowski ordered tuple. It is for this reason that the meta-definitions below usually have just one "parameter" set (since any finite ordered tuple of parameter sets can be encoded as a single set using the Kuratowski ordered tuple set).

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To formulate category theory, we could start over and list a first-order theory that begins with categories (and produces set theory as a special case). Alternatively, we could use a first-order theory of sets and classes, such as von Neumann – Bernays – Gödel theory. However, most mathematicians stick with Zermelo-Fraenkel set theory as their first-order theory, and they work with category theory as a second-order theory. Statements about classes and categories are formulated in a meta-metalanguage, since these are theory are really statements about both sets (formulated using the first-order language of sets) and predicates for sets (stated in the metalanguage).

Definition 2.1. A **string** is an ordered pair (s, n) of a nonnegative integer n and of an ordered n-tuple s. A **predicate** is an ordered pair (s, n) of a nonnegative integer n and an ordered n-tuple s of symbols from the signature of Zermelo-Frankel set theory satisfying the production rules / formal grammar of Zermelo-Frankel set theory, i.e., a string such that, for every model of Zermelo-Frankel set theory with assignments of sets to all free variables, Tarski's inductive procedure for truth values gives a well-defined truth value to the string. For every nonnegative integer n, for every ordered pair $(P(x_1, \dots, x_n), Q(y_1, \dots, y_n))$ of predicates together with an ordered n-tuple of variables that contain all free variables of the predicate (and none of the bound variables), the pair is **Lindenbaum-Tarski equivalent** if (and only if), for every ordered n-tuple $(z_1, \dots, z_n)$ of sets, the predicate $P(z_1, \dots, z_n)$ holds if and only if $Q(z_1, \dots, z_n)$ holds, i.e.,

$$\forall\ (z_1, \dots, z_n)\ (P(z_1, \dots, z_n) \Leftrightarrow Q(z_1, \dots, z_n)).$$

Similarly, for every positive integer n, for every ordered pair

$$(((a_1, \dots, a_{n-1}), P(x_1, \dots, x_n)), ((b_1, \dots, b_{n-1}), Q(y_1, \dots, y_n)))$$

each of whose two entries is itself an ordered pair of an ordered $(n-1)$-tuple of sets and a predicate as above, the pair is **Lindenbaum-Tarski equivalent** if (and only if), for every set z, the predicate $P(a_1, \dots, a_{n-1}, z)$ holds if and only if the predicate $Q(b_1, \dots, b_{n-1}, z)$ holds.

Definition 2.2. A (Zermelo-Fraenkel) **class** C is a Lindenbaum-Tarski equivalence class $[(a, P(x, t))]$ of a set a and of a predicate P with an ordered pair (x, t) of variables that contain all free variables of P (and no bound variables). For every class $C = [(a, P(x, t))]$, for every set s, the set s is a **member** of the class C if (and only if) the predicate $P(a, s)$ holds. In particular, two classes are equal if and only if they have the same members (extensionality). For every class $C = [(a, P(x, t))]$, for every class $D = [(b, Q(y, u))]$, the class C is a **subclass** of D if (and only if), for every set s, if $P(a, s)$ holds then $Q(b, s)$ holds. In particular, when $P(x, t)$ is in the tautological Lindenbaum-Tarski equivalence class of predicates, e.g., $(x = x) \wedge (t = t)$, then the corresponding class is the **class of all sets**, denoted **Set**; this is the unique class such that every class is a subclass of **Set**. For the predicate $P(x, t)$ of $t \in x$, for every set a, the class $[(a, P(x, t))]$ is the **class of the set** a, denoted $\mathbf{Set}_{\in a}$.

For every class C, for every class D, the **Cartesian product class** is the class $C \times D$ whose members are those Kuratowski ordered pairs (x, t) such that x is member of C and such that t is a member of D. For every class C, the **diagonal class of** C is the subclass of $C \times C$ whose members are Kuratowski ordered pairs (s, s) such that s is a member of C; this class is denote Δ_C. For every class C, the class C is a **class of ordered pairs** if (and only if) the class C is a subclass of the Cartesian product class $\mathbf{Set} \times \mathbf{Set}$. For every class of ordered pairs D, the **transpose class** $D^\dagger$ is the class of ordered pairs whose members are all ordered pairs (t, x) such that (x, t) is a member of D.

For every class C, for every class D, the class D is a C-**class** if (and only if), D is a class of Cartesian products such that for every set x such that there exists a set t with (x, t) a member of D, the set x is a member of C, i.e., every member of D is of the form (x, t) for some member x of C and for some set t. In this case, for every member x of C, the **fiber class** of D over x is the class whose members are all sets t such that (x, t) is a member of D; this is denoted $D_{x,\bullet}$.

For every class C, for every C-class F, the class F is a C-**function** if (and only if), for every member x of C there exists a unique set t such that (x, t) is a member of F. For every class C, for every C-class F, for every member x of C, for every set t, the set t is the F-**value** of x if (and only if) the ordered pair (x, t) is a member of F. In this case, we denote t by $F(x)$. For every class C, for every C-class F, the **image class** of F is the class whose members are all sets t such that there exists a member x of C with t equal to $F(x)$. More generally, for every class C, for every class D, for every C-function F, this is a C-function **to the class** D if (and only if) the image of F is a subclass of D.

For every class C, for every class D, for every C-morphism F to D, for every object t of D, the t-**fiber** of F is the subclass of C whose members are all members x of C such that $F(x)$ equals t. For every class C, for every class D, for every class E, for every C-morphism F to D, for every E-morphism G to D, the **fiber product** of G and F is the subclass of the product class $C \times E$ whose members are all members (s, t) of $C \times E$ such that $F(s)$ equals $G(t)$; this is denoted $C \times_{F,D,G} E$, or just $C \times_D E$ when F and G are understood.

For every class C, for every class D, for every class E, for every C-function F to D, for every D-function G to E, the **composite** C-function to E of G and F is the unique C-function $G \circ F$ to E such that for every member s of C and for every member u of E, the ordered pair (s, u) is a member of $G \circ F$ if and only if there exists a member t of D such that t equals $F(s)$ and u equals $G(t)$.

Exercise 2.3. Formulate and prove the statement that for every class C there is a unique left-right identity C-function to C. Also formulate and prove associativity for composition of class functions. Prove that for every ordered pair (a, b) of sets, for every function f from a to b considered as a subset of the Cartesian product $a \times b$, the class $\mathbf{Set}_{\in f}$ is the unique $\mathbf{Set}_{\in a}$-function to $\mathbf{Set}_{\in b}$ whose members are those ordered pairs (x, y) with x an element of a, with y an element of b, and with y equals $f(x)$. Prove that every $\mathbf{Set}_{\in a}$-functo to $\mathbf{Set}_{\in b}$ equals $\mathbf{Set}_{\in f}$ for a unique function f from a to b.

3. Categories

Using the second-order notion of class, we can formulate the second-order notion of category.

Definition 3.1. For every ordered triple $\mathcal{C} = (\mathrm{Obj}_{\mathcal{C}}, \mathrm{Hom}_{\mathcal{C}}, \mathsf{comp}_{\mathcal{C}})$ of a class $\mathrm{Obj}_{\mathcal{C}}$, of a $\mathrm{Obj}_{\mathcal{C}} \times \mathrm{Obj}_{\mathcal{C}}$-function $\mathrm{Hom}_{\mathcal{C}}$ that associates to every ordered pair (a, b) of members of $\mathrm{Obj}_{\mathcal{C}}$ a set denoted $\mathrm{Hom}_{\mathcal{C}}(a,b) = \mathcal{C}^a_b$, and of a $\mathrm{Obj}_{\mathcal{C}} \times \mathrm{Obj}_{\mathcal{C}} \times \mathrm{Obj}_{\mathcal{C}}$-function $\mathsf{comp}_{\mathcal{C}}$ that associates to every ordered triple (a,b,c) of members of $\mathrm{Obj}(\mathcal{C})$ a set function,

$$\mathsf{comp}^{a,b,c}_{\mathcal{C}} : \mathcal{C}^b_c \times \mathcal{C}^a_b \to \mathcal{C}^a_c,$$

equivalently, that associates to (a,b,c) a subset of the Cartesian product set $\left(\mathcal{C}^b_c \times \mathcal{C}^a_b\right) \times \mathcal{C}^a_c$ whose projection to the first factor is a bijection, the ordered triple $\mathcal{C} = (\mathrm{Obj}_{\mathcal{C}}, \mathrm{Hom}_{\mathcal{C}}, \mathsf{comp}_{\mathcal{C}})$ is a **category** if (and only if) both, for every member a of $\mathrm{Obj}_{\mathcal{C}}$ there exists an element $\mathrm{Id}^{\mathcal{C}}_a$ of $\mathrm{Hom}_{\mathcal{C}}(a,a)$ such that both $f \circ \mathrm{Id}^{\mathcal{C}}_a$ equals f for every member b of $\mathrm{Obj}_{\mathcal{C}}$ and for every element f of $\mathrm{Hom}_{\mathcal{C}}(a,b)$ and such that $\mathrm{Id}^{\mathcal{C}}_a \circ g$ equals g for every member b of $\mathrm{Obj}_{\mathcal{C}}$ and for every element g of $\mathrm{Hom}_{\mathcal{C}}(b,a)$, and for every ordered quadruple (a,b,c,d) of members of $\mathrm{Obj}_{\mathcal{C}}$, the following compositions of set function are both equal,

$$\mathrm{Hom}_{\mathcal{C}}(c,d) \times \mathrm{Hom}_{\mathcal{C}}(b,c) \times \mathrm{Hom}_{\mathcal{C}}(a,b) \xrightarrow{\mathrm{Id} \times \circ_{a,b,c}} \mathrm{Hom}_{\mathcal{C}}(c,d) \times \mathrm{Hom}_{\mathcal{C}}(a,c) \xrightarrow{\circ_{a,c,d}} \mathrm{Hom}_{\mathcal{C}}(a,d),$$

$$\mathrm{Hom}_{\mathcal{C}}(c,d) \times \mathrm{Hom}_{\mathcal{C}}(b,c) \times \mathrm{Hom}_{\mathcal{C}}(a,b) \xrightarrow{\circ_{b,c,d} \times \mathrm{Id}} \mathrm{Hom}_{\mathcal{C}}(b,d) \times \mathrm{Hom}_{\mathcal{C}}(a,b) \xrightarrow{\circ_{a,b,d}} \mathrm{Hom}_{\mathcal{C}}(a,d).$$

