So far we have studied individual vector spaces & their subspaces. We will now study maps between them.

§ Linear maps (3A)

As before $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and all vector spaces are defined over $\mathbb{F}$ unless stated otherwise.

Def: A <u>linear map</u> from V to W is a function $T: V \to W$ such that:

- (additivity) $T(u+v) = T(u) + T(v)$ $\forall u, v \in V$
- (homogeneity) $T(\lambda u) = \lambda T(u)$ $\forall \lambda \in \mathbb{F}$, $u \in V$.

We will denote the set of linear maps $V \to W$ by $\mathcal{L}(V, W)$. We also write $\mathcal{L}(V) := \mathcal{L}(V, V)$.

Ex: • The constant map

$$\begin{aligned} V &\longrightarrow W \\ v &\longmapsto 0 \end{aligned}$$ that sends every vector to 0

is called the zero map and is sometimes simply denoted by $0 \in \mathcal{L}(V,W)$.

• The linear map $\begin{aligned} V &\longrightarrow V \\ v &\longmapsto v \end{aligned}$ is called the identity map. It's usually denoted by $I \in \mathcal{L}(V)$.

• Define $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ by $Dp = p'$ for any $p \in \mathcal{P}(\mathbb{R})$. We know from calculus that $(p+q)' = p' + q'$ and $(\lambda p)' = \lambda p'$, $\lambda \in \mathbb{R}$, meaning that D is indeed linear.

• Similarly $Tp = \int_0^1 p(x)\,dx$ is a linear map $\mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}$.

• $T : \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$ defined by

$(Tp)(x) = x^2 p(x)$ is linear.

- Recall $\mathbb{F}^\infty = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{F}\ \forall i\}$.

$$T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

is a linear map $\mathbb{F}^\infty \to \mathbb{F}^\infty$.

- $$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$(x,y,z) \longmapsto (2x - y + 3z, 7x + 5y - 6z)$$

is linear.

- $$\mathbb{R}^2 \longrightarrow \mathbb{R}$$

$(x,y) \longmapsto x^2 + y^2$ is <u>not</u> linear

- Fix $q \in \mathcal{P}(\mathbb{R})$. Then

$$\mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$$

$$p \longmapsto p \circ q$$

is linear.

<u>Prop</u> (linear map lemma)

Suppose $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are bases for V and W, respectively. Then there exists a <u>unique</u> linear map $T: V \to W$ such that $Tv_k = w_k \ \forall k$

<u>Proof</u>: <u>Existence</u>: For any $v \in V$ write $v = \sum_{i=1}^{n} c_i v_i$ & <u>define</u>

$$Tv = \sum_{i=1}^{n} c_i w_i .$$

We then have $Tv_k = w_k$ by def for any $k \in \{1, \ldots, n\}$. To show T is linear, pick $u \in V$ & write $u = \sum_{i=1}^{n} b_i v_i$. Then

$$u + v = \sum_{i=1}^{n} b_i v_i + \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} (b_i + c_i) v_i$$

so

$$T(u+v) = \sum_{i=1}^{n} (b_i + c_i) w_i = \sum_{i=1}^{n} b_i w_i + \sum_{i=1}^{n} c_i w_i$$

$$= Tu + Tv.$$

If $\lambda \in \mathbb{F}$ then

$$\lambda v = \lambda \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} (\lambda c_i) v_i, \text{ so}$$

$$T(\lambda v) = \sum_{i=1}^{n} (\lambda c_i) w_i = \lambda \sum_{i=1}^{n} c_i w_i = \lambda T v.$$

<u>Uniqueness</u>: If $S \in \mathcal{L}(V,W)$ is any other linear map w/ $S(v_k) = w_k$ $\forall k$ we have

$$Tv = \sum_{i=1}^{n} c_i w_i = \sum_{i=1}^{n} c_i S(v_i)$$

$$= \sum_{i=1}^{n} S(c_i v_i) = S\left(\sum_{i=1}^{n} c_i v_i\right) = Sv$$

for any $v \in V$, so $T = S$. $\square$

It turns out that the set of linear maps $\mathcal{L}(V,W)$ is itself a vector space.

