MAT 322/523 Lec 26 W May 1

We have spent a lot of time discussing manifolds. In the last lecture we will briefly talk about topology. How do we distinguish manifolds from each other? Differential forms can in fact help us answer this question.

Poincaré lemma (§39)

$\omega \in \Omega^k(M)$.

- closed : $d\omega = 0$
- exact : $\exists \eta \in \Omega^{k-1}(M) : \omega = d\eta$.

Clearly exactness $\Rightarrow$ closedness:

$$d\omega = d(d\eta) = 0 \quad \text{because } d^2 = 0.$$

When is a closed form exact? The answer turns out to be intimately connected with topology.

<u>Def</u> $A \overset{\text{open}}{\subset} \mathbb{R}^n$, $B \overset{\text{open}}{\subset} \mathbb{R}^m$

$g, h : A \to B$ smooth.

We say that g and h are <u>homotopic</u> if there's a smooth fcn

$$H : A \times [0,1] \to B$$

s.t. $H|_{A \times \{0\}} = g$, $H|_{A \times \{1\}} = h$.

<u>Rmk</u>: We think of the $[0,1]$-factor as being parametrized by time.

$H = H(x,t)$, and at time $t=0$ & $t=1$ resp. we have the smooth maps g and h, resp.

For each t we have a smooth map

$$H_t : A \to B$$
$$x \mapsto H(x,t)$$

Thm: $A \overset{\text{open}}{\subset} \mathbb{R}^n$, $B \overset{\text{open}}{\subset} \mathbb{R}^m$

$g, h : A \to B$ smooth and homotopic.

Then $\exists$ a linear map

$$P : \Omega^{k+1}(B) \longrightarrow \Omega^k(A)$$

such that

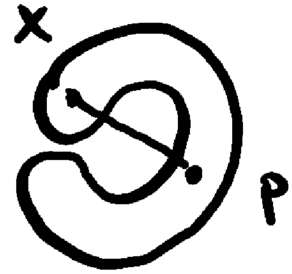
$$\boxed{dP - Pd = h^* - g^*}$$

$$\begin{tikzcd}
\Omega^k(A) \arrow[r, "d"] & \Omega^{k+1}(A) \arrow[r, "d"] & \Omega^{k+2}(A) \\
\Omega^k(B) \arrow[r, "d"] & \Omega^{k+1}(B) \arrow[r, "d"] \arrow[ul, "P"] \arrow[u, "h^*-g^*"] & \Omega^{k+2}(B) \arrow[ul, "P"]
\end{tikzcd}$$

$\boxed{\text{This diagram commutes}}$.

Def: An open set $A \subset \mathbb{R}^n$ is called star-shaped iff $\exists$ point $p \in A$ such that $\forall x \in A$, the straight line connecting x and p belongs to A.

star-shaped



not star-shaped

Thm (Poincaré lemma)

Let $A \overset{\text{open}}{\subset} \mathbb{R}^n$ be star-shaped. If $\omega \in \Omega^k(A)$ is closed, then its exact.

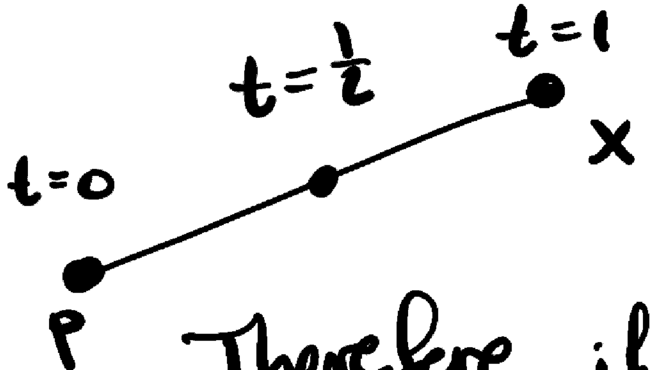
Proof sketch: If A is star-shaped wrt the point $p \in A$ then we first show that the two maps

$$\begin{aligned} id: A &\to A \\ x &\mapsto x \end{aligned} \qquad \begin{aligned} c: A &\to A \\ x &\mapsto p \end{aligned}$$

are homotopic:

$$\begin{aligned} H: A \times [0,1] &\longrightarrow A \\ (x,t) &\longmapsto t\, id(x) + (1-t)c(x) \\ &\qquad = tx + (1-t)p \end{aligned}$$

For all t, this lies on the line segment conn. x and p



Therefore if $f \in \Omega^0(A)$ we have

$$P(df) = id^* f - c^* f = f \circ id - f \circ c$$
$$= f - f(p)$$

so if f closed we get

$$0 = f(x) - f(p) \quad \forall x \in A.$$

$\Rightarrow f$ constant on A.

