MAT 310 L6 Thu Feb 13

Recall: • $\mathcal{L}(V,W) = \{T: V \to W \text{ linear}\}$

• For $T \in \mathcal{L}(V,W)$,
$\text{null } T = \{v \in V \mid Tv = 0\} \subset V.$

Prop: null $T \subset V$ is a subspace.

Proof: We obviously have $0 \in \text{null } T$ since we proved last time $T(0) = 0$.
If $v, w \in \text{null } T$, then

$$T(v+w) = Tv + Tw = 0 + 0 = 0$$

so $v+w \in \text{null } T$. If $\lambda \in \mathbb{F}$ then

$$T(\lambda v) = \lambda Tv = \lambda \cdot 0 = 0,$$

so $\lambda v \in \text{null } T$. This shows

- $0 \in \text{null } T$
- null T closed under addition & scalar multiplication

so it's a subspace. $\square$

Def: A function $T: V \to W$ is **injective** if $Tv = Tw \Rightarrow v = w$.

Prop: Let $T \in \mathcal{L}(V, W)$. Then T is injective $\Leftrightarrow$ null $T = \{0\}$.

Proof: $\Rightarrow$: Assume T is injective and let $v \in$ null T be arbitrary. We need to show $v = 0$.

$$T(v) = 0 = T(0)$$

so injectivity implies $v = 0$.

$\Leftarrow$: Assume null $T = \{0\}$ and $v, w \in V$ are such that $Tv = Tw$.

Then $T(v) - T(w) = 0$

$$\Leftrightarrow T(v) + T(-w) = 0$$
$$\Leftrightarrow T(v - w) = 0$$

So $v - w \in$ null $T = \{0\} \Rightarrow v - w = 0$
$\Leftrightarrow v = w$, so T is injective. $\square$

Def: Let $T \in \mathcal{L}(V,W)$. The range of T is defined as

$$\text{range } T = \{Tv \mid v \in V\} \subset W.$$

Rem: More commonly, the range is called the "image", and denoted by $\operatorname{im} T$.

Ex:
- Let $0 \in \mathcal{L}(V,W)$ be the zero map. Then $\text{range } 0 = \{0\}$.
- Let $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x,y) = (2x, 5y, x+y)$.

$$\text{range } T = \{(2x, 5y, x+y) \mid (x,y) \in \mathbb{R}^2\}$$

- Recall $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ differentiation: $Dp = p'$. Then

$$\text{range } D = \{p' \mid p \in \mathcal{P}(\mathbb{R})\}$$

By integration, we see that any polynomial is the derivative of some other polynomial, so $\text{range } D = \mathcal{P}(\mathbb{R})$.

Prop: Let $T \in \mathcal{L}(V,W)$. Then $\operatorname{range} T \subset W$ is a subspace.

Proof: Since $T(0)=0$, $0 \in \operatorname{range} T$.
If $u,v \in \operatorname{range} T$ it means
$u = T(u')$, $v = T(v')$ for some $u',v' \in V$. Therefore
$$u+v = T(u') + T(v') = T(u'+v')$$
so $u+v \in \operatorname{range} T$.
If $\lambda \in \mathbb{F}$ then
$$\lambda u = \lambda T(u') = T(\lambda u')$$
so $\lambda u \in \operatorname{range} T$. As $0 \in \operatorname{range} T$, and $\operatorname{range} T$ is closed under addition & scalar multiplication, $\operatorname{range} T \subset W$ is a subspace. $\square$

Def: $T: V \to W$ is *surjective* if $\operatorname{range} T = W$. I.e., if $\forall w \in W$ $\exists v \in V$: $w = Tv$.

Ex: Surjectivity depends on the target space of a linear map.

Differentiation $Dp = p'$ considered as a linear map $\mathcal{P}_5(\mathbb{R}) \to \mathcal{P}_5(\mathbb{R})$ is not surjective, since any degree 5 polynomial is not the derivative of any polynomial of degree ≤ 5.

D considered as a linear map $\mathcal{P}_5(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$ is surjective.

Thm (Fundamental theorem of linear maps). Assume V is fin dim, and $T \in \mathcal{L}(V, W)$. Then range T is fin dim, and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Proof: Let $u_1, \dots, u_m \in V$ be a basis of null $T \subset V$.

It can be completed to a basis $u_1, \dots, u_m, v_1, \dots, v_n$ of V.

We will show that $Tv_1, \dots, Tv_n$ is a basis for range T.

<u>Linear independence</u>: Assume

$$0 = \sum_{i=1}^{n} a_i T v_i = T\left(\sum_{i=1}^{n} a_i v_i\right)$$ for some

$a_i \in \mathbb{F}$ $\forall i$. We need to show $a_i = 0$ $\forall i$

The eqn above means $\sum_{i=1}^{n} a_i v_i \in \operatorname{null} T$

So $\exists b_i$: $\sum_{i=1}^{n} a_i v_i = \sum_{j=1}^{m} b_j u_j$ since

$u_1, \dots, u_m$ is a basis for null T.

