MAT 310 L8 Thu Feb 20

§ Invertability & isomorphisms (3D)

<u>Def</u>: A linear map $T \in \mathcal{L}(V,W)$ is called <u>invertible</u> if $\exists S \in \mathcal{L}(W,V)$ such that

$$ST = I_V \quad \text{and} \quad TS = I_W$$

identity maps on V and W, resp

S is called an <u>inverse</u> of T (by symmetry T is an inverse of S)

<u>Prop</u>: If an inverse to a linear map exists, it must be unique.

<u>Proof</u>: Let $T \in \mathcal{L}(V,W)$ and assume $S_1, S_2 \in \mathcal{L}(W,V)$ are inverses:

$$TS_i = I_W \quad S_iT = I_V \quad i = 1,2.$$

Then

$$S_1 = S_1 \cdot I_V = S_1(TS_2) = (S_1T)S_2$$
$$= I_W S_2 = S_2 \qquad \square$$

If $T \in \mathcal{L}(V,W)$ has an inverse we know it's unique, and so we will denote it by $T^{-1} \in \mathcal{L}(W,V)$.

Theorem: $T \in \mathcal{L}(V,W)$ is invertible if and only if T is injective & surjective.

Proof: $\Longrightarrow$: Assume $T^{-1} \in \mathcal{L}(W,V)$ exists. Then we first show injectivity. It suffices to show null $T = \{0\}$. Therefore assume $v \in$ null T. Because T^{-1} is linear $T^{-1}(0) = 0$. Therefore

$0 = T^{-1}(0) = T^{-1}(T(v)) = I_V(v) = v$

$\Rightarrow$ null $T = \{0\}$.

Next we show surjectivity, i.e. range $T = W$. Let $w \in W$, then

$w = I_W(w) = T(T^{-1}(w)) = T(v)$

So $w \in$ range T for any $w \in W$, and so T must be surjective.

$\Leftarrow$: Assume null $T = \{0\}$ and range $T = W$. For any $w \in W$ it means $\exists v \in V : w = T(v)$, moreover this v is <u>unique</u> because

$T(v_1) = w = T(v_2) \Leftrightarrow T(v_1 - v_2) = 0$

$\Rightarrow v_1 = v_2$ by injectivity.

Define $S: W \to V$ as $S(w) = v$ where v is as above. Have to show

① $ST = I_V$ and $TS = I_W$

② S is linear

① By def $w = T(S(w))$ so $TS = I$.

$S(T(v)) =$ unique element $\bar{v} \in V : T(\bar{v}) = T(v)$

so must have $\bar{v} = v \Rightarrow ST(v) = v$.

②: Let $w_1, w_2 \in W$. Then by def

$S(w_1 + w_2) =$ unique element $v : T(v) = w_1 + w_2$.

$S(w_1) =$ — " ———— $v_1 : T(v_1) = w_1$

$S(w_2) =$ — " ———— $v_2 : T(v_2) = w_2$

So

$T(S(w_1) + S(w_2)) = T(v_1 + v_2) = T(v_1) + T(v_2)$

$= T(v) = T(S(w_1 + w_2))$

$\Rightarrow S(w_1 + w_2) = S(w_1) + S(w_2)$ by inj of T. Scalar mult is similar. $\square$

<u>EX</u>: • The linear map

$T: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$, $Tp = x^2 p(x)$

is injective: If $Tp = x^2 p(x) = 0$

for all x we must have $p = 0$.

So $\operatorname{null} T = \{0\}$.

But T is not surjective, since

eg $g(x) = x+1$ is not in range T.

- $\sigma: \mathbb{F}^\infty \to \mathbb{F}^\infty$

  $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$

is surjective: For any $(c_1, c_2, \dots)$

we have e.g.

$(c_1, c_2, \dots) = \sigma(0, c_1, c_2, \dots)$.

σ is not injective: The element

$(1, 0, 0, \dots) \in \operatorname{null} \sigma$ so $\operatorname{null} \sigma \neq \{0\}$.

<u>Prop.</u> If V, W are fin. dim. v. sp.

$\dim V = \dim W$, and $T \in \mathcal{L}(V, W)$.

Then T injective $\Leftrightarrow$ T surjective
$\Leftrightarrow$ T invertible

<u>Proof</u>: By def T invertible $\Leftrightarrow$
T inj + surj. So it suffices to prove the first equivalence

<u>$\Rightarrow$</u>: Assume null $T = \{0\}$
$\Leftrightarrow$ dim null $T = 0$. Then by the fundamental thm of lin maps
dim W = dim range T, & because range $T \subset W$ is a subspace, we must have range $T = W$

<u>$\Leftarrow$</u>: This is similar: Assume range $T = W$ then fund. thm. of lin maps implies dim null $T = 0$
$\Leftrightarrow$ null $T = \{0\}$. $\square$

Prop: If V, W are fin dim vector spaces of the same dim, and $T \in \mathcal{L}(V,W)$, $S \in \mathcal{L}(W,V)$ then

$$ST = I_V \iff TS = I_W.$$

Proof: $\Longrightarrow$: Assume $ST = I_V$. Then if $v \in \text{null } T$ we have

$$v = I_V(v) = ST(v) = S(0) = 0$$

so $\text{null } T = \{0\} \iff T$ inj $\iff T$ invertible w/ inverse $T^{-1} \in \mathcal{L}(W,V)$. Then

$$ST = I_V \iff STT^{-1} = T^{-1}$$
$$\iff S = T^{-1}$$

so $TS = TT^{-1} = I_W$

$\Longleftarrow$: This is similar. $\square$

Def: ① An isomorphism is an invertible linear map.

② Two vector spaces V, W are **isomorphic** ($V \cong W$) if there is an isomorphism $T: V \to W$.

Thm: If V, W are fin dim V sp, then $V \cong W \iff \dim V = \dim W$

Proof: $\Longrightarrow$: Let $T: V \to W$ be an iso. Fund thm of lin maps implies

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$
$$= 0 + \dim W = \dim W$$

$\Longleftarrow$: Pick bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ of V and W, respectively. Define $T: V \to W$ to be the linear map

$$T(v) = T\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i w_i.$$

Assume $v \in \operatorname{null} T$. Then

$$T(v) = \sum_{i=1}^{n} c_i w_i = 0 \Rightarrow c_i = 0 \ \forall i$$

since $(w_1, \ldots, w_n)$ is lin indep.

$$\Rightarrow v = \sum_{i=1}^{n} c_i v_i = 0 \text{ so } \operatorname{null} T = \{0\}$$

$\Leftrightarrow T$ inj $\Leftrightarrow T$ surj $\Leftrightarrow T$ invertible.

$\square$

Rmk: This result means that any fin dim vector space V is isomorphic to $\mathbb{F}^{\dim V}$.

Recall that if V, W are fin dim, and we have picked bases for V and W, the assignmen

$$T \mapsto \mathcal{M}(T) \quad (= \text{the matrix of } T)$$

defines a linear map

$$\mathcal{M} : \mathcal{L}(V, W) \longrightarrow \mathbb{F}^{m,n}$$

Prop: Assume we have picked bases for V and W, then $\mathcal{M}$ is an isomorphism.

Proof: Recall if $(v_1, \dots, v_n)$ and $(w_1, \dots, w_m)$ are bases for V, W then

$$\mathcal{M}(T) = \begin{pmatrix} | & & | \\ T(v_1) & \cdots & T(v_n) \\ | & & | \end{pmatrix}.$$

Then if $T \in \operatorname{null} \mathcal{M}$, we have $T(v_i) = 0 \ \forall i \in \{1, \dots, n\}$ so for any $v \in V$ write $v = \sum_{i=1}^{n} c_i v_i$. Then

$$T(v) = T\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i T(v_i) = 0$$

so $T =$ the zero map, and hence $\operatorname{null} \mathcal{M} = \{0\}$.

For surjectivity, let $A \in \mathbb{F}^{m,n}$.

We define a linear map $T: V \to W$ by

$T(v_k) = \sum_{j=1}^{m} A_{j,k} w_j$ & we can easily check that this is linear. By constr, we have

$$\mathcal{M}(T) = \begin{pmatrix} | & & | \\ T(v_1) & \cdots & T(v_n) \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = A.$$

$\square$

<u>Prop</u>: Assume V, W are fin dim v.sp. Then $\mathcal{L}(V,W)$ is fin dim &

$\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$

<u>Proof</u>: Previous thm says $\mathcal{L}(V,W) \cong \mathbb{F}^{\dim W, \dim V}$ &

$$\dim \mathbb{F}^{\dim W . \dim V} = (\dim V)(\dim W)$$

$\square$