In this case, the members of $\mathrm{Obj}_{\mathcal{C}}$ are **objects** of $\mathcal{C}$, each set $\mathrm{Hom}_{\mathcal{C}}(a,b)$ is a **Hom set** of $\mathcal{C}$ whose elements are **morphisms** of $\mathcal{C}$, and each set function $\mathsf{comp}^{a,b,c}_{\mathcal{C}}$ is **composition** of morphisms of $\mathcal{C}$. A morphism f is **invertible** or an isomorphism if (and only if) there exists a morphism g such that both $g \circ f$ and $f \circ g$ are defined and equal identity morphisms, i.e., g is both a left inverse and a right inverse for f.

For every category $\mathcal{C}$, the **source class function** is the $\mathrm{Hom}_{\mathcal{C}}$-function to $\mathrm{Obj}_{\mathcal{C}}$ associating the object a to every ordered triple $((a,b),f)$ of an ordered pair (a,b) of objects of $\mathcal{C}$ and a $\mathcal{C}$-morphism f from a to b; this is denoted by $\mathrm{source}_{\mathcal{C}}$ or just $s_{\mathcal{C}}$. For every category $\mathcal{C}$, the **target class function** is the $\mathrm{Hom}_{\mathcal{C}}$-function to $\mathrm{Obj}_{\mathcal{C}}$ associating the object b to every ordered triple $((a,b),f)$ of an ordered pair (a,b) of objects of $\mathcal{C}$ and a $\mathcal{C}$-morphism f from a to b; this is denoted by $\mathrm{target}_{\mathcal{C}}$ or just $t_{\mathcal{C}}$. In other words, the product class function $(s_{\mathcal{C}}, t_{\mathcal{C}})$ from $\mathrm{Hom}_{\mathcal{C}}$ to $\mathrm{Obj}_{\mathcal{C}} \times \mathrm{Obj}_{\mathcal{C}}$ is the usual domain class function of the $\mathrm{Obj}_{\mathcal{C}} \times \mathrm{Obj}_{\mathcal{C}}$-class function $\mathrm{Hom}_{\mathcal{C}}$.

For every category $\mathcal{C} = (\mathrm{Obj}_{\mathcal{C}}, \mathrm{Hom}_{\mathcal{C}}, \mathsf{comp}_{\mathcal{C}})$, the **opposite category** is the category $\mathcal{C}^{\mathrm{opp}}$ whose objects are the objects of $\mathcal{C}$, such that for every ordered pair (a,b) of objects of $\mathcal{C}$ the Hom set $\mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(b,a)$ equals the Hom set $\mathrm{Hom}_{\mathcal{C}}(a,b)$, and such that the composition in $\mathcal{C}^{\mathrm{opp}}$ of f with g equals the composition $\mathsf{comp}^{a,b,c}_{\mathcal{C}}(g,f)$.

A category $\mathcal{C}$ is **strictly small** if (and only if) the class of objects equals $\mathbf{Set}_{\in O}$ for some set O.

Definition 3.2. For every category $\mathcal{C} = (\mathrm{Obj}_{\mathcal{C}}, \mathrm{Hom}_{\mathcal{C}}, \mathsf{comp}_{\mathcal{C}})$ and for every category $\mathcal{D} = (\mathrm{Obj}_{\mathcal{D}}, \mathrm{Hom}_{\mathcal{D}}, \mathsf{comp}_{\mathcal{D}})$, an ordered pair $F = (F_{\mathrm{Obj}}, F_{\mathrm{Hom}})$ of a $\mathrm{Obj}_{\mathcal{C}}$-function F_{Obj} to $\mathrm{Obj}_{\mathcal{D}}$ and of a $\mathrm{Obj}_{\mathcal{C}} \times \mathrm{Obj}_{\mathcal{C}}$-function F_{Hom} that associates to every ordered pair (a,b) of members of $\mathrm{Obj}_{\mathcal{C}}$ a set function,

$$F^a_b : \mathcal{C}^a_b \to \mathcal{D}^{F(a)}_{F(b)},$$

is a **covariant functor** if (and only if) both, for every object a of $\mathcal{C}$, the value under F^a_a of $\mathrm{Id}^{\mathcal{C}}_a$ equals $\mathrm{Id}^{\mathcal{D}}_{F(a)}$, and, for every ordered triple (a,b,c) of objects of $\mathcal{C}$ and for every element (g,f) of $\mathcal{C}^b_c \times \mathcal{C}^a_b$, the $\mathcal{D}$-morphism $F^a_c(g \circ f)$ equals the $\mathcal{D}$-composition of $F^b_c(g)$ with $F^a_b(f)$.

In particular, for every category $\mathcal{C}$, the **identity functor** from $\mathcal{C}$ to $\mathcal{C}$ is the functor such that, for every object a of $\mathcal{C}$, the value under the functor equals a, and such that, for every ordered pair (a, b) of objects of $\mathcal{C}$, the set function from $\mathcal{C}_b^a$ to $\mathcal{C}_b^a$ under the functor is the identity set function.

For every category $\mathcal{C}$, for every category $\mathcal{D}$, for every category $\mathcal{E}$, for every covariant functor F from $\mathcal{C}$ to $\mathcal{D}$, for every covariant functor G from $\mathcal{D}$ to $\mathcal{E}$, the **composite covariant functor** from $\mathcal{C}$ to $\mathcal{E}$ of G and F is the covariant functor $G \circ F$ such that both, for every object a of $\mathcal{C}$, the value under $G \circ F$ equals $G(F(a))$, and, for every ordered pair (a, b) of objects of $\mathcal{C}$, the set function $(G \circ F)_b^a$ from $\mathcal{C}_b^a$ to $\mathcal{E}_{G(F(b))}^{G(F(a))}$ equals the $\mathcal{D}$-composition of $G_{F(b)}^{F(a)}$ with F_b^a.

Definition 3.3. For every category $\mathcal{C}$, for every category $\mathcal{D}$, a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor from $\mathcal{C}^{\text{opp}}$ to $\mathcal{D}$.

Definition 3.4. For every category $\mathcal{C}$, for every category $\mathcal{D}$, for every covariant functor F from $\mathcal{C}$ to $\mathcal{D}$, for every covariant functor G from $\mathcal{C}$ to $\mathcal{D}$, a **natural transformation** from F to G is a $\text{Obj}_{\mathcal{C}}$-function θ associating to every object a of $\mathcal{C}$ an element θ_a of $\mathcal{D}_{G(a)}^{F(a)}$ such that, for every ordered pair (a, b) of objects of $\mathcal{C}$, for every element u of $\mathcal{C}_b^a$, the $\mathcal{D}$-composite $\theta_b \circ F_b^a(u)$ equals the $\mathcal{D}$-composite $G_b^a(u) \circ \theta_a$. A natural transformation is a **natural equivalence** (or **natural isomorphism**) if (and only if) the morphism associated to each object is an isomorphism. In particular, for every category $\mathcal{C}$, for every category $\mathcal{D}$, and for every covariant functor F from $\mathcal{C}$ to $\mathcal{D}$, the **identity natural transformation** from F to itself is the natural transformation that associates to every object a of $\mathcal{C}$ the identity morphism $\text{Id}_{F(a)}^{\mathcal{D}}$; this is denoted by $\text{Id}_F^{\mathcal{C},\mathcal{D}}$.

For every category $\mathcal{C}$, for every category $\mathcal{D}$, for every ordered triple (F, G, H) of covariant functors from $\mathcal{C}$ to $\mathcal{D}$, for every natural transformation θ from F to G, for every natural transformation η from G to H, the (vertical) **composite natural transformation** $\eta \circ \theta$ from F to H is the natural transformation such that for every object a of $\mathcal{C}$, the associated $\mathcal{D}$-morphism from $F(a)$ to $H(a)$ equals the $\mathcal{D}$-composition of η_a with θ_a.