Therefore $\sum_{j=1}^{m} b_j u_j - \sum_{i=1}^{n} a_i v_i = 0$

$\Rightarrow b_j = 0 = a_i$ $\forall i j$ because

$u_1, \dots, u_m, v_1, \dots, v_n$ is lin indep.

<u>Spans</u>: Let $w \in$ range T. Then by def $\exists v \in V$: $w = Tv$. Write

$$v=\sum_{j=1}^{m} b_j u_j + \sum_{i=1}^{n} a_i v_i$$ & note

$\sum_{j=1}^{m} b_j u_j \in \operatorname{null} T$, so

$$w = Tv = \underbrace{T\left(\sum_{j=1}^{m} b_j u_j\right)}_{=0} + T\left(\sum_{i=1}^{n} a_i v_i\right)$$

$$= \sum_{i=1}^{n} a_i T(v_i),$$ so $w \in \operatorname{span}(Tv_1,\dots,Tv_n)$

so $W = \operatorname{span}(Tv_1,\dots,Tv_n)$

$\square$

<u>Prop</u>: Let V and W be fin dim such that $\dim V > \dim W$. Then no linear map $V \to W$ can be injective.

<u>Proof</u>: Recall that injectivity is equivalent to $\operatorname{null} T = \{0\}$ which is equivalent to $\dim \operatorname{null} T = 0$.

range $T \subset W$ subspace, and W

fin dim $\Rightarrow$ $\dim \operatorname{range} T \leq \dim W$.
Therefore if T is injective

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} W$$

$$= \dim \operatorname{range} W \leq \dim W$$

which contradicts the assumption
$\dim V > \dim W$. $\square$

<u>Prop</u>: Assume that V and W are fin dim, and $\dim V < \dim W$. Then any linear map $V \to W$ can not be surjective.

<u>Proof</u>: Assume $T: V \to W$ is surjective. By the fundamental thm of linear maps

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

$$= \dim \operatorname{null} T + \dim W$$

$$\geq \dim W,$$ which is a contradiction. $\square$

<u>Corollary</u>: Let V, W be fin dim. If $T: V \to W$ is bijective (injective & surjective), then $\dim V = \dim W$.

We have already in previous linear algebra courses studied the relation between linear maps $\mathbb{R}^n \to \mathbb{R}^m$ and systems of linear equations, so let's recall:

Given a homogeneous system of linear equations

$$\begin{cases} A_{1,1}x_1 + \cdots + A_{1,n}x_n = 0 \\ A_{2,1}x_1 + \cdots + A_{2,n}x_n = 0 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ A_{m,1}x_1 + \cdots + A_{m,n}x_n = 0 \end{cases} \qquad (*)$$

We may def a linear map

$T: \mathbb{F}^n \to \mathbb{F}^m$ by

$$(x_1, \dots, x_n) \mapsto \left(\sum_{k=1}^{n} A_{1,k}x_k, \dots, \sum_{k=1}^{n} A_{m,k}x_k \right).$$

The null space of T is precisely the set of solutions to (*). If $\operatorname{null} T$ contains more elements than only 0, it means that (*) has a non-zero solution, which is to say $\dim \operatorname{null} T > 0$
$\Rightarrow T$ not injective.

<u>Prop</u>: A homogeneous system of linear equations w/ more variables than equations has a non-zero solution.

<u>Proof</u>: $\dim \mathbb{F}^n = n = \#$ variables
$\dim \mathbb{F}^m = m = \#$ eqns
If $m < n \Rightarrow T$ not injective by a previous result. $\square$

Similarly we may consider the system of linear equations

$$\begin{cases} \sum_{i=1}^{n} A_{1,k} x_k = c_1 \\ \vdots \\ \sum_{i=1}^{n} A_{m,k} x_k = c_m \end{cases} \quad (**)$$

and we may ask if there exists any solution for a given $(c_1, \dots, c_m) \in \mathbb{F}^m$.

For which $(c_1, \dots, c_m) \in \mathbb{F}^m$ is $(**)$ solvable? In terms of the linear map T: For which $(c_1, \dots, c_m) \in \mathbb{F}^m$ does the eqn $T(x_1, \dots, x_n) = (c_1, \dots, c_m)$ have a solution $(x_1, \dots, x_n) \in \mathbb{F}^n$?

<u>Prop</u>: If there are more equations than variables in a system of linear eqns, then there are some constant terms such that no solution exists.

<u>Proof</u>: $\dim \mathbb{F}^n = n = \#\text{variables}$

$\dim \mathbb{F}^m = m = \#\text{eqns}$

$n > m \Rightarrow T$ not surjective $\Rightarrow \text{range } T \neq W$

So $(c_1, \dots, c_m) \in W \setminus \text{range } T$ is such that there is no Solution to the eqn $T(x_1,\dots,x_n)=(c_1,\dots,c_m)$. $\square$