MAT 310 L4 Tue Feb 6

Recall: • $v_1, \dots, v_n$ lin indep iff

$$a_1 v_1 + \cdots + a_n v_n = 0 \Rightarrow a_i = 0 \ \forall i$$

• $V = \mathrm{span}(v_1, \dots, v_n)$ iff $\forall u \in V$ $\exists b_1, \dots, b_n \in \mathbb{F}$: $u = b_1 v_1 + \cdots + b_n v_n$.

§ Bases

Def: A basis of V is a linearly independent spanning set of vectors

Ex: (a) $(1,0), (0,1)$ is a basis for $\mathbb{F}^2$

(b) $(1,0,\dots,0), \dots, (0,\dots,0,1)$ is a basis for $\mathbb{F}^n$. ("standard basis")

(c) $(1,2), (1,0)$ is a basis for $\mathbb{F}^2$.

(d) $1, z, \dots, z^m$ is a basis (the "standard basis") for $\mathcal{P}_m(\mathbb{F})$.

<u>Prop</u>: A list $v_1, \dots, v_n$ is a basis for V iff $\forall u \in V$ can be written uniquely on the form

$$u = a_1 v_1 + \dots + a_n v_n.$$

<u>Proof</u>: <u>$\Longrightarrow$</u>: Assume $v_1, \dots, v_n$ is a basis. Since it spans every $u \in V$ can be written as

$$u = a_1 v_1 + \dots + a_n v_n.$$

If it could also be written as

$$u = b_1 v_1 + \dots + b_n v_n$$

we have

$$0 = u - u = (a_1 - b_1) v_1 + \dots + (a_n - b_n) v_n$$

$\Rightarrow a_i = b_i \ \forall i$ because of lin indep

<u>$\Longleftarrow$</u>: Since every $u \in V$ has a unique representation

$u = a_1 v_1 + \dots + a_n v_n$ it means $v_1, \dots, v_n$ spans.

In particular we always have
$0 = 0 \cdot v_1 + \cdots + 0 \cdot v_n$ & becuse this is the unique way of representing 0, $v_1, \dots, v_n$ must also be lin indep. $\square$

<u>Prop</u>: Every spanning list can be reduced to a basis for V.

<u>Proof</u>: Let $V = \text{span}(v_1, \dots, v_n)$, and call this spanning list B. Define

$$B' := \{ v_k \mid v_k \notin \text{span}(v_1, \dots, v_{k-1}) \}$$

(Note $\text{span}() = \{0\}$ by def.)

Then by the linear dep lemma from last time $\text{span}\, B = \text{span}\, B' = V$ & B' is now lin indep. Hence B' is a basis. $\square$

<u>Theorem</u>: Every fin dim vector space has a basis.

<u>Proof</u>: By def, every fin dim vector space has a spanning set. A basis is now obtained by reducing it as in the previous proposition. □

We can also construct a basis by extending a lin indep list;

<u>Prop</u>: Every lin indep set of vectors in a fin. dim. vector space can be extended to a basis

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<u>Proof</u>: If $u_1, \dots, u_m$ is linearly indep & $V = \text{span}(w_1, \dots, w_n)$ then $V = \text{span}(u_1, \dots, u_m, w_1, \dots, w_n)$. The

reduction in the above prop now produces a basis for V, and it will not remove any of the u_i's because of lin independence. $\square$

<u>Thm</u>: If V is fin dim and $U \subset V$ is a subspace, then there exists a subspace $W \subset V$ such that $V = U \oplus W$.

<u>Proof</u>: V fin dim $\Rightarrow$ U fin dim. So we pick a basis $u_1, \dots, u_m$ of U. Extend $u_1, \dots, u_m$ to a basis $u_1, \dots, u_m, w_1, \dots, w_n$ for V.

By construction $w_1, \dots, w_n$ is lin indep (else it wouldn't be part of a basis for V).

Define $W = \text{span}(w_1, \dots, w_n)$. We now need to show

$V = U + W$ and $U \cap W = \{0\}$.

Any $v \in V$ can be written as

$$v = \underbrace{a_1u_1 + \cdots + a_mu_m}_{\in U} + \underbrace{b_1w_1 + \cdots + b_nw_n}_{\in W}$$

So $v = u + w \in U + W$ and so $V = U + W$ since $U + W \subset V$ is already known to be a subspace.

Next assume $\exists v \in U \cap W$ then write

$$v = a_1u_1 + \cdots + a_mu_m$$

$$v = b_1w_1 + \cdots + b_nw_n$$

for some $a_i, b_j \in \mathbb{F}$ $\forall i, j$. Then

$$0 = v - v = \sum_{i=1}^{m} a_iu_i - \sum_{j=1}^{n} b_jw_j$$

so $a_i = 0$ $\forall i$ and $b_j = 0$ $\forall j$ by lin indep of $u_1, \ldots, u_m, w_1, \ldots, w_n$

$\Rightarrow v = \{0\}$ so $U \cap W = \{0\}$ $\square$

§ Dimension

Prop: Any two bases of a fin dim vector space have the same length.