Next if $\omega \in \Omega^k(A)$ $(k>0)$

$$dP\omega - Pd\omega = id^*\omega - c^*\omega \quad (+)$$

$$id^*\omega = \omega$$
$$c^*\omega = c^*\left(\sum_I \omega_I dx_I\right)$$

$$= \sum_I (\omega_I \circ c)\, dc_I$$

but $dc_I = 0$ because c is the constant function.

If ω closed we get

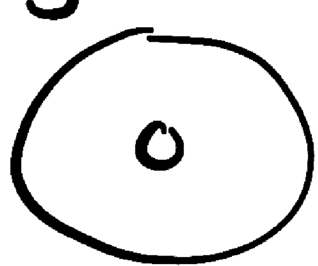
$$d(P\omega) = \omega \quad \text{which}$$

precisely means that ω is exact.

$\square$

<u>Rmk</u> Poincaré's lemma works in greater general than star-shaped open sets. In fact it works for any contractible open set.
(This means diffeomorphic to a ball!)

It fails e.g. for open annuli in $\mathbb{R}^2$



de Rham Cohomology (§40)

We have already seen previously that $d^2 = 0$. For each $k \geq 0$, $\Omega^k(M)$ is a vector space.

Define

$$\Omega^*(M) := \bigoplus_{k=0}^{n} \Omega^k(M)$$

This is also a vector space. We say that $\Omega^*(M)$ is a <u>graded</u> vector space. $\Omega^k(M) \subset \Omega^*(M)$ are the "degree" k forms.

[In fact, equipping $\Omega^*(M)$ with wedge product, defines a graded algebra.]

More importantly, equipping the graded vector space $\Omega^*(M)$ w/ the differential d defines a cochain complex $(\Omega^*(M), d)$

$$\boxed{\Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \cdots}$$

$$\boxed{d_{i+1} \circ d_i = 0}$$

In more algebraic terms $d^2 = 0$ is equivalent to

$\operatorname{im} d_i \subset \ker d_{i+1}$ for all i

<u>Def</u>: If $C^* = \bigoplus_{i=0}^{\infty} C_i$ is a graded vector space, and

$d_i : C_i \longrightarrow C_{i+1} \quad \forall i$

linear map s.t. $d_{i+1} \circ d_i = 0$, the <u>i-th cohomology group</u> is defined as the vector sp

$$\boxed{H^i(C, d) := \ker d_i / \operatorname{im} d_{i-1}}$$

Def: The i-th de Rham cohomology group is defined as

$$H^i(\Omega^*(M), d)$$

$$= H^i_{dR}(M) := \ker d_i / \mathrm{im}\, d_{i-1}$$

$$= \frac{\{\text{closed } i\text{-forms}\}}{\{\text{exact } i\text{-forms}\}}$$

Ex: If $U \overset{\text{open}}{\subset} \mathbb{R}^n$ is star-shaped we have by Poincaré lemma that

- Any closed 0-form is constant (fcn w/ $\frac{df}{dx} = 0$ is const)
- Any closed k-form ($k > 0$) is exact.

$$H^0_{dR}(U) = \{\text{closed } 0\text{-forms}\}$$
$$= \{\text{constant functions}\}$$
$$\cong \mathbb{R} \quad \text{as a vector space}$$
$$H^k_{dR}(U) = \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}} \cong 0$$

Therefore $H^*_{dR}(U) = \begin{cases} \mathbb{R}, & * = 0 \\ 0, & \text{else} \end{cases}$

Recall that if $A \overset{\text{open}}{\subset} \mathbb{R}^n$, $B \overset{\text{open}}{\subset} \mathbb{R}^m$

$g : A \to B$ smooth, then

$g^* : \Omega^k(B) \to \Omega^k(A)$

commutes with d: $g^*(d\omega) = dg^*\omega$

This realizes g^* as a "cochain map" $g^* : \Omega^*(B) \to \Omega^*(A)$. It's a general fact that it can be defined on the de Rham cohomology groups

$g^* : H^*_{dR}(B) \to H^*_{dR}(A)$.