For every category $\mathcal{C}$, for every category $\mathcal{D}$, for every category $\mathcal{E}$, for every ordered pair (F, G) of covariant functors from $\mathcal{C}$ to $\mathcal{D}$, for every ordered pair (H, I) of covariant functors from $\mathcal{D}$ to $\mathcal{E}$, for every natural transformation θ from F to G, for every natural transformation η from H to I, the **horizontal composition natural transformation** of η and θ, or **Godement product**, is the natural transformation $\eta * \theta : H \circ F \to I \circ G$ associating to every object a of $\mathcal{C}$ the $\mathcal{E}$-morphism,

$$\eta_{G(a)} \circ_{\mathcal{C}} H_{F(a),G(a)}(\theta_a) = (\eta * \theta)_a = I_{F(a),G(a)}(\theta_a) \circ \eta_{F(a)}.$$

This is associative in both θ and η separately.

For every category $\mathcal{C}$, for every category $\mathcal{D}$, for every category $\mathcal{E}$, for every ordered pair (F, G) of covariant functors from $\mathcal{C}$ to $\mathcal{D}$, for every covariant functor H from $\mathcal{D}$ to $\mathcal{E}$, for every natural transformation θ from F to G, the H-**pushforward natural transformation** is $H_* \theta = \text{Id}_H^{\mathcal{D},\mathcal{E}} * \theta$, associating to every object a of $\mathcal{A}$ the $\mathcal{E}$-morphism $H_{F(a),G(a)}(\theta_a)$. Similarly, for every category $\mathcal{B}$, for every category $\mathcal{C}$, for every category $\mathcal{D}$, for every covariant functor E from $\mathcal{B}$ to $\mathcal{C}$, for every ordered pair (F, G) of covariant functors from $\mathcal{C}$ to $\mathcal{D}$, for every natural transformation θ

from F to G, the E-**pullback natural transformation**, $E^*\theta = \theta * \mathrm{Id}_E^{\mathcal{B},\mathcal{E}}$ associates to everyupd object b of $\mathcal{B}$ the $\mathcal{D}$-morphism $\theta_{E(b)}$. Of course the Godement product can be expanded in terms of pushforward, pullback and vertical composition,

$$G^*\eta \circ H_*\theta = \eta * \theta = I_*\theta \circ F^*\eta.$$

Definition 3.5. For every category $\mathcal{C}$, for every category $\mathcal{D}$, an **adjoint pair** of covariant functors between $\mathcal{C}$ and $\mathcal{D}$ is an ordered pair $((L, R), (\theta, \eta))$ of an ordered pair pair of covariant functors,

$$L : \mathcal{C} \to \mathcal{D},$$
$$R : \mathcal{D} \to \mathcal{C},$$

and a pair of natural transformations of covariant functors,

$$\theta : \mathrm{Id}_{\mathcal{C}} \Rightarrow R \circ L,\ \ \theta(a) : a \to R(L(a)),$$
$$\eta : L \circ R \Rightarrow \mathrm{Id}_{\mathcal{D}},\ \ \eta(b) : L(R(b)) \to b,$$

such that the following composition of natural transformations equals Id_R, respectively equals Id_L,

$$(*_R) : R \overset{\theta\circ R}{\Rightarrow} R \circ L \circ R \overset{R\circ\eta}{\Rightarrow} R,$$
$$(*_L) : L \overset{L\circ\theta}{\Rightarrow} L \circ R \circ L \overset{\eta\circ L}{\Rightarrow} R.$$

For every object a of $\mathcal{C}$ and for every object b of $\mathcal{D}$, define set maps,

$$H_R^L(a, b) : \mathrm{Hom}_{\mathcal{D}}(L(a), b) \to \mathrm{Hom}_{\mathcal{C}}(a, R(b)),$$
$$(L(a) \xrightarrow{\phi} b) \mapsto \left(a \xrightarrow{\theta(a)} R(L(a)) \xrightarrow{R(\phi)} R(b)\right),$$

and

$$H_L^R(a, b) : \mathrm{Hom}_{\mathcal{C}}(a, R(b)) \to \mathrm{Hom}_{\mathcal{D}}(L(a), b),$$
$$(a \xrightarrow{\psi} R(b)) \mapsto \left(L(a) \xrightarrow{L(\psi)} L(R(b)) \xrightarrow{\eta(b)} b\right).$$

Exercise 3.6. For L, R, θ and η as above, prove that the conditions $(*_R)$ and $(*_L)$ hold if and only if, for every object a of $\mathcal{C}$ and for every object b of $\mathcal{D}$, $H_R^L(a, b)$ and $H_L^R(a, b)$ are inverse bijections.

Exercise 3.7. Prove that both $H_R^L(a, b)$ and $H_L^R(a, b)$ are binatural in a and b.

Exercise 3.8. For functors L and R, and for binatural inverse bijections $H_R^L(a, b)$ and $H_L^R(a, b)$ between the bifunctors

$$\mathrm{Hom}_{\mathcal{D}}(L(a), b), \mathrm{Hom}_{\mathcal{C}}(a, R(b)) : \mathcal{C} \times \mathcal{D} \to \mathbf{Sets},$$

prove that there exist unique θ and η extending L and R to an adjoint pair such that H_R^L and H_L^R agree with the binatural inverse bijections defined above.

Exercise 3.9. Let (L, R, θ, η) be an adjoint pair as above. Let a covariant functor

$$\widetilde{R} : \mathcal{D} \to \mathcal{C},$$

and natural transformations,

$$\widetilde{\theta} : \mathrm{Id}_{\mathcal{C}} \Rightarrow \widetilde{R} \circ L, \widetilde{\eta} : L \circ \widetilde{R} \Rightarrow \mathrm{Id}_{\mathcal{D}},$$

be natural transformations such that $(L, \widetilde{R}, \widetilde{\theta}, \widetilde{\eta})$ is also an adjoint pair. For every object b of $\mathcal{D}$, define $I(b)$ in $\text{Hom}_{\mathcal{D}}(R(b), \widetilde{R}(b))$ to be the image of Id_b under the composition,

$$\text{Hom}_{\mathcal{D}}(b, b) \xrightarrow{\text{Hom}_{\mathcal{D}}(\theta(b), b)} \text{Hom}_{\mathcal{D}}(L(R(b)), b) \xrightarrow{H_L^{\widetilde{R}}(R(b), b)} \text{Hom}_{\mathcal{D}}(R(b), \widetilde{R}(b)).$$

Similarly, define $J(b)$ in $\text{Hom}_{\mathcal{D}}(\widetilde{R}(b), R(b))$, to be the image of Id_b under the composition,

$$\text{Hom}_{\mathcal{D}}(b, b) \xrightarrow{\text{Hom}_{\mathcal{D}}(\widetilde{\theta}(b), b)} \text{Hom}_{\mathcal{D}}(L(\widetilde{R}(b)), b) \xrightarrow{H_L^{R}(\widetilde{R}(b), b)} \text{Hom}_{\mathcal{D}}(\widetilde{R}(b), R(b)).$$

Prove that I and J are the unique natural transformations of functors,

$$I : R \Rightarrow \widetilde{R}, \ \ J : \widetilde{R} \Rightarrow R,$$

such that $\widetilde{\theta}$ equals $(I \circ L) \circ \theta$, θ equals $(J \circ L) \circ \widetilde{\theta}$, $\widetilde{\eta}$ equals $\eta \circ (L \circ I)$, and η equals $\widetilde{\eta} \circ (L \circ J)$. Moreover, prove that I and J are inverse natural equivalences. In this sense, every extension of a functor L to an adjoint pair (L, R, θ, η) is unique up to unique natural isomorphisms (I, J). Formulate and prove the symmetric statement for all extensions of a functor R to an adjoint pair (L, R, θ, η).

Exercise 3.10. For every adjoint pair (L, R, θ, η), prove that also $(R^{\text{opp}}, L^{\text{opp}}, \eta^{\text{opp}}, \theta^{\text{opp}})$ is an adjoint pair.

Exercise 3.11. Formulate the corresponding notions of adjoint pairs when L and R are contravariant functors (just replace one of the categories by its opposite category).

Exercise 3.12. For every ordered triple of categories, $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ for all covariant functors,

$$L' : \mathcal{C} \to \mathcal{D}$$
$$R' : \mathcal{D} \to \mathcal{C},$$

for all natural transformations that form an adjoint pair,

$$\theta' : \text{Id}_{\mathcal{C}} \Rightarrow R'L',$$
$$\eta' : L'R' \Rightarrow \text{Id}_{\mathcal{D}},$$

for all covariant functors,

$$L'' : \mathcal{D} \to \mathcal{E},$$
$$R'' : \mathcal{E} \to \mathcal{D},$$

and for all natural transformations that form an adjoint pair,

$$\theta'' : \text{Id}_{\mathcal{D}} \Rightarrow R''L'',$$
$$\eta'' : L''R'' \Rightarrow \text{Id}_{\mathcal{E}},$$

define covariant functors

$$L : \mathcal{C} \to \mathcal{E}, \ \ R : \mathcal{E} \to \mathcal{C}$$

by $L = L'' \circ L'$, $R = R' \circ R''$, define the natural transformation,

$$\theta : \text{Id}_{\mathcal{C}} \Rightarrow R \circ L,$$

to be the composition of natural transformations,

$$\text{Id}_{\mathcal{C}} \overset{\theta'}{\Rightarrow} R' \circ L' \overset{R' \circ \theta'' \circ L'}{\Rightarrow} R' \circ R'' \circ L'' \circ L',$$

and define the natural transformation,

$$\eta : L \circ R \Rightarrow \mathrm{Id}_{\mathcal{E}},$$

to be the composition of natural transformations,

$$L'' \circ L' \circ R' \circ R'' \stackrel{L''\circ\eta'\circ R''}{\Rightarrow} L'' \circ R'' \stackrel{\eta''}{\Rightarrow} \mathrm{Id}_{\mathcal{E}}.$$

Prove that L, R, θ and η form an adjoint pair of functors. This is the **composition** of (L', R', θ', η') and $(L'', R'', \theta'', \eta'')$.

