A new operation on differential forms

Une nouvelle opération sur les formes différentielles

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C. R. Acad. Sci. Paris, 244 (1957), 426-428.

Abstract

Generalization of certain calculus of Jacobson[1] and Tate[2] regrading on differential forms in characteristic $p \neq 0$. Application to the theory of algebraic curves and abelian varieties.

1. We will denote by K a commutative algebra with unity over a field k of characteristic $p \neq 0$. We refer to the Cartan-Chevalley Seminar[3] for the definition of the module $\Omega^1(K)$ of the k-differentials of K of degree 1 and we will denote by $\Omega^*(K)$ the exterior algebra of the K-module $\Omega^1(K)$. In the ring $\Omega^*(K)$, we define in the usual way a derivation d of degree 1 with square zero extending the map $x \to dx$ from K to $\Omega^1(K)$. We denote by $H^*(K)$ the homology of the complex $(\Omega^*(K), d)$.

2. Since K has characteristic $p \neq 0$, one can define on the set $W_m(K)$ of systems $(x_0, \ldots, x_{m-1})$ of elements of K the structure of a commutative ring by means of the polynomial formula of Witt[4]; we can define a homomorphism F from $W_m(K)$ into itself, a homomorphism R_m from $W_m(K)$ into $W_{m-1}(K)$ and an additive map V_m from $W_m(K)$ into $W_{m+1}(K)$ by the formulae

$$F(x_0, \ldots, x_{m-1}) = (x_0^p, \ldots, x_{m-1}^p), \tag{1}$$

$$R_m(x_0, \ldots, x_{m-1}) = (x_0, \ldots, x_{m-2}), \tag{2}$$

$$V_m(x_0, \ldots, x_{m-1}) = (0, x_0, \ldots, x_{m-1}). \tag{3}$$

The differential $\partial\mathbf{x}$ of the element $\mathbf{x} = (x_0, \ldots, x_{m-1})$ will be the element $\sum_{i=0}^{m-1} x^{p^{m-i-1}-1} dx_i$ of $\Omega^1(K)$. The map $\mathbf{x} \to \partial\mathbf{x}$ is additive and we have the formula

$$\partial(\mathbf{x}.\mathbf{y}) = x_0^{p^{m-1}}.\partial\mathbf{y} + \partial\mathbf{x}.y_0^{p^{m-1}}. \tag{4}$$

3. If we take $m = 2$ in the above, and if we take into the account of the universal definition of $\Omega^*(K)$, we see that there exists a homomorphism φ_1 from the ring $\Omega^*(K)$ into the ring $H^*(K)$ which associates to x and dx respectively the cohomology class of x^p and $x^{p-1}dx$.

Theorem 1. If k is contained in the subring K^p of K formed by x^p with $x \in K$, and if the ring K has a p-base (i.e. a family of elements c_i such that the monomials $\prod_i c_i^{\alpha_i}$ with $0 \leq \alpha_i < p$ form a basis of the K^p-module K), then the homomorphism φ_1 is a bijection from $\Omega^*(K)$ onto $H^*(K)$.

In what follows, we will limit ourselves to the particular case where we know that there always exists a p-basis in K.

Under these conditions, let $\omega \in \Omega^*(K)$ be such that $d\omega = 0$; we denote by $C(\omega)$ the differential form such that $\varphi_1 C(\omega)$ is the cohomology class of ω. We have the following formulae:

$$
\tag{5}
\begin{cases}
C(\omega+\omega') = C(\omega) + C(\omega') \\
C(x^p\omega) = xC(\omega) \\
C(dx) = 0 \\
C(x^{p-1}dx) = dx \\
C(\frac{dx}{x}) = \frac{dx}{x} \\
C(\partial\mathbf{x}) = \partial R_m \mathbf{x}
\end{cases}
$$

for $x \in K$, $\mathbf{x} \in W_m(K)$ and $\omega, \omega' \in \Omega^*(K)$. Moreover if D is a linear form on the K-module $\Omega^1(K)$ (in other words a k-derivation of the ring K), we have

$$
\tag{6}
\langle C(\omega), D\rangle^p = \langle \omega, D^p\rangle - D^{p-1}.\langle \omega, D\rangle.
$$

for $\omega \in \Omega^1(K)$ such that $d\omega = 0$.

Theorem 2. For $\omega \in \Omega^1(K)$ to be of the form dx/x with $x \in K$, it is necessary and sufficient that $d\omega = 0$ and $C(\omega) = 0$.

