MAT 322/523 Lec 25 M Apr 29

Recall:

- M compact oriented k-mfd in $\mathbb{R}^n$
  $\{\alpha_i : U_i \to V_i\}_{i\in I}$ orientation
  $M \subset \bigcup_{i\in I} V_i$. Pick p.o.u. wrt this cover $\{\phi_i\}_{i=1}^N$
  $\omega \in \Omega^k(M)$, then

$$\boxed{\int_M \omega = \sum_{i=1}^N \int_M \phi_i \omega}$$

- Stokes thm:
  M cpt, oriented k-mfd
  ∂M equipped with the induced ori (coming from the "outward normal first"-rule)
  $\omega \in \Omega^{k-1}(M)$

$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

Recall $I^k = [0,1]^k \subset \mathbb{R}^k$

$\mathrm{int}\, I^k = (0,1)^k$

$\partial I^k = I^k \setminus \mathrm{int}\, I^k$

Lma $\boxed{k>1}$

Let $I^k \subset U \overset{\text{open}}{\subset} \mathbb{R}^k$, $\eta \in \Omega^{k-1}(U)$

Assume $\eta(x) = 0$ for $x \in \partial I^k$ unless $x \in \mathrm{int}\, I^{k-1} \times \{0\}$.

$$\int_{\mathrm{int}\, I^k} d\eta = (-1)^k \int_{\mathrm{int}\, I^{k-1}} b^* \eta$$

$$\boxed{\begin{aligned} b : I^{k-1} &\longrightarrow I^k \\ (x_1, \dots, x_{k-1}) &\longmapsto (x_1, \dots, x_{k-1}, 0) \end{aligned}}$$

Proof: Step 1: $\int_{\mathrm{int}\, I^k} d\eta = (-1)^k \int_{I^{k-1}} f \circ b$

Claim:

$$(-1)^k \int_{I^{k-1}} f \circ b = (-1)^k \int_{I^{k-1}} b^* \eta$$

Proof of claim:

$$b^*\eta = b^*(f \, dx_{I_j})$$

$$= b^*(f \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k)$$

$$= (f \circ b)(\det Db_{I_j}) \, db_1 \wedge \cdots \wedge \widehat{db_j} \wedge \cdots \wedge db_k$$

$$Db = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{pmatrix}$$

So $\det Db_{I_j} = \begin{cases} 0 & j < k \\ 1 & j = k. \end{cases}$

$$\Rightarrow b^*\eta = (f \circ b) \, db_1 \wedge \cdots \wedge \widehat{db_k}$$

$$= (f \circ b) \, du_1 \wedge \cdots \wedge du_{k-1}$$

$\square$

Thm $\boxed{k>1}$ M cpt oriented k-mfd in $\mathbb{R}^n$. Assume ∂M is given the induced orientation if its non-empty.

$$M \subset U \overset{\text{open}}{\subset} \mathbb{R}^n, \quad \omega \in \Omega^k(U)$$

$$\int_M d\omega = \int_{\partial M} \omega \quad \text{if } \partial M \neq \emptyset$$

$$\left(\int_M d\omega = 0 \quad \text{if } \partial M = \emptyset \right)$$

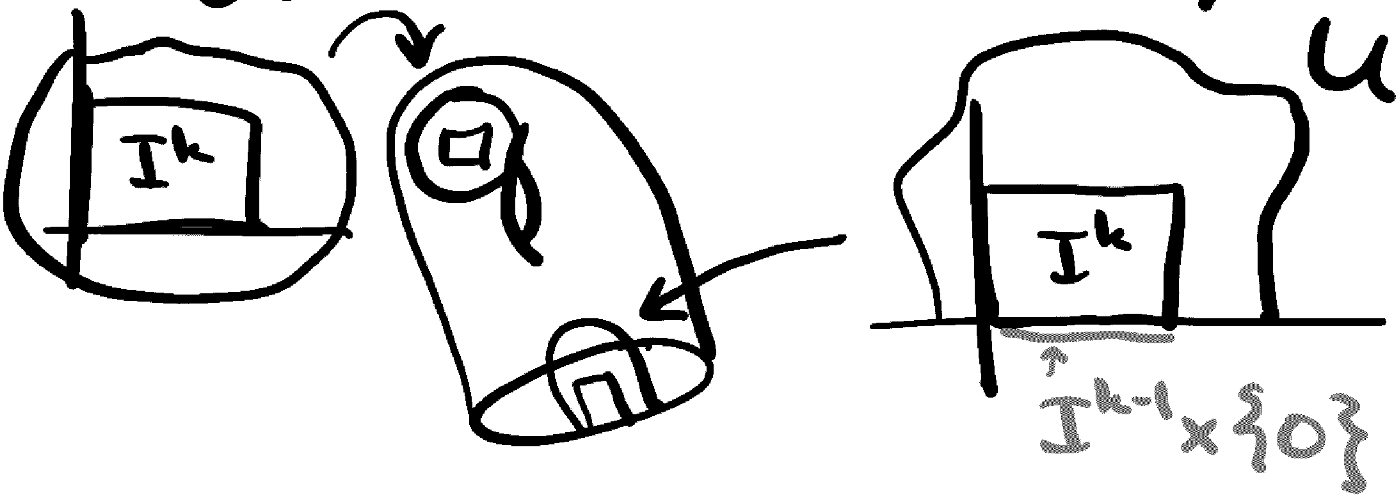
The essence of the theorem was already seen when we proved it for cubes. We sketch the general proof.

Proof sketch:

① Possibly after rescaling/translation we can assume that every coord chart

$\alpha : U \to V$ is so that

- $I^k \subset U$ and
- $I^{k-1} \times \{0\} \subset U \cap \partial \mathbb{H}^k$ (if $U \overset{\text{open}}{\subset} \mathbb{H}^k$)



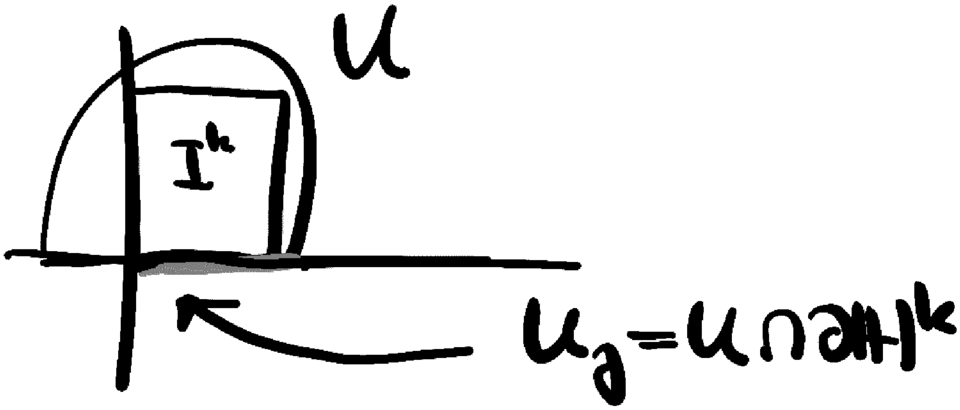
By linearity of integrals, d and pullbacks, it suffices again to prove it for $\omega = f\,dx_{J_j}$.

For a chart $u : U \to V$ around $x \in \partial M$

$$\int_{U,\alpha} d\omega = \int_{\text{int } I^k} d(f\,dx_{J_j})$$

$$= (-1)^k \int_{\text{int } I^{k-1}} b^*(f\,dx_{J_j})$$

Previous thm



$I^{k-1} \subset U_\partial \overset{\text{open}}{\subset} \mathbb{R}^{k-1}$.

Let $\beta : U_\partial \longrightarrow V_\partial$, $V_\partial = V \cap \partial M$ be the restricted coord chart. Now, by def β belongs to the induced ori on ∂M if k is even; else if k is odd we have to take the opposite ori. This leads to

$$\int_{U_\partial, \beta} \omega = (-1)^k \int_{\operatorname{int} I^{k-1}} \beta^* \omega = \int_{U_\alpha} d\omega$$

Ori reversal due to dim

Previous calculation

Now patch everything together with a partition of unity

$$\rightsquigarrow \int_M d\omega = \int_{\partial M} \omega$$

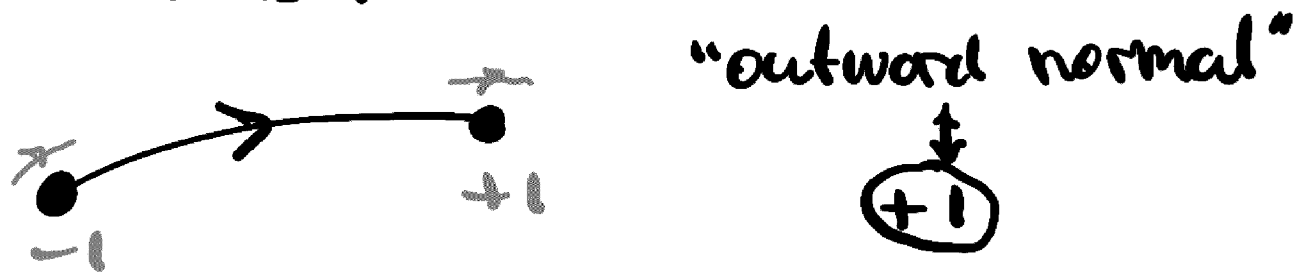
$\square$

<u>Rmk:</u> • We excluded the $k=1$ case for simplicity, because we have not discussed orientations on 0-manifolds (= discrete set of points)

• An orientation on a 0-mfd M is a function

$$\varepsilon : M \to \{\pm 1\}.$$

• If M is an oriented 1-mfd with boundary, then the induced orientation is as follows:



Vector calculus

Thm (Divergence theorem)

Let M be a compact n-mfld w/ ∂ in $\mathbb{R}^n$. Suppose $M \subset U \overset{\text{open}}{\subset} \mathbb{R}^n$ and G is a v.f. on U. Let N be the outwards pointing normal vector field to M, along ∂M. Then

$$\int_M \operatorname{div} G \, dV = \int_{\partial M} \langle G, N \rangle \, dV$$

Proof: Given the v.f. G we def an $(n-1)$-form by

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} G_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

components of G

This is $\beta_{n-1} G$ where

$$\beta_{n-1} : \text{vector fields} \longrightarrow (n-1)\text{-forms}$$

that we saw in L21

Then wlog assume ∂M is covered by a single coord chart $\alpha : \mathcal{U} \to U$ (else use a partition of unity + linearity)

$$\int_{\partial M} \omega = \int_{\mathcal{U}} \alpha^* \omega$$

$$= \sum_{i=1}^{n} (-1)^{i+1} \int_{\mathcal{U}} (G_i \circ \alpha) \det \frac{\partial(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_n)}{\partial(x_1, \ldots, \hat{x}_i, \ldots, x_n)} (t)$$

The rest is linear algebra:

If $v_1, \ldots, v_{n-1}$ are lin indep vectors in $\mathbb{R}^n$ let $i \in \{1, \ldots, n-1\}$ & def

$$X_i := [v_1 \cdots v_{n-1}] \text{ w/ row } i \text{ removed}$$

Then $C = \sum_{i=1}^{n} c_i e_i$

$$c_i = (-1)^{i-1} \det X_i$$

is s.t. $(C, v_1, \ldots, v_{n-1})$ is a right-handed basis of $\mathbb{R}^n$, and

$$\|C\| = V([v_1 \cdots v_{n-1}])$$

Back to (+) we have that

$$C = \sum_{i=1}^{n} (-1)^{i+1} \det \frac{\partial(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_n)}{\partial(x_1, \dots, \hat{x}_i, \dots, x_n)} \partial_i$$

is the outwards normal. So

$$N(\alpha(x)) = \left(\alpha(x), \frac{C(x)}{\|C(x)\|}\right)$$

$$\int_{\partial M, \alpha} \langle G, N \rangle \, dV = \int_U \langle G \circ \alpha, N \circ \alpha \rangle V(D\alpha)$$

$$= \int_U \left\langle G \circ \alpha, \frac{C}{\|C\|} \right\rangle \|C\| = \int_U \langle G \circ \alpha, C \rangle$$

$$= \int_U \sum_{i=1}^{n} (-1)^{i+1} (G_i \circ \alpha) \det \frac{\partial(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_n)}{\partial(x_1, \dots, \hat{x}_i, \dots, x_n)}$$

Next by Stokes thm

$\int_{\partial M} \omega = \int_M d\omega$, and we know from a previous calculation (see L21) that

$$d\omega = (\operatorname{div} G)\, dx_1 \wedge \cdots \wedge dx_n$$

$\square$