Exercise 3.13. If $\mathcal{C}$ equals $\mathcal{D}$, if L' and R' are the identity functors, and if θ' and η' are the identity natural transformations, prove that (L, R, θ, η) equals $(L'', R'', \theta'', \eta'')$. Similarly, if $\mathcal{D}$ equals $\mathcal{E}$, if L'' and R'' are the identity functors, and if θ'' and η'' are the identity natural transformations, prove that (L, R, θ, η) equals (L', R', θ', η'). Finally, prove that composition of three adjoint pairs is associative.

Definition 3.14. For every category $\mathcal{C}$, for every category $\mathcal{D}$, for every adjoint pair $(L : \mathcal{C} \to \mathcal{D}, R : \mathcal{D} \to \mathcal{C}, \theta : \mathrm{Id}_{\mathcal{C}} \Rightarrow R \circ L, \eta : L \circ R \Rightarrow \mathrm{Id}_{\mathcal{D}})$, the adjoint pair is a **strict equivalence** from $\mathcal{C}$ to $\mathcal{D}$ if (and only if) both θ is a natural equivalence and η is a natural equivalence.

Exercise 3.15. Prove that identity adjoint pairs are strict equivalences. Prove that the composition adjoint pair of strict equivalences is a strict equivalence. For every strict equivalence from $\mathcal{C}$ to $\mathcal{D}$ as above, prove that also $(R, L, \eta^{-1}, \theta^{-1})$ is a strict equivalence from $\mathcal{D}$ to $\mathcal{C}$ that is a left-right inverse of the original strict equivalence.

Definition 3.16. For every category $\mathcal{C}$, for every category $\mathcal{D}$, for every covariant functor F from $\mathcal{C}$ to $\mathcal{D}$, the covariant functor F is **full**, respectively **faithful**, **fully faithful**, if (and only if), for every ordered pair (a, b) of objects of $\mathcal{C}$, the set map $F^a_b : \mathcal{C}^a_b \to \mathcal{D}^{F(a)}_{F(b)}$ is surjective, respectively injective, bijective. The covariant functor F is **essentially surjective** if (and only if), every object of $\mathcal{D}$ is $\mathcal{D}$-isomorphic to the F-value of an object of $\mathcal{C}$. A covariant functor is a **weak equivalence** if (and only if) it is both fully faithful and essentially surjective.

Exercise 3.17. Prove that each of the functors in a strict equivalence is a weak equivalence. Prove that every composition of weak equivalences is a weak equivalence.

Exercise 3.18. Let $\mathcal{C}$ and $\mathcal{D}$ be strictly small categories. Prove that for every weak equivalence L from $\mathcal{C}$ to $\mathcal{D}$ there exists a strict equivalence (L, R, θ, η) from $\mathcal{C}$ to $\mathcal{D}$, and this strict equivalence is unique up to isomorphism (which is not necessarily unique). Thus, using Hilbert's formulation of the Axiom of Choice, using the Axiom of Choice in von Neumann – Bernays – Gödel theory, or using large cardinal axioms / Grothendieck universes, every weak equivalence should arise (non-uniquely) from a strict equivalence.

4. Categories Fibered in Groupoids

A groupoid is a category such that every morphism in the category is invertible. Loosely speaking, the groupoids are themselves "objects" of a 2-"category" whose

1-morphisms are covariant functors and whose 2-morphisms are natural equivalences. A lax functor to the 2-category of groupoids from a category with a specified Grothendieck topology is a stack if it satisfies the sheaf axioms for morphisms and for objects with respect to a specified Grothendieck topology.

Definition 4.1. For every category $\mathcal{C}$ that has all finite fiber products, a $\mathcal{C}$-**category** is a functor of categories, $p : \mathcal{X} \to \mathcal{C}$. A 1-**morphism** from a $\mathcal{C}$-category $p : \mathcal{X} \to \mathcal{C}$ to a $\mathcal{C}$-category $q : \mathcal{Y} \to \mathcal{C}$ is a functor $\zeta : \mathcal{X} \to \mathcal{Y}$ such that the composite functor $q \circ \zeta$ equals p.

For a pair of 1-morphisms of $\mathcal{C}$-categories, $\zeta : \mathcal{X} \to \mathcal{Y}$ and $\eta : \mathcal{X} \to \mathcal{Y}$, a 2-**morphism** from ζ to η is a natural transformation $\alpha : \zeta \Rightarrow \eta$ such that the pushforward natural transformation, $q_*\alpha : q \circ \zeta \Rightarrow q \circ \eta$, equals the identity natural transformation $\mathrm{Id}_p : p \Rightarrow p$, i.e., for every object x of $\mathcal{X}$, the morphism $q(\alpha_x)$ from $q(\zeta(x)) = p(x)$ to $q(\eta(x)) = p(x)$ equals $\mathrm{Id}_{p(x)}$. For a 1-morphism $\theta : \mathcal{X} \to \mathcal{Y}$ and a 2-morphism $\beta : \eta \Rightarrow \theta$, the **vertical composition** of α and β is the usual composite natural transformation $\beta \circ \alpha : \zeta \Rightarrow \theta$, i.e., for every object x of $\mathcal{X}$, the morphism $(\beta \circ \alpha)_x : \zeta(x) \to \theta(x)$ in $\mathcal{Y}$ equals the composition $\beta_x \circ \alpha_x$.

If there exists an inverse natural transformation, then the natural transformation is a **natural isomorphism**. A 2-morphism that is a natural isomorphism is a 2-**isomorphism**, and the inverse natural isomorphism is automatically a 2-morphism.

For a $\mathcal{C}$-category $r : \mathcal{Z} \to \mathcal{C}$, for 1-morphisms $\zeta' : \mathcal{Y} \to \mathcal{Z}$ and $\eta' : \mathcal{Y} \to \mathcal{Z}$, and for a 2-morphism $\alpha' : \zeta' \Rightarrow \eta'$, the **horizontal composition** or **Godement product** is the natural transformation $\alpha' * \alpha : \zeta' \circ \zeta \Rightarrow \eta' \circ \eta$ that associates to every object x of $\mathcal{X}$ the morphism in $\mathcal{Z}$,

$$\alpha'_{\eta(x)} \circ \zeta'_{\zeta(x),\eta(x)}(\alpha_x) = (\alpha' * \alpha)_x = \eta'_{\zeta(x),\eta(x)}(\alpha_x) \circ \alpha'_{\zeta(x)}.$$

In particular, the **pullback** $\zeta^*\alpha'$ of a natural transformation $\alpha' : \zeta' \Rightarrow \eta'$ with respect to a covariant functor ζ is defined to be $\alpha' * \mathrm{Id}_\zeta$. Similarly, the **pushforward** $\zeta'_*\alpha$ of a natural transformation $\alpha : \zeta \Rightarrow \eta$ with respect to a covariant functor ζ' is defined to be $\mathrm{Id}_{\zeta'} * \alpha$. Conversely, the general form of the Godement product equals the vertical composition of pushforwards and pullbacks,

$$\alpha' * \alpha = \eta^*\alpha' \circ \zeta'_*\alpha = \eta'_*\alpha \circ \zeta^*\alpha'.$$

Exercise 4.2. Check that the identity functor $\mathrm{Id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ is a $\mathcal{C}$-category that is final among all $\mathcal{C}$-categories.