<u>Def</u>: Suppose $S, T \in \mathcal{L}(V,W)$. Then $S+T \in \mathcal{L}(V,W)$ and $\lambda T \in \mathcal{L}(U,W)$ for $\lambda \in \mathbb{F}$ are defined by

$$(S+T)(v) = S(v) + T(v)$$
$$(\lambda T)(v) = \lambda T(v). \qquad \forall v \in V.$$

Prop: $\mathcal{L}(V,W)$ is a vector space with the addition & scalar mult defined above. $\lrcorner$

Def: If $T \in \mathcal{L}(U,V)$, $S \in \mathcal{L}(V,W)$ we may define their product $ST \in \mathcal{L}(U,W)$ to be the linear map
$$(ST)(u) = S(Tu).$$

Remark: ST is just usual composition of linear maps. $\lrcorner$

Prop. The product of linear maps satisfy the following properties:

- (associativity) $(T_1 T_2) T_3 = T_1 (T_2 T_3)$ for any linear maps $V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \xrightarrow{T_3} V_4$.
- (identity) $T I_V = I_W T = T$ for any $T \in \mathcal{L}(V,W)$, where $I_V \in \mathcal{L}(V)$, $I_W \in \mathcal{L}(W)$ are the identity maps on V and W,

respectively.

- (distributive properties)

$$(S_1+S_2)T = S_1T + S_2T$$

$$S(T_1+T_2) = ST_1 + ST_2$$

$T, T_1, T_2 \in \mathcal{L}(U,V)$, $S, S_1, S_2 \in \mathcal{L}(V,W)$.

<u>Ex</u> The product of linear maps is not commutative. Recall $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$, $Dp = Dp'$ and $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ $(Tp)(x) = x^2 p(x)$ from before. Then

$$((DT)p)(x) = D(x^2 p(x)) = x^2 p'(x) + 2x p(x)$$

$$(TDp)(x) = (Tp')(x) = x^2 p'(x).$$

<u>Ex</u>: $T, S \in \mathcal{L}(\mathbb{F}^2)$ defined by

$$T(x,y) = (y,x), \quad S(x,y) = (x+y, y)$$

then

$(TS)(x,y) = T(x+y,y) = (y, x+y)$

$(ST)(x,y) = S(y,x) = (y+x, x)$

<u>Prop</u>: If $T \in \mathcal{L}(V,W)$ then $T(0)=0$.

<u>Proof</u>: $T(0) = T(0+0) = T(0)+T(0)$

Add $-T(0)$ to both sides:

$$0 = T(0). \qquad \square$$

<u>§ Null space & range (3B)</u>

<u>Def</u>: Let $T \in \mathcal{L}(V,W)$. The <u>null space</u>, denoted by null T, is the subset of V

$$\text{null } T = \{v \in V \mid Tv = 0\}.$$

<u>Rem.</u>: The more common term of the null space is "kernel", and the notation $\ker T$.

Ex: • Let $0 \in \mathcal{L}(V,W)$ be the zero map. null $0 = V$ since everything maps to 0.

• Let $\varphi \in (\mathbb{R}^3, \mathbb{R})$ be def by
$\varphi(x,y,z) = x+2y+3z$.

null $\varphi = \{(x,y,z) \in \mathbb{R}^3 \mid x+2y+3z = 0\}$

this is a plane in $\mathbb{R}^3$

• Consider $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$. Then

null $D = \{p \in \mathcal{P}(\mathbb{R}) \mid p' = 0\}$

$= \{$constant polynomials$\} = \mathcal{P}_0(\mathbb{R})$.

• $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$, $(Tp)(x) = x^2 p(x)$,

null $T = \{p \in \mathcal{P}(\mathbb{R}) \mid x^2 p(x) = 0 \ \forall x \in \mathbb{R}\}$.

$= \{0\}$

• $T \in \mathcal{L}(\mathbb{F}^\infty)$, $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$

null $T = \{(x_1, x_2, \ldots) \in \mathbb{F}^\infty \mid (x_2, x_3, \ldots) = 0\}$

$= \{(x_1, 0, 0, \ldots) \mid x_1 \in \mathbb{F}\}$.