Thm: $A \overset{\text{open}}{\subset} \mathbb{R}^n$, $B \overset{\text{open}}{\subset} \mathbb{R}^m$
$g: A \to B$, $h: B \to A$ smooth
such that $h \circ g$ is homotopic to id_A. Then

$$H^*_{dR}(A) \cong H^*_{dR}(B)$$

Rmk: If g is a diffeomorphism then $h = g^{-1}$ is already equal to id. This leads to:

$$M \underset{\text{diffeo}}{\cong} N \Longrightarrow H^*_{dR}(M) \cong H^*_{dR}(N)$$

or equivalently

$$H^*_{dR}(M) \not\cong H^*_{dR}(N) \Longrightarrow M \not\cong N$$

One of the most powerful computational tools of de Rham cohomology is the Mayer-Vietoris exact sequence.

<u>Def</u>: A sequence

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

of vector spaces is called <u>short exact</u> if f is injective, g is surjective and $\mathrm{im}\, f = \ker g$.

<u>Lma</u>: Any short exact sequence

$$0 \to (U, d_U) \xrightarrow{f} (V, d_V) \xrightarrow{g} (W, d_W) \to 0$$

of cochain complexes (f, g should also commute w/ the relevant differentials) induces an exact sequence

$$\begin{aligned} 0 \to H^0(U) \to H^0(V) \to H^0(W) \\ \to H^1(U) \to H^1(V) \to H^1(W) \\ \to H^2(U) \to \cdots \end{aligned}$$

If $A, B \overset{\text{open}}{\subset} \mathbb{R}^n$ w/ $A \cap B \neq \emptyset$ and $M := A \cup B$, then we have inclusions

$$U \cap V \underset{\imath_V}{\overset{\imath_U}{\rightrightarrows}} U \sqcup V \xrightarrow{\text{incl.}} M$$

<u>Thm</u> (Mayer–Vietoris) There's a short exact sequence:

$$0 \to \Omega^*(M) \xrightarrow{\text{restr.}} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{f} \Omega^*(U \cap V) \to 0$$

where $f(\omega, \tau) = \tau - \omega$.

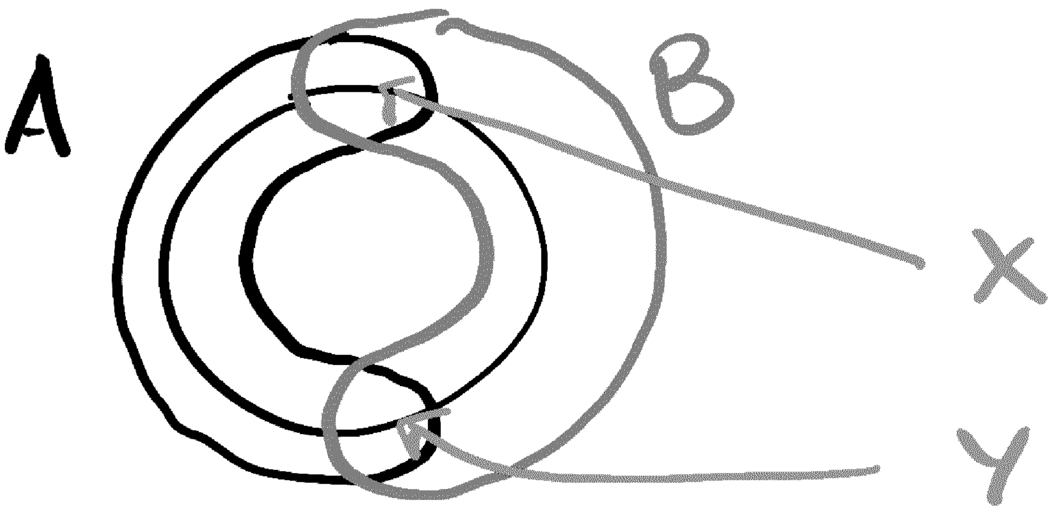
This induces a long exact sequence in homology:

$$0 \to H^0_{dR}(M) \to H^0_{dR}(U) \oplus H^0_{dR}(V) \to H^0_{dR}(U \cap V) \to H^1_{dR}(M) \to \cdots$$

which is a <u>very</u> powerful

Computational tool!

<u>Ex</u>: $M = S^1$ unit circle in $\mathbb{R}^2$.



$$U = A \cap M, \quad V = B \cap M$$

Want to compute $H^*_{dR}(S^1)$.
Using the more general Poincaré lemma we know

$$H^*_{dR}(U) = H^*_{dR}(V) \cong \begin{cases} \mathbb{R}, & \text{else} \\ 0, & \text{else} \end{cases}$$

$$H^*_{dR}(U \cap V) = H^*_{dR}(X \cup Y)$$
$$= H^*_{dR}(X) \oplus H^*_{dR}(Y)$$

fact since $X \cap Y = \emptyset$

$$\cong \begin{cases} \mathbb{R}^2, & * = 0 \\ 0, & \text{else} \end{cases}$$

since each X, Y is an open interval.

$$0 \to H^0_{dR}(S^1) \to \overbrace{H^0_{dR}(U) \oplus H^0_{dR}(V)}^{\cong \mathbb{R}^2}$$

$$\to \underbrace{H^0_{dR}(X) \oplus H^0_{dR}(Y)}_{\cong \mathbb{R}^2} \to H^1_{dR}(S^1)$$

$$\to H^1_{dR}(U) \oplus H^1_{dR}(V) \cong 0$$

$$\to \underbrace{H^1_{dR}(X) \oplus H^1_{dR}(Y)}_{\cong 0} \to 0$$

$$0 \to H^0_{dR}(S^1) \xrightarrow{f} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2$$

$$\xrightarrow{h} H^1_{dR}(S^1) \to 0$$

where f injective, h surjective

$\operatorname{im} f = \ker g$, $\operatorname{im} g = \ker h$

Now by the rank nullity thm.

$$H^0_{dR}(S^1) \cong \ker f \oplus \operatorname{im} f$$
$$\cong \operatorname{im} f \cong \ker g$$

the map g is

$$g : H^0_{dR}(U) \oplus H^0_{dR}(V) \longrightarrow H^0_{dR}(X) \oplus H^0_{dR}(Y)$$
$$(\omega, \tau) \longmapsto (\tau - \omega, \tau - \omega)$$

these are closed, hence constant functions

$$\ker g = \{(\omega, \tau) \in H^0_{dR}(U) \oplus H^0_{dR}(V) \mid \omega = \tau\}$$

is 1-dimensional

$$\Rightarrow \boxed{H^0_{dR}(S^1) \cong \mathbb{R}.}$$

Rank-nullity again:

$$\ker g \oplus \operatorname{im} g \cong \mathbb{R}^2 \Rightarrow \operatorname{im} g \cong \mathbb{R}$$

$$\ker h \oplus \operatorname{im} h \cong \operatorname{im} g \oplus H^1_{dR}(S^1)$$
$$\cong \mathbb{R}^2$$

$$\Rightarrow \boxed{H^1_{dR}(S^1) \cong \mathbb{R}}$$

Can show by induction
(and similar techniques as above)
that

$$H^*_{dR}(S^n) \cong \begin{cases} \mathbb{R}, & * = 0, n \\ 0, & \text{else} \end{cases}$$

So for instance

$\boxed{S^n \text{ is not diffeomorphic to } \mathbb{R}^n}$!

Can also show

$$H^*_{dR}(T^2) \cong \begin{cases} \mathbb{R}, & * = 0, 2 \\ \mathbb{R}^2, & * = 1 \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow \quad T^2 \not\cong S^2$$