Definition 4.3. A contravariant **lax functor** from $\mathcal{C}$ to the 2-category of categories is an ordered triple $(F_{\mathrm{Obj}}, F_{\mathrm{Hom}}, F_{\mathsf{comp}})$ whose first entry is an $\mathrm{Obj}_\mathcal{C}$-class $\mathrm{Obj}_\mathcal{X}$ together with both an $\mathrm{Obj}_\mathcal{X} \times_{\mathrm{Obj}_\mathcal{C}} \mathrm{Obj}_\mathcal{X}$-function assigning to every object T of $\mathcal{C}$ and to every ordered pair (a, b) of members of the T-fiber $\mathrm{Obj}_{\mathcal{X}_T}$ of a set $\mathrm{Hom}_{\mathcal{X}_T}(a, b)$ and a $\mathrm{Obj}_\mathcal{X} \times_{\mathrm{Obj}_\mathcal{C}} \mathrm{Obj}_\mathcal{X} \times_{\mathrm{Obj}_\mathcal{C}} \mathrm{Obj}_\mathcal{X}$-function associating to every object T of $\mathcal{C}$ and to every ordered triple (a, b, c) of members of the T-fiber $\mathrm{Obj}_{\mathcal{X}_T}$ of a set function

$$\mathrm{Hom}_{\mathcal{X}_T}(b, c) \times \mathrm{Hom}_{\mathcal{X}_T}(a, b) \to \mathrm{Hom}_{\mathcal{X}_T}(a, c),$$

that form a category $\mathcal{X}_T$ for every object T of $\mathcal{C}$, whose second entry is a $\mathrm{Hom}_\mathcal{C}$-class whose fiber over every $\mathcal{C}$-morphism, $S \xrightarrow{f} T$, is a class that is a covariant functor $\mathcal{X}_f : \mathcal{X}_T \to \mathcal{X}_S$ (required to be the identity functor for each identity morphism), and whose third entry is a $\mathrm{Hom}_\mathcal{C} \times_{s_\mathcal{C},\mathrm{Obj}_\mathcal{C},t_\mathcal{C}} \mathrm{Hom}_\mathcal{C}$-class whose fiber over every pair of composable $\mathcal{C}$-morphisms, $(R \xrightarrow{g} S, S \xrightarrow{f} T)$, is a natural equivalent $\theta_{f,g} : g^* \circ f^* \Rightarrow (f \circ g)^*$ (required to be the identity natural equivalence when at least one of f or g is an identity morphism) such that for every triple of composable $\mathcal{C}$-morphisms, $(Q \xrightarrow{h} R, R \xrightarrow{g} S, S \xrightarrow{f} T)$, the following compositions of natural equivalences from $h^* \circ g^* \circ f^*$ to $(f \circ g \circ h)^*$ are equal,

$$\theta_{f,g\circ h} \circ f_* \theta_{g,h} = \theta_{h,f\circ g} \circ h^* \theta_{f,g}.$$

A 1-**morphism** from a lax functor $T \mapsto \mathcal{X}_T$ to a lax functor $T \mapsto \mathcal{Y}_T$ is an ordered pair whose first entry is a $\mathrm{Obj}_{\mathcal{C}}$-class whose fiber over every object T of $\mathcal{C}$ is a covariant functor, $\phi_T : \mathcal{X}_T \to \mathcal{Y}_T$ and whose second entry is a $\mathrm{Hom}_{\mathcal{C}}$-class whose fiber over every $\mathcal{C}$-morphism, $S \xrightarrow{f} T$, is a natural equivalence $\alpha_f : \phi_S \circ f^*_{\mathcal{X}} \Rightarrow f^*_{\mathcal{Y}} \circ \phi_T$, such that for every pair of composable $\mathcal{C}$-morphisms, $(R \xrightarrow{g} S, S \xrightarrow{f} T)$, the following compositions of natural isomorphism from $\phi_R \circ g^*_{\mathcal{X}} \circ f^*_{\mathcal{X}}$ to $(f \circ g)^*_{\mathcal{Y}} \circ \phi_T$ are equal,

$$\theta^{\mathcal{Y}}_{f,g} \circ (\mathrm{Id}_{g^*_{\mathcal{Y}}} * \alpha_f) \circ (\alpha_g * \mathrm{Id}_{f^*_{\mathcal{X}}}) = \alpha_{f\circ g} \circ (\mathrm{Id}_{\phi_R} * \theta^{\mathcal{X}}_{f,g}).$$

Definition 4.4. For every lax functor $T \mapsto \mathcal{X}_T$, the **associated $\mathcal{C}$-category** $\mathcal{X}$ has objects that are pairs (T, x) of a $\mathcal{C}$-object T and an $\mathcal{X}_T$-object x, it has morphisms from an object (S, y) to an object (T, x) that are pairs (f, ι) of a $\mathcal{C}$-morphism $S \xrightarrow{f} T$ and an $\mathcal{X}_S$-morphism $y \xrightarrow{\iota} f^*x$, and the functor $p : \mathcal{X} \to \mathcal{C}$ sends each object (T, x) to T and each morphism (f, ϕ) to f. Composition in the category uses the natural isomorphisms θ.

For every 1-morphism of lax functors as above, define $\phi : \mathcal{X} \to \mathcal{Y}$ to be the 1-morphism of $\mathcal{C}$-categories sending each $\mathcal{X}$-object (T, x) to the $\mathcal{Y}$-object $(T, \phi_T(x))$ and sending each $\mathcal{X}$-morphism (f, ι) to the $\mathcal{Y}$-morphism $(f, \phi_S(\iota))$.

Exercise 4.5. Check that these definitions are well-defined. Formulate the notion of 2-morphisms for lax functors, and define the 2-morphisms between the associated $\mathcal{C}$-categories. Check that the final $\mathcal{C}$-category, $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, is equivalent to the $\mathcal{C}$-category associated to the lax functor sending every object T of $\mathcal{C}$ to a final category, i.e., a category with a unique object where the only morphism is the identity morphism.

Definition 4.6. For a pair of 1-morphisms of $\mathcal{C}$-categories, $\zeta : \mathcal{X} \to \mathcal{Z}$ and $\eta : \mathcal{Y} \to \mathcal{Z}$, a 2-**commutative diagram** from ζ to η is a $\mathcal{C}$-category, $m : \mathcal{W} \to \mathcal{C}$, a pair of 1-morphisms of $\mathcal{C}$-categories, $\pi : \mathcal{W} \to \mathcal{X}$ and $\rho : \mathcal{W} \to \mathcal{Y}$, and a 2-isomorphism of $\mathcal{C}$-categories, $\alpha : \zeta \circ \pi \Rightarrow \eta \circ \rho$. A 2-commutative diagram as above is a 2-**fibered product** if for every 2-commutative diagram from ζ to η, $(\pi' : \mathcal{W}' \to \mathcal{X}, \rho' : \mathcal{W}' \to \mathcal{Y}, \alpha' : \zeta \circ \pi' \Rightarrow \eta \circ \rho')$, there exists a triple, $(\xi : \mathcal{W}' \to \mathcal{W}, \beta : \pi \circ \xi \Rightarrow \pi', \gamma : \rho' \Rightarrow \rho \circ \xi)$, unique up to unique isomorphisms, of a 1-morphism ξ of $\mathcal{C}$-categories, a 2-isomorphism β, and a 2-isomorphism γ such that the vertical composition $\eta * \gamma \circ \alpha' \circ \zeta_* \beta$ equals the pullback $\xi^* \alpha$.

Exercise 4.7. For every 2-fibered product as in the definition, show that the pair of 1-morphisms, $(\zeta \circ \mathrm{pr}_1, \eta \circ \mathrm{pr}_2) : \mathcal{X} \times_{\mathcal{C}} \mathcal{Y} \to \mathcal{Z} \times_{\mathcal{C}} \mathcal{Z}$ and $\Delta_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z} \times_{\mathcal{C}} \mathcal{Z}$, extends to a 2-fibered product $(\pi, \rho) : \mathcal{W} \to \mathcal{X} \times_{\mathcal{C}} \mathcal{Y}$ and $\zeta \circ \pi : \mathcal{W} \to \mathcal{Z}$ and an appropriate choice of 2-morphism defined using α. Thus, every 2-fibered product is equivalent to a 2-fibered product where part of the pair 1-morphisms is a diagonal 1-morphism $\Delta_{\mathcal{Z}}$.

Notation 4.8. For every pair of 1-morphisms of $\mathcal{C}$-categories, $\zeta : \mathcal{X} \to \mathcal{Z}$ and $\eta : \mathcal{Y} \to \mathcal{Z}$, denote by $\mathcal{X} \times_{\zeta,\mathcal{Z},\eta} \mathcal{Y}$ the category whose objects are triples (x, y, ϕ) of an $\mathcal{X}$-object x, a $\mathcal{Y}$-object y such that $p(x)$ equals $q(y)$, and a $\mathcal{Z}$-isomorphism $\phi : \zeta(x) \xrightarrow{\cong} \eta(y)$ whose image in $\mathcal{C}$ is the identity morphism from $p(x)$ to $q(y)$. A morphism in $\mathcal{X} \times_{\zeta,\mathcal{Z},\eta} \mathcal{Y}$ from (x, y, ϕ) to (x', y', ϕ') is a pair (χ, υ) of an $\mathcal{X}$-morphism $\chi : x \to x'$ and a $\mathcal{Y}$-morphism $\upsilon : y \to y'$ whose images in $\mathcal{C}$ are equal and such that the composite $\mathcal{Z}$-morphism $\phi' \circ \zeta_{x,x'}(\chi)$ equals $\eta_{y,y'}(\upsilon) \circ \phi$. The first projection,

$$\mathrm{pr}_1 : \mathcal{X} \times_{\zeta,\mathcal{Z},\eta} \mathcal{Y} \to \mathcal{X},$$

sends each object (x, y, ϕ) to x and sends each morphism (χ, υ) to χ. The second projection,

$$\mathrm{pr}_2 : \mathcal{X} \times_{\zeta,\mathcal{Z},\eta} \mathcal{Y} \to \mathcal{Y},$$

sends each object (x, y, ϕ) to y and sends each morphism (χ, υ) to υ. The natural transformation from $\zeta \circ \mathrm{pr}_1$ to $\eta \circ \mathrm{pr}_2$ associates to each object (x, y, ϕ) the $\mathcal{Z}$-isomorphism $\phi : \zeta(x) \to \eta(y)$.