Proof: Let B_1 and B_2 be two bases.

B_1 lin indep & B_2 spans

$\Rightarrow$ length $B_1 \leq$ length B_2

But B_2 lin indep & B_1 spans

$\Rightarrow$ length $B_2 \leq$ length B_1

$\Rightarrow$ length $B_1 =$ length B_2 $\square$

Def: The dimension of a fin dim vector space V is defined as

$\dim V =$ length of any basis for V.

Ex: (a) $(1,0,\dots,0),\dots,(0,\dots,0,1)$ Standard basis of $\mathbb{F}^n$ has length n so $\dim \mathbb{F}^n = n$

(b) $1, z, \dots, z^m$ standard basis of $\mathcal{P}_m(\mathbb{F})$ has length $m+1$ so $\dim \mathcal{P}_m(\mathbb{F}) = m+1$.

Prop: If $U \subset V$ is a subspace of a fin dim vector space V, then $\dim U \leq \dim V$.

Proof: A previous result guarantees U is fin dim. Pick bases $u_1,\dots,u_m$ and $v_1,\dots,v_n$ of U and V.

$v_1,\dots,v_n$ spans V & $u_1,\dots,u_m$ is lin independent in V

$\Rightarrow \dim U = m \leq n = \dim V$

$\square$

Prop: V fin dim vector space.

Every lin indep list in V of length $\dim V$ is a basis.

<u>Proof</u>: A lin indep list can be extended to a basis. But if the list is already of length $\dim V$, no vectors are needed to be added & the lin indep list was already a basis to begin with. $\square$

<u>Prop</u>: If V is fin dim and $U \subset V$ such that $\dim U = \dim V$, then $U = V$.

<u>Proof</u>. If $u_1, \ldots, u_m$ is a basis for U, its lin indep in V. & since $m = \dim U = \dim V$, it has the correct length (in V), so it's a basis in V, and

$$V = \operatorname{span}(u_1, \ldots, u_m) = U$$

$\square$

Similar to the above, we also have:

Prop: If V is fin dim, and $V = \text{span}(v_1, \ldots, v_m)$ where $m = \dim V$, then $v_1, \ldots, v_m$ is a basis for V.

Theorem. Let V_1, V_2 be subspaces of a fin dim vector space V. Then

$$\dim V_1 + V_2 = \dim V_1 + \dim V_2 - \dim V_1 \cap V_2$$

Proof: Let $v_1, \ldots, v_m$ be a basis for $V_1 \cap V_2$. This list is lin indep in both V_1 and V_2, so we can find extensions to bases

$v_1, \ldots, v_m, u_1, \ldots, u_n$ of V_1

$v_1, \ldots, v_m, w_1, \ldots, w_\ell$ of V_2

So
$$\begin{aligned} \dim V_1 \cap V_2 &= m \\ \dim V_1 &= m+n \\ \dim V_2 &= m+\ell \end{aligned}$$

We will show $v_1, \ldots, v_m, u_1, \ldots, u_n, w_1, \ldots, w_\ell$ is a basis for $V_1 + V_2$ which would conclude the proof.

It is obvious that

$$V_1+V_2 = \operatorname{span}(v_1,\dots,v_m,u_1,\dots,u_n,w_1,\dots,w_\ell)$$

& so it suffices to show it's lin indep. Assume

$$\sum_{p=1}^{m} a_p v_p + \sum_{q=1}^{n} b_q u_q + \sum_{r=1}^{\ell} c_r w_r = 0. \quad (*)$$

Then

$$\sum_{r=1}^{\ell} c_r w_r = -\sum_{p=1}^{m} a_p v_p - \sum_{q=1}^{n} b_q u_q$$

By assumtion $\sum_{r=1}^{\ell} c_r w_r \in V_2$, but in this eqn RHS $\in V_1$, so

$$\sum_{r=1}^{\ell} c_r w_r \in V_1 \cap V_2 \Rightarrow$$

$$\sum_{r=1}^{\ell} c_r w_r = \sum_{s=1}^{m} d_s v_s \text{ for some } d_s \in \mathbb{F}$$

$$\Leftrightarrow \sum_{s=1}^{m} d_s v_s - \sum_{r=1}^{\ell} c_r w_r = 0$$

implies $d_s = 0 \ \forall s$ and $c_r = 0 \ \forall r$ because $v_1, \dots, v_m, w_1, \dots, w_\ell$ lin indep in V_2. Then (*) becomes

$$\sum_{p=1}^{m} a_p v_p + \sum_{q=1}^{n} b_q u_q = 0$$

Which implies $a_p = 0 \ \forall p$ and $b_q = 0 \ \forall q$ Since $v_1, \dots, v_m, u_1, \dots, u_n$ is lin indep in V_1. This shows $v_1, \dots, v_m, u_1, \dots, u_n, w_1, \dots, w_\ell$ is lin indep in $V_1 + V_2$ & hence a basis.

$\square$