The condition is necessary according to one of the formulae (5). To show the sufficiency, we reduce to the case where K has finite degree over $k(K^p)$; in this case, Theorem 2 easily follows from the following theorem which is the analogue of a known theorem of E. Noether in Galois theory:

Theorem 3. Let K and L be two fields of characteristic $p \neq 0$ and such that $K \supset L \supset K^p$ and $[K : L] < \infty$. Suppose given for any L-derivation D of K an additive operator $\rho(D)$ of a K-vector space V such that

$$
\tag{7}
\rho(xD).v = x.(\rho(D).v),
$$

$$
\tag{8}
\rho(D).xv = Dx.v + x.(\rho(D).v) \quad (x \in K, v \in V)
$$

and so that ρ is a p-Lie ring homomorphism. Under these conditions, any basis of the L-vector space V_0 formed from the elements of V annihilated by all the $\rho(D)$ is a basis of the K-vector space V.

The proof is based on the theory of simple algebras and on the following lemma:

Lemma. If $(D_i)_{1\leq i\leq n}$ is a basis of the K-module of L-derivations of K, then any endomorphism of the L-vector space K can be written uniquely in the form

$$
\sum_{0\leq\alpha_i<p} c_{\alpha_1,\ldots,\alpha_n} D_1^{\alpha_1}\cdots D_n^{\alpha_n} \quad \text{with} \quad c_{\alpha_1,\ldots,\alpha_n} \in K.
$$

4. The applications to algebraic geometry of the above rely on the following theorems:

Theorem 4. Let X be a normal and complete curve defined over an algebraically closed field k of characteristic $p \neq 0$. For any rational differential form ω on X and any $x \in X$, we have

$$
\tag{9}
\operatorname{res}_x\left(C\left(\omega\right)\right) = \left(\operatorname{res}_x \omega\right)^p
$$

Moreover, the k-vector space $\Omega^1(k(X))$[1] is the direct sum of the subspace $\bigcup_{m\geq 0} \partial(W_m(k(X)))$ and the subspace spanned by the df/f with nonzero $f \in k(X)$.

From (9) we deduce a very easy proof of the residue formula.

[1] We denote by $k(X)$ the field of rational functions on the variety X.

Corollary. Let φ be the canonical map of the normal and complete curve X into its Jacobian J and let h be the map of the cohomology group $H^1(X, \mathcal{O}_X)$ of X (with values in the sheaf of local rings), into the space of invariant vector fields on J which is transposed from the map $\omega \to \varphi^{-1}\omega$ on the differential forms[2]. If F is the endomorphism of $H^1(X, \mathcal{O}_X)$ induced from the map $f \to f^p$ on $\mathcal{O}_{X,x}$, we have

$$h(F(a)) = h(a)^p \quad [a \in H^1(X, \mathcal{O}_X)]. \tag{10}$$

In other words, the Hesse-Witt matrix A is that of the map $D \to D^p$ on the Lie algebra of the Jacobian J of X.

Theorem 5. Let X be a normal and complete variety defined over the algebraically closed field k of characteristic $p \neq 0$ and let Q be the space of rational differential forms on X with positive divisor. The additive subgroup G of Q, defined by the conditions $d\omega = 0$ and $C(\omega) = \omega$ is canonically isomorphic to the group of divisor classes of order p on X. Moreover, if Ω is finite dimensional over k and if we have $d\omega = 0$ for all $\omega \in \Omega$, then the space Ω is the direct sum of the subspace spanned by G and the subspace $\Omega \cap \left(\bigcup_{m \geq 0} \partial \left(W_m \left(k \left(X \right) \right) \right) \right)$.

We have an analogous statement with invariant forms when X is a commutative algebraic group, and this extends a result of Barsotti on abelian varieties.

Moreover Theorem 4 shows that if X is a normal and complete curve of genus g and if σ is the rank of the matrix $A.A^p \ldots A^{p^{g-1}}$, under Hasse-Witt[5] notations, there are $p^{n\sigma}$ classes of divisors of order p^n on X, and that $\sigma = g$ if and only if X does not have an exact differential of the first kind.

References

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[4] Ernst Witt. Zyklische körper und algebren der charakteristik p vom grad p^n. struktur diskret bewerteter perfekter körper mit vollkommenem restklassenkörper der charakteristik p. *J. Reine Angew. Math.*, 176:126–140, 1936.

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[2] We put in duality, by means of the residues, the space $H^1(X, \mathcal{O}_X)$ with the space of forms of the first kind on X.