Exercise 4.9. Check that the diagram defined above is a 2-commutative diagram, and even a 2-fiber product. Thus the 2-category of $\mathcal{C}$-categories has 2-fiber products.

Exercise 4.10. Check that for 1-morphisms of lax functors from $\mathcal{C}$ to the 2-category of categories, the 2-fiber product of the corresponding 1-morphisms of $\mathcal{C}$-categories equals the $\mathcal{C}$-category associated to the lax functor sending every object T of $\mathcal{C}$ to the 2-fiber product category $\mathcal{X}_T \times_{\mathcal{Z}_t} \mathcal{Y}_T$.

Definition 4.11. A $\mathcal{C}$-category, $p : \mathcal{X} \to \mathcal{C}$, is **fibered in groupoids** or a $\mathcal{C}$-**groupoid** if it satisfies the following two axioms. First, for every triple of $\mathcal{X}$-objects, (x, y, z), for every pair of $\mathcal{X}$-morphisms, $(\alpha : x \to z, \beta : y \to z)$, and for every $\mathcal{C}$-morphism $g : p(x) \to p(y)$ such that $p(\alpha)$ equals $p(\beta) \circ g$, there exists a unique $\mathcal{X}$-morphism $\gamma : x \to y$ such that $p(\gamma)$ equals g and such that α equals $\beta \circ \gamma$. The morphism γ is the **lift** of g relative to (α, β). Second, for every pair (y, f) of an $\mathcal{X}$-object y and a $\mathcal{C}$-morphism $f : T \to p(y)$, there exists a $\mathcal{X}$-morphism $\phi : x \to y$ such that the $\mathcal{C}$-object $p(x)$ equals T, and such that the $\mathcal{C}$-morphism $p(\phi)$ equals f. The pair (x, ϕ) is an f-**pullback** of y.

A **clivage normalisé** is an assignment to every pair (y, f) of an f-pullback, denoted $f^*y = (\phi, i)$, such that for every pair $(y, \mathrm{Id}_{p(y)})$ the assignment is $\mathrm{Id}^*_{p(y)}y = (\mathrm{Id}_y, \mathrm{Id}_{p(y)})$.

A 1-**morphism** between $\mathcal{C}$-groupoids is a 1-morphisms between $\mathcal{C}$-categories, and a 2-**morphism** between 1-morphisms of $\mathcal{C}$-groupoids is a 2-morphism between the 1-morphisms of $\mathcal{C}$-categories, i.e., the 2-category of $\mathcal{C}$-groupoids is full inside the 2-category of $\mathcal{C}$-categories.

Definition 4.12. For every $\mathcal{C}$-category $p : \mathcal{X} \to \mathcal{C}$ and for every $\mathcal{C}$-object T, the **fiber** $\mathcal{X}_T$ of p over T is the subcategory of $\mathcal{X}$ of those objects x such that the $\mathcal{C}$-object $p(x)$ equals T, and whose morphisms from each $\mathcal{X}_T$-object x to a $\mathcal{X}_T$-object y are those $\mathcal{X}$-morphisms from x to y whose image under p equals Id_T.

Exercise 4.13. For every $\mathcal{C}$-groupoid $p : \mathcal{X} \to \mathcal{C}$ and for every $\mathcal{C}$-object T of $\mathcal{C}$, prove that the fiber over T is a groupoid, i.e., every morphism is an isomorphism. For every clivage normalisé of p, for every $\mathcal{C}$-morphism $S \xrightarrow{f} T$, prove that the clivage extends to a **pullback functor** from the fiber over T to the fiber over S. This need not be *strictly* contravariant, i.e., for a morphism $g : R \to S$ in $\mathcal{C}$, the pullback functor along $f \circ g$ need not equal the composition of the pullback functors along f and along g. However, prove that there is a unique natural equivalence between these functors, and prove that the natural equivalences are strictly compatible with composition of three morphisms of $\mathcal{C}$.

Exercise 4.14. Check that for 1-morphisms of $\mathcal{C}$-groupoids, $\zeta : \mathcal{X} \to \mathcal{Z}$ and $\eta : \mathcal{Y} \to \mathcal{Z}$, the 2-fiber product $\mathcal{X} \times_{\zeta,\mathcal{Z},\eta} \mathcal{Y}$ is also a $\mathcal{C}$-groupoid.

Exercise 4.15. For a lax functor from $\mathcal{C}$ to the 2-category of categories, prove that the associated $\mathcal{C}$-category is a $\mathcal{C}$-groupoid if and only if, the category $\mathcal{X}_T$ of each $\mathcal{C}$-object T is a groupoid. In this case, prove that the functors f^* also define a clivage normalisé. Prove that for every 1-morphism between lax functors from $\mathcal{C}$ to the 2-category of groupoids, the associated 1-morphism $\mathcal{C}$-groupoids respects the specified clivages. Using the fibers defined above, this gives an equivalence between lax functors from $\mathcal{C}$ to the 2-category of groupoids and $\mathcal{C}$-categories with specified clivages normalisé.

Exercise 4.16. As a special case of the above, for every contravariant functor $F : \mathcal{C} \to \mathbf{Sets}$, for every object T of $\mathcal{C}$, define $\widetilde{F}_T$ to be the groupoid whose objects are elements of the set $F(T)$ and where the only morphisms are identity morphisms (a groupoid where every Hom set is either empty or a singleton is called a **setoid**). For every morphism of $\mathcal{C}$, $S \xrightarrow{f} T$, define $f^*_{\widetilde{F}} : \widetilde{F}_T \to \widetilde{F}_S$ to be the unique functor that equals $F(f)$ on objects. Check that this extends uniquely to a lax functor from $\mathcal{C}$ to the 2-category of groupoids, and hence there is an associated category $\widetilde{F}$ fibered in groupoids over $\mathcal{C}$ with a specified clivage normalisé. Check that this defines an equivalence between contravariant set-valued functors on $\mathcal{C}$, with natural transformations as morphisms, and $\mathcal{C}$-setoids with specified clivage normalisé. In particular, for every object X of $\mathcal{C}$, associated to the contravariant Yoneda functor $h_X : \mathcal{C}^{\text{opp}} \to \mathbf{Sets}$, there is a $\mathcal{C}$-setoid $\widetilde{h}_X$. Show that this gives a *faithful* rule associating $\widetilde{h}_X$ to each object of $\mathcal{C}$ and associating to every $\mathcal{C}$-morphism $Y \xrightarrow{q} X$ the morphism of $\mathcal{C}$-setoids associated to the natural transformation $h_q : h_Y \Rightarrow h_X$. Show that this rule is also *essentially full*, i.e., every 1-morphism from $\widetilde{h}_Y$ to $\widetilde{h}_X$ is 2-equivalent to the 1-morphism coming from a unique morphism of $\mathcal{C}$, $Y \xrightarrow{q} X$. This is one incarnation of Yoneda's lemma. Also check that this rule sends fiber products of morphisms in $\mathcal{C}$ to 2-fiber products of the associated 1-morphisms of $\mathcal{C}$-setoids.

Definition 4.17. For a $\mathcal{C}$-groupoid $\mathcal{Y}$, a **$\mathcal{C}$-object over $\mathcal{Y}$** is a pair (Y, ζ) of a $\mathcal{C}$-object Y and a 1-morphism of $\mathcal{C}$-groupoids, $\eta : \widetilde{h}_Y \to \mathcal{Y}$. A 1-morphism between $\mathcal{C}$-groupoids, $\zeta : \mathcal{X} \to \mathcal{Y}$, is **representable** if for every $\mathcal{C}$-object over $\mathcal{Y}$, (Y, η), there exists a $\mathcal{C}$-morphism, $f : X \to Y$, a 1-morphism of $\mathcal{C}$-groupoids, $\theta : \widetilde{h}_X \to \mathcal{X}$, and a 2-morphism of $\mathcal{C}$-groupoids, $\alpha : \zeta \circ \theta \Rightarrow \eta \circ \widetilde{h}_f$, forming a 2-fibered product. For a property P of $\mathcal{C}$-morphisms that is preserved by base change, **property P holds** for a representable 1-morphism of $\mathcal{C}$-groupoids, $\zeta : \mathcal{X} \to \mathcal{Y}$, if for every $\mathcal{C}$-object over $\mathcal{Y}$, (Y, η), the morphism $f : X \to Y$ has property P.

If the diagonal 1-morphism $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_{\mathcal{C}} \mathcal{X}$ is representable, then $\mathcal{X}$ is **diagonal representable**.

Exercise 4.18. Use Exercise 4.7 to prove that $\mathcal{X}$ is diagonal representable if and only if every 1-morphism $\theta : \widetilde{h}_X \to \mathcal{X}$ is representable. In particular, conclude that for every $\mathcal{C}$-object Y, the Yoneda $\mathcal{C}$-groupoid $\widetilde{h}_Y$ is diagonal representable.

The remaining axioms for a stack are sheaf axioms with respect to a Grothendieck pretopology.

Definition 4.19. For a category $\mathcal{C}$ with finite fiber products, a **Grothendieck pretopology** on $\mathcal{C}$ is a specification for every object T of $\mathcal{C}$ of which sets of morphisms in $\mathcal{C}$ to T, $\mathfrak{U} = (f_\lambda : U_\lambda \to T)_{\lambda\in\Lambda}$, are *covering families* satisfying the following axioms.

