<u>Recall</u>: • If $T: \mathbb{F}^n \longrightarrow \mathbb{F}^m$ is defined by

$$T(x_1, \dots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \dots, \sum_{k=1}^{n} A_{m,k} x_k\right)$$

then

$$\text{null } T = \{(x_1, \dots, x_n) \mid T(x_1, \dots, x_n) = 0\}$$

$$= \left\{ (x_1, \dots, x_n) \,\middle|\, \begin{cases} A_{1,1} x_1 + \dots + A_{1,n} x_n = 0 \\ \quad \vdots \qquad\qquad\qquad \vdots \qquad \vdots \\ A_{m,1} x_1 + \dots + A_{m,n} x_n = 0 \end{cases} (*) \right\}$$

T injective $\iff$ null $T = \{0\}$

$\iff$ system of lin equs $(*)$ has no non-zero solutions.

In particular: If there's more variables than equations, $n > m$ so T can not be injective.

$\Rightarrow$ there exists a non-zero solution.

We can also consider the range:

$$\text{range } T = \{T(x_1, \dots, x_n) \mid (x_1, \dots, x_n) \in \mathbb{F}^n\}$$

$(c_1, \dots, c_m) \in \text{range } T$ is equivalent to

$$\begin{cases} A_{1,1}x_1 + \dots + A_{1,n}x_n = c_1 \\ \quad \vdots \qquad\qquad \vdots \qquad \vdots \\ A_{m,1}x_1 + \dots + A_{m,n}x_n = c_m \end{cases} \qquad (**)$$

having a solution.

T surjective $\Leftrightarrow$ range $T = \mathbb{F}^m$

$\Leftrightarrow$ system of lin eqns $(**)$ has a solution for every choice of right hand side.

In particular: If there are more eqns than variables, $m > n$ so T can not be surjective, so there's some $(d_1, \dots, d_m) \in \mathbb{F}^m$ so that

$$\begin{cases} A_{1,1}x_1 + \dots + A_{1,n}x_n = d_1 \\ \quad \vdots \qquad\qquad \vdots \qquad \vdots \\ A_{m,1}x_1 + \dots + A_{m,n}x_n = d_m \end{cases}$$

has <u>no</u> solutions.

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§Matrices (3C)

<u>Def</u>: Let m,n be non-negative integers. An $(m \times n)$-matrix is a rectangular array of elements of $\mathbb{F}$ w/ m rows & n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

$A_{j,k} \in \mathbb{F}$ is the element in row j & column k.

<u>Def</u>: Let V and W be vector spaces with bases $v_1, \dots, v_n$ and $w_1, \dots, w_m$, resp. Let $T \in \mathcal{L}(V,W)$. The matrix of T is the $(m \times n)$-matrix $\mathcal{M}(T)$ with entries $A_{j,k}$ defined by

$$Tv_k = A_{1,k} w_1 + \cdots + A_{m,k} w_m$$

We have already seen the special case of this def for $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$

where $\mathbb{F}^n$ and $\mathbb{F}^m$ are equipped w/ their standard bases. Then

$$\mathcal{M}(T) = \begin{pmatrix} | & & | \\ Tv_1 & \cdots & Tv_n \\ | & & | \end{pmatrix}$$

Ex: $T: \mathbb{F}^2 \longrightarrow \mathbb{F}^3$,

$T(x,y) = (x+3y, 2x+5y, 7x+9y)$ where we pick the standard bases for $\mathbb{F}^2, \mathbb{F}^3$.

$T(1,0) = (1,2,7)$, $T(0,1) = (3,5,9)$

$$\mathcal{M}(T) = \begin{pmatrix} | & | \\ T(1,0) & T(0,1) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$

Ex Consider $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{F}), \mathcal{P}_2(\mathbb{F}))$

$Dp = p'$, where $\mathcal{P}_3(\mathbb{F})$ and $\mathcal{P}_2(\mathbb{F})$ are equipped w/ their standard bases.

Then $D(1) = 0 + 0 \cdot x + 0 \cdot x^2$

$D(x) = 1 + 0 \cdot x + 0 \cdot x^2$

$D(x^2) = 0 + 2 \cdot x + 0 \cdot x^2$

$$D(x^3) = 0 + 0 \cdot x + 3 \cdot x^2$$

So $\mathcal{M}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

Addition & scalar multiplication of matrices are done componentwise:

Let $A = (A_{j,k})_{j,k}$ and $C = (C_{j,k})_{j,k}$ be matrices of the same size. Then

$A + C = (A_{j,k} + C_{j,k})_{j,k}$. Meaning

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} A_{1,1} + C_{1,1} & \cdots & A_{1,n} + C_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} + C_{m,1} & \cdots & C_{m,n} + A_{m,n} \end{pmatrix}.$$

If $\lambda \in \mathbb{F}$ is a scalar, then

$\lambda A = (\lambda A_{j,k})_{j,k}$, meaning

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}$$

Prop: Let $T, S \in \mathcal{L}(V,W)$ and $\lambda \in \mathbb{F}$.

