

Topological Methods in the Quest for Periodic Orbits

Lecture Notes
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Joa Weber
Unicamp

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To the cycles of life
unimaginable in variety
stunning surprises

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