(i) Every family consisting of a single isomorphism is a covering family.

(ii) For every covering family $\mathfrak{U}$ of an object T and for every morphism $h : S \to T$, every pullback family $h^*\mathfrak{U} = (S \times_T U_\lambda \xrightarrow{\mathrm{pr}_1} S)_{\lambda\in\Lambda}$ is a covering family of S.

(iii) For every covering family $(f_\lambda : U_\lambda \to T)_{\lambda\in\Lambda}$ of an object T, for every covering family $(f_{\lambda,\mu} : U_{\lambda,\mu} \to U_\lambda)_{\mu\in M_\lambda}$ of the object U_λ for each $\lambda \in \Lambda$, the family $(f_\lambda \circ f_{\lambda,\mu} : U_{\lambda,\mu} \to T)_{\lambda\in\Lambda,\mu\in M_\lambda}$ is a covering family of T.

For covering families $\mathfrak{U} = (f_\lambda : U_\lambda \to T)_{\lambda\in\Lambda}$ and $\mathfrak{V} = (g_\mu : V_\mu \to T)_{\mu\in M}$ of T, a **refinement** from $\mathfrak{V}$ to $\mathfrak{U}$ is a pair $e_\bullet := (r, (e_\mu)_{\mu\in M})$ of a set function $r : M \to \Lambda$ and a collection $(e_\mu : V_\mu \to U_{r(\mu)})_{\mu\in M}$ of morphisms of $\mathcal{C}$ such that $f_{r(\mu)} \circ e_\mu$ equals g_μ for every $\mu \in M$. Composition of covering families are defined in the usual way.

Exercise 4.20. Check that the covering family $(\mathrm{Id}_T : T \to T)_{*\in\{*\}}$ of T is a final object in the category of covering families of T with refinements as morphisms. More precisely, for every covering $\mathfrak{U} = (f_\lambda : U_\lambda \to T)_{\lambda\in\Lambda}$, check that that is a unique refinement from $\mathfrak{U}$ to the final object,

$$f_\bullet := (\mathrm{const} : \Lambda \to \{*\}, (f_\lambda : U_\lambda \to T)_{\lambda\in\Lambda}).$$

Also, for every pair of covering families of T, $\mathfrak{U} = (f_\lambda : U_\lambda \to T)_{\lambda\in\Lambda}$ and $\mathfrak{V} = (g_\mu : V_\mu \to T)_{\mu\in M}$, check that family of fiber products $U_\lambda \times_T V_\mu$ with morphisms

$$e_{\lambda,\mu} = f_\lambda \circ \mathrm{pr}_1 = g_\mu \circ \mathrm{pr}_2$$

gives a covering family of T, $(e_{\lambda,\mu} : U_\lambda \times_T V_\mu \to T)_{(\lambda,\mu)\in\Lambda\times M}$, together with refinements,

$$\mathrm{pr}_1 : \mathfrak{U} \times_T \mathfrak{V} \to \mathfrak{U}, (\mathrm{pr}_1 : \Lambda \times M \to \Lambda, (\mathrm{pr}_1 : U_\lambda \times_T V_\mu \to U_\lambda)_{(\lambda,\mu)\in\Lambda\times M}),$$

$$\mathrm{pr}_2 : \mathfrak{U} \times_T \mathfrak{V} \to \mathfrak{V}, (\mathrm{pr}_2 : \Lambda \times M \to M, (\mathrm{pr}_2 : U_\lambda \times_T V_\mu \to V_\mu)_{(\lambda,\mu)\in\Lambda\times M}),$$

that altogether give a product of the coverings of $\mathfrak{U}$ and $\mathfrak{V}$ in the category of coverings of T. So the category of covering families of T with refinements as morphisms has finite products.

Since the category of covering families of T has finite products, there are coskeleton functors. In particular, associated to every covering family $\mathfrak{U}$ of T there is a simplicial object $\mathfrak{U}_\bullet$ in the category of covering families of T. With the standard conventions, $\mathfrak{U}_\bullet$ has $\mathfrak{U}_0 := \mathfrak{U}$, has $\mathfrak{U}_1 = \mathfrak{U}\times\mathfrak{U} = \mathfrak{U}^2$, and has $\mathfrak{U}_n := \mathfrak{U}\times\mathfrak{U}_{n-1} = \mathfrak{U}^{n+1}$ for every integer $n \geq 1$. Each face map $d_{n,i} : \mathfrak{U}_n \to \mathfrak{U}_{n-1}$ for $0 \leq i \leq n$ is the corresponding projection,

$$\mathrm{pr}_{1,\dots,i,i+2,\dots,n+1} = (\mathrm{pr}_1, \dots, \mathrm{pr}_i, \mathrm{pr}_{i+2}, \dots, \mathrm{pr}_{n+1}) : \mathfrak{U}^{n+1} \to \mathfrak{U}^n.$$

Each degeneracy map $s_{n,i} : \mathfrak{U}_n \to \mathfrak{U}_{n+1}$ for $0 \le i \le n$ is the corresponding diagonal morphism,

$$\Delta_{1,\dots,i+1,i+1,\dots,n+1} = (\mathrm{pr}_1, \dots, \mathrm{pr}_{i+1}, \mathrm{pr}_{i+1}, \dots, \mathrm{pr}_{n+1}) : \mathfrak{U}^{n+1} \to \mathfrak{U}^{n+2}.$$

Definition 4.21. For every $\mathcal{C}$-groupoid $p : \mathcal{X} \to \mathcal{C}$, for every $\mathcal{C}$-object T, for every covering family $\mathfrak{U} = (f_\lambda : U_\lambda \to T)_{\lambda \in \Lambda}$ of T, a **$\mathfrak{U}$-object** of $\mathcal{X}$ is a collection $x_{\mathfrak{U}} = (x_\lambda)_{\lambda\in\Lambda}$ of an object x_λ of the fiber of p over U_λ for every λ, with morphisms defined in the obvious way, i.e., the category of $\mathfrak{U}$-objects is the product over all $\lambda \in \Lambda$ of the fiber of p over U_λ. For every covering $\mathfrak{V} = (g_\mu : U_\mu \to T)_{\mu \in M}$ of T, and for every refinement from $\mathfrak{V}$ to $\mathfrak{U}$, $e_\bullet = (r, (e_\mu)_{\mu\in M})$, an **$e_\bullet$-morphism** to the $\mathfrak{U}$-object $x_{\mathfrak{U}}$ from a $\mathfrak{V}$-object $y_{\mathfrak{V}} = (y_\mu)_{\mu\in M}$ is a family of morphisms $\delta_\bullet = (\delta_\mu : y_\mu \to x_{r(\mu)})_{\mu\in M}$ in $\mathcal{X}$ that are e_μ-pullbacks of $x_{r(\mu)}$. In particular, for the identity refinement of $\mathfrak{U}$, this defines the notion of a morphism of $\mathfrak{U}$-objects. Notice, for every $\mathfrak{U}$-object $x_{\mathfrak{U}}$, for every refinement $e_\bullet$, existence of pullbacks for objects in a category fibered in groupoids guarantees the existence of a $\mathfrak{V}$-object $y_{\mathfrak{V}}$ and an $e_\bullet$-morphism from $y_{\mathfrak{V}}$ to $x_{\mathfrak{U}}$; this is called a **$e_\bullet$-pullback** of $x_{\mathfrak{U}}$.

A **$\mathfrak{U}$-descent datum** of objects of $\mathfrak{C}$ is a pair $(x_{\mathfrak{U}}, \phi_\bullet)$ of a $\mathfrak{U}$-object $x_{\mathfrak{U}}$, and for one (hence every) pair of pullbacks of $x_{\mathfrak{U}}$,

$$(\epsilon_{1,\bullet} : \mathrm{pr}_1^* x_{\mathfrak{U}} \to x_{\mathfrak{U}}, \epsilon_{2,\bullet} : \mathrm{pr}_2^* x_{\mathfrak{U}} \to x_{\mathfrak{U}}),$$

relative to the projections, $\mathrm{pr}_i : \mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$, $i = 1, 2$, a specified isomorphism of $\mathfrak{U} \times \mathfrak{U}$-objects,

$$\phi_\bullet : \mathrm{pr}_1^* x_{\mathfrak{U}} \xrightarrow{\cong} \mathrm{pr}_2^* x_{\mathfrak{U}},$$

that satisfies the following *cocycle condition.* For one (hence every) choice of pullbacks to $\mathfrak{U} \times \mathfrak{U} \times \mathfrak{U}$-objects,

$$(\epsilon_{i,j;k,\bullet} : \mathrm{pr}_{i,j}^* \mathrm{pr}_k^* x_{\mathfrak{U}} \to \mathrm{pr}_k^* x_{\mathfrak{U}})_{1\le i<j\le 3, 1\le k\le 2},$$

with the unique lifts of the identity map of $\mathfrak{U}\times\mathfrak{U}\times\mathfrak{U}$ to isomorphisms of $\mathfrak{U}\times\mathfrak{U}\times\mathfrak{U}$-objects guaranteed by the axioms of a category fibered in groupoids,

$$\mathrm{nat}_{1,2;1,\bullet} : \mathrm{pr}_{1,2}^* \mathrm{pr}_1^* x_{\mathfrak{U}} \xrightarrow{\cong} \mathrm{pr}_{1,3}^* \mathrm{pr}_1^* x_{\mathfrak{U}},$$

$$\mathrm{nat}_{1,2;2,\bullet} : \mathrm{pr}_{1,2}^* \mathrm{pr}_2^* x_{\mathfrak{U}} \xrightarrow{\cong} \mathrm{pr}_{2,3}^* \mathrm{pr}_1^* x_{\mathfrak{U}},$$

$$\mathrm{nat}_{1,3;2,\bullet} : \mathrm{pr}_{1,3}^* \mathrm{pr}_2^* x_{\mathfrak{U}} \xrightarrow{\cong} \mathrm{pr}_{2,3}^* \mathrm{pr}_2^* x_{\mathfrak{U}},$$

the following two composite isomorphisms of $\mathfrak{U} \times \mathfrak{U} \times \mathfrak{U}$-objects are required to be equal,

$$\mathrm{nat}_{1,3;2,\bullet} \circ \mathrm{pr}_{1,3}^* \phi_\bullet \circ \mathrm{nat}_{1,2;1,\bullet} = \mathrm{pr}_{2,3}^* \phi_\bullet \circ \mathrm{nat}_{1,2;2,\bullet} \circ \mathrm{pr}_{1,2}^* \phi_\bullet.$$

For every refinement of covers of T, $e_\bullet : \mathfrak{V} \to \mathfrak{U}$ with the corresponding refinement $e_\bullet \times e_\bullet : \mathfrak{V}\times\mathfrak{V} \to \mathfrak{U}\times\mathfrak{U}$, for every $\mathfrak{U}$-descent datum $(x_{\mathfrak{U}}, \phi^x)$ and for every $\mathfrak{V}$-descent datum $(y_{\mathfrak{V}}, \phi^y)$, an **$e_\bullet$-morphism** of descent data from $(y_{\mathfrak{V}}, \phi^y)$ to $(x_{\mathfrak{U}}, \phi^x)$ is an $e_\bullet$-morphism $\delta_\bullet : y_{\mathfrak{V}} \to x_{\mathfrak{U}}$ such that for one (hence every) choice of pullbacks,

$$(\epsilon_{1,\bullet}^X : \mathrm{pr}_1^* x_{\mathfrak{U}} \to x_{\mathfrak{U}}, \epsilon_{2,\bullet}^x : \mathrm{pr}_2^* x_{\mathfrak{U}} \to x_{\mathfrak{U}}),$$

$$(\epsilon_{1,\bullet}^y : \mathrm{pr}_1^* y_{\mathfrak{V}} \to y_{\mathfrak{V}}, \epsilon_{2,\bullet}^y : \mathrm{pr}_2^* y_{\mathfrak{V}} \to y_{\mathfrak{V}}),$$

the following $e_\bullet \times e_\bullet$-morphisms from $\mathrm{pr}_1^* y_{\mathfrak{V}}$ to $\mathrm{pr}_2^* x_{\mathfrak{U}}$ are equal,

$$\phi_\bullet \circ \mathrm{pr}_1^* \delta_\bullet = \mathrm{pr}_s^* \delta_\bullet \circ \phi.$$

By the axioms of a category fibered in groupoids, for every $\mathfrak{U}$-descent datum $(x_{\mathfrak{U}}, \phi^x_\bullet)$, for every refinement of covers of T, $e_\bullet : \mathfrak{V} \to \mathfrak{U}$, there exists a $\mathfrak{V}$-descent datum $(y_{\mathfrak{V}}, \phi^y_\bullet)$ and a $e_\bullet$-morphism of descent datum from $(y_{\mathfrak{V}}, \phi^y_\bullet)$ to $(x_{\mathfrak{U}}, \phi^x_\bullet)$; this is called a **$e_\bullet$-pullback** of $(x_{\mathfrak{U}}, \phi_\bullet)$.

For a cover of T, $\mathfrak{U} = (f_\lambda : U_\lambda \to T)_{\lambda\in\Lambda}$, with its unique refinement $f_\bullet$ to the final cover of T, $(T \xrightarrow{\mathrm{Id}_T} T)$, an **effectivization** of a $\mathfrak{U}$-descent datum is a descent datum for the final cover of T and a $f_\bullet$-morphism between the descent data. Since the descent datum over the final object is equivalent to an object of the fiber of p over T, such an object is often called an **effectivization**. If there exists an effectivization, then the descent datum is **effective**.

Exercise 4.22. Up to working with normalized objects instead of unnormalized objects (and invoking the Axiom of Choice), check that the category of descent data relative to a covering family $\mathfrak{U}$ of T is weakly equivalent to the "fiber" of p over the simplicial object $\mathfrak{U}_\bullet$ of the category of covers of T. The formulation above requires less notation.

Definition 4.23. For a category $\mathcal{C}$ with finite fiber products and a specified Grothendieck pretopology, a $\mathcal{C}$-category fibered in groupoids is a **stack** if every descent datum of objects in $\mathcal{C}$ for every cover is effective and every morphism of descent data relative to a fixed cover is induced by a morphism of the effectivizations. More succinctly, the fiber of p over each object T of $\mathcal{C}$ is (weakly) equivalent to the fiber of p over the simplicial object $\mathfrak{U}_\bullet$ for every covering family $\mathfrak{U}$ of each object T of $\mathcal{C}$. A **1-morphism** of stacks is a 1-morphism of $\mathcal{C}$-categories. A **2-morphism** between 1-morphisms of stacks is a 2-morphism between the 1-morphisms of $\mathcal{C}$-categories, i.e., the 2-category of stacks is a full subcategory of the 2-category of $\mathcal{C}$-categories.

Exercise 4.24. Check that for 1-morphisms of $\mathcal{C}$-stacks, $\zeta : \mathcal{X} \to \mathcal{Z}$ and $\eta : \mathcal{Y} \to \mathcal{Z}$, the 2-fiber product $\mathcal{X} \times_{\zeta,\mathcal{Z},\eta} \mathcal{Y}$ is also a $\mathcal{C}$-stack.

Exercise 4.25. Check that for a set-valued, contravariant functor $F : \mathcal{C} \to \mathbf{Sets}$, the category fibered in groupoids $\widetilde{F}$ is a stack if and only if F is a sheaf for the Grothendieck topology in the usual sense. Again, one incarnation of the Yoneda embedding is an equivalence of sheaves with stacks that are fibered in setoids.

Definition 4.26. A category $\mathcal{C}$ that has finite fiber products together with a Grothendieck pretopology satisfies the **Yoneda sheaf condition** if the Yoneda contravariant functor of each object X of $\mathcal{C}$ is a sheaf for the Grothendieck topology, i.e., if the category $\widetilde{h}_X$ fibered in groupoids over $\mathcal{C}$ is a stack over $\mathcal{C}$ with its specified Grothendieck pretopology.

Exercise 4.27. For every scheme S, check the Yoneda sheaf condition for the category $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ of S-schemes with the fppf Grothendieck pretopology (or even the fpqc Grothendieck topology).

Definition 4.28. Let S be a scheme. Let $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ denote the category of S-schemes with the fppf Grothendieck pretopology. A stack $\mathcal{X}$ over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ is an **S-algebraic space** if every fiber groupoid is a setoid, if the diagonal 1-morphism is representable and if there exists an S-scheme X and a 1-morphism $\eta : \widetilde{h}_X \to \mathcal{X}$ (necessarily representable since the diagonal is representable) that is faithfully flat, and étale.

Remark 4.29. Please note that the diagonal morphism is not assumed to be a closed immersion, nor a locally closed immersion, nor even quasi-compact. So algebraic spaces as above need not be separated, locally separated, or quasi-separated. The original references by Mike Artin and Donald Knutson did assume separatedness hypotheses.

Definition 4.30. A 1-morphism between categories fibered in groupoids over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$, $\zeta : \mathcal{X} \to \mathcal{Y}$, is **representable by algebraic spaces** if for every S-scheme Y and for every 1-morphism between categories fibered in groupoids over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$, $\eta_Y : \widetilde{h}_Y \to \mathcal{Y}$, there exists an S-algebraic space X, a 1-morphism $f : X \to \widetilde{h}_Y$, a 1-morphism $\eta_X : X \to \mathcal{X}$, and a 2-morphism $\alpha : \zeta \circ \eta_X \Rightarrow \eta_Y \circ f$, forming a 2-fibered product diagram. For a property P of morphisms of S-algebraic spaces that is preserved by base change, **property** P **holds** for a representable 1-morphism $\zeta : \mathcal{X} \to \mathcal{Y}$ between categories fibered in groupoids if for every 1-morphism $\eta_Y : \widetilde{h}_Y \to \mathcal{Y}$ as above, the morphism $f : X \to Y$ has property P.

Definition 4.31. A stack $\mathcal{X}$ over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ is an **algebraic** S**-stack**, respectively a **Deligne-Mumford** S**-stack**, if the diagonal 1-morphism is representable by algebraic spaces and if there exists an S-scheme X and a 1-morphism $\eta : \widetilde{h}_X \to \mathcal{X}$ (necessarily representable by algebraic spaces) that is faithfully flat and smooth, resp. that is faithfully flat and étale.