1. $\mathcal{M}(T+S) = \mathcal{M}(T) + \mathcal{M}(S)$
2. $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

We denote the set of $(m \times n)$-matrices by $\mathbb{F}^{m,n}$.

Prop: $\mathbb{F}^{m,n}$ with addition & scalar mult. as above is a vector space of dimension $m \cdot n$.

Proof: Exercise. Show that the matrices $\delta_{j,k} \in \mathbb{F}^{m,n}$ where

$\delta_{j,k}$ has entry 1 in position (j,k) & all other entries $=0$, forms a basis for $\mathbb{F}^{m,n}$! $\square$

<u>Def</u>: If $A \in \mathbb{F}^{m,n}$ and $B \in \mathbb{F}^{n,p}$ then $AB \in \mathbb{F}^{m,p}$ is the matrix w/ entry (j,k) equal to

$$(AB)_{j,k} = \sum_{r=1}^{n} A_{j,r} B_{r,k}.$$

<u>Ex</u>: $$\underset{1\times 2}{\begin{pmatrix} 0 & -1 \end{pmatrix}} \underset{2\times 2}{\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}} = \underset{1\times 2}{\begin{pmatrix} 0 & -1 \end{pmatrix}}$$

<u>Prop</u>: Let $T \in \mathcal{L}(U,V)$, $S \in \mathcal{L}(V,W)$. Then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

remember "product" of lin maps is just composition.

<u>Ex</u>: Matrix multiplication is not commutative:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

<u>Def</u>: Let $A \in \mathbb{F}^{m,n}$.

- The column rank of A is the dimension of the span of the columns.
- The row rank of A is the dimension of the span of the rows.

<u>Def</u>: The transpose of $A \in \mathbb{F}^{m,n}$ is the matrix $A^t \in \mathbb{F}^{n,m}$ def by

$$(A^t)_{j,k} = A_{k,j}.$$

i.e. it's the matrix obtained by exchanging cols/rows.

<u>Prop</u>: Let $A, B \in \mathbb{F}^{m,n}$ and $C \in \mathbb{F}^{n,p}$, then

1. $(A+B)^t = A^t + B^t$

2. $(\lambda A)^t = \lambda A^t$

3. $(AC)^t = C^t A^t$.

Proof: 1. and 2.: Exercise

3. By def

$$\left((AC)^t\right)_{j,k} = (AC)_{k,j} = \sum_{r=1}^{n} A_{k,r} C_{r,j}$$

$$\left(C^t A^t\right)_{j,k} = \sum_{r=1}^{n} (C^t)_{j,r} (A^t)_{r,k} \quad \Box$$

Prop: Let $A \in \mathbb{F}^{m,n}$ and assume A has col. rank $c \geq 1$. Then there exist matrices $C \in \mathbb{F}^{m,c}$, $R \in \mathbb{F}^{c,n}$ such that $A = CR$.

Proof: Let $A_{\cdot,k} \in \mathbb{F}^{m,1}$ denote the k-th column of A.

$$A = \begin{pmatrix} | & & | \\ A_{\cdot,1} & \cdots & A_{\cdot,n} \\ | & & | \end{pmatrix}.$$ Then we

of course know $A_{\bullet,1}, \dots, A_{\bullet,k}$ spans $\mathrm{span}(A_{\bullet,1}, \dots, A_{\bullet,k})$, but they might not be lin indep. We reduce it to a basis $a_1, \dots, a_c$.

Define $C = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_c \\ | & & | \end{pmatrix}$. By def

$A_{\bullet,k} \in \mathrm{span}(a_1, \dots, a_c)$,

$A_{\bullet,k} = \sum_{i=1}^{c} b_i^k a_i$. Let R be the matrix with $R_{\bullet,k} = \begin{pmatrix} b_1^k \\ \vdots \\ b_c^k \end{pmatrix}$.

$$CR = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_c \\ | & & | \end{pmatrix} \begin{pmatrix} b_1^1 & \cdots & b_1^n \\ \vdots & \ddots & \vdots \\ b_c^1 & \cdots & b_c^n \end{pmatrix}$$

and one can check that $(CR)_{\bullet,k} = A_{\bullet,k}$ so $CR = A$. $\square$

<u>Prop</u>: Let $A \in \mathbb{F}^{m,n}$. Then col rank = row rank

<u>Proof</u>: Write $A = CR$ as in the prop above, where $C \in \mathbb{F}^{m,c}$, $R \in \mathbb{F}^{c,n}$.
By 3.51(b) in the textbook, each row of A is a linear comb of the rows of R, and because R has c rows we must have row rank(A) $\leq c =$ col rank(A). Also note

col rank$(A) =$ row rank(A^t). So

col rank$(A) =$ row rank(A^t)
$\leq$ col rank$(A^t) =$ row rank(A)
$\Rightarrow$ col rank$(A) =$ row rank(A) $\square$

<u>Def</u>: $A \in \mathbb{F}^{m,n}$

rank$(A) =$ col rank$(A) =$